The maximum of $|z^2\wp(z)|$ on the Voronoi cell of a lattice

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1. Introduction

Over a number field the elliptic logarithm method for integral points needs, at each archimedean place, a bound

$$|x(P)|\,|\psi|^2 \le c \tag{1.1}$$

for an elliptic logarithm ψ of a point P. In December 2013, working on an implementation of the method proposed for Sage, Cremona found that the bound used there holds only for points with $|x(P)|$ bounded below, and that over a general number field it was not clear to him that the other points lie in a finite region. He was working, he wrote, on a bound "(which certainly exists)" for this quantity, "valid for all complex points with any embedding" [14]. This note gives such a bound, in a form a program can evaluate.

With Λ the period lattice at the place and $\wp = \wp_\Lambda$, the point with elliptic logarithm z has $x = \wp(z) - b_2/12$ (the curve becomes $Y^2 = 4X^3 - g_2X - g_3$ with $X = x + b_2/12$), so

$$|x(P)|\,|\psi|^2 \le |F(\psi)| + \tfrac{1}{12}|b_2|\,|\psi|^2, \qquad F(z) = F_\Lambda(z) = z^2\wp(z). \tag{1.2}$$

Since F is not Λ-periodic, what (1.1) can say depends on the representative of ψ modulo Λ. Sage's `elliptic_logarithm()`, which uses the method of [2], returns at a complex embedding the representative whose coordinates in the basis `normalised_basis()` lie in $[0,1)$. At a real embedding it returns a real number in $[0,\Omega)$, with $\Omega > 0$ the real period (the first entry of `basis()`), plus half the imaginary period on the bounded real component. Either way the closure of the possible values contains a pole of $\wp$ other than 0, and F is unbounded on them. We take instead a representative nearest to 0, that is, ψ in the closed Voronoi cell $V = V_\Lambda = \{z \in \mathbb{C} : |z| \le |z-\omega| \text{ for all } \omega \in \Lambda\}$. Its only lattice point is 0, and $F(z) \to 1$ as $z \to 0$, so F is holomorphic on V.

Theorem 1.1. *Let $\Lambda \subset \mathbb{C}$ be a lattice, ω_1 a shortest nonzero vector of Λ and ω_2 a shortest vector of $\Lambda \smallsetminus \mathbb{Z}\omega_1$, chosen with $\mathrm{Re}(\omega_2/\omega_1) \ge 0$, and let v be the circumcenter of 0, ω_1, ω_2. Then*

$$\max_{z\in V}|F(z)| = \max\{|F(\tfrac{\omega_1}{2})|,\ |F(\tfrac{\omega_2}{2})|,\ |F(v)|\},$$

and the maximum is $|F(v)|$ when $|\mathrm{Im}(\omega_2/\omega_1)| \ge \frac{5}{2}$. In particular it is $\Gamma(\frac14)^4/(32\pi) = 1.71879\ldots$ for $\mathbb{Z}+\mathbb{Z}i$ and $\Gamma(\frac13)^6/(2^{14/3}\pi^2) = 1.47458\ldots$ for $\mathbb{Z}+\mathbb{Z}e^{\pi i/3}$.

The point v is a vertex of V, and $\omega_1/2$, $\omega_2/2$ are the midpoints of the edges of V that bisect 0 and ω_1, ω_2 (Section 2). For the two special lattices $\wp(v) = 0$, and the values are the classical ones of $|F|$ at a half-period [3, 23.5.3, 23.5.7].

Corollary 1.2. *Let E be an elliptic curve over a number field K, $\sigma\colon K \to \mathbb{C}$ an embedding, real or complex, and (w_1, w_2) the basis `E.period_lattice(σ).normalised_basis()`, so*

that $\tau = w_1/w_2$ *has* $\operatorname{Im}\tau > 0$, $|\operatorname{Re}\tau| \le \frac12$ *and* $|\tau|^2 \ge |\operatorname{Re}\tau|$. *Let* $\varepsilon = 1$ *if* $\operatorname{Re}\tau \ge 0$ *and* $\varepsilon = -1$ *otherwise, let* v *be the circumcenter of* 0, w_2, εw_1, *and set*

$$M_\sigma = \max\{|F(\tfrac{w_2}{2})|,\ |F(\tfrac{w_1}{2})|,\ |F(\tfrac{w_1-\varepsilon w_2}{2})|,\ |F(v)|\}, \qquad R = |v| = \frac{|w_2|\,|\tau|\,|\tau-\varepsilon|}{2\operatorname{Im}\tau},$$

where $F(z) = z^2\wp(z) = z^2\bigl(x(z) + b_2/12\bigr)$, *with* $x(z)$ *the* x*-coordinate of the point with elliptic logarithm* z. *Then every point* $P \ne O$ *of* $E(\mathbb{C})$, *with* ψ *an elliptic logarithm of* P *nearest to* 0, *satisfies*

$$|x(P)|\,|\psi|^2 \le M_\sigma + \frac{|b_2|}{12}R^2. \tag{1.3}$$

Here x, b_2 *and* $E(\mathbb{C})$ *are taken through* σ, *and* $R \le c_8 = \max\{|w_1|, |w_2|, |w_1+w_2|\}$.

Proof. The triangle 0, w_2, εw_1 has no obtuse angle: with $t = \varepsilon\tau$ and $a = \operatorname{Re} t = |\operatorname{Re}\tau|$, the inner products of its sides at 0, at w_2 and at εw_1 are $a \ge 0$, $1-a > 0$ and $|t|^2 - a \ge 0$, in units of $|w_2|^2$. Let p be the vertex where its two shortest sides meet, and ω_1, ω_2 those sides as vectors from p, with $|\omega_1| \le |\omega_2|$. The side facing p is the longest, so $|\omega_2 - \omega_1| \ge |\omega_2|$, which gives $\operatorname{Re}(\omega_2/\omega_1) \le \frac12$; and the angle at p is not obtuse, so $\operatorname{Re}(\omega_2/\omega_1) \ge 0$. Thus ω_1, ω_2 is a basis of Λ with $|\omega_2/\omega_1| \ge 1$ and $0 \le \operatorname{Re}(\omega_2/\omega_1) \le \frac12$, hence a pair as in Theorem 1.1 ($|m\omega_1 + n\omega_2|^2 \ge (m^2 - |mn|)|\omega_1|^2 + n^2|\omega_2|^2$, which is at least $|\omega_1|^2$ always and at least $|\omega_2|^2$ when $n \ne 0$). Up to sign, ω_1 and ω_2 are two of w_2, w_1, $w_1 - \varepsilon w_2$, and the circumcenter of 0, ω_1, ω_2 is $v - p$ with $p \in \Lambda$, so $|v-p| = R$ and $\wp(v-p) = \wp(v)$. By Theorem 1.1 and the evenness of F, $\max_V |F| \le M_\sigma$. Equality holds, as the three half-periods are midpoints of edges of V and v is a vertex of V, by Lemma 2.1 for the pair ω_1, ω_2. Also $|\psi| \le R$ on V by Lemma 2.1; now apply (1.2). The radius R is the largest distance from a point of $\mathbb{C}$ to Λ, and rounding coordinates shows that it is at most $(|w_1| + |w_2|)/2 \le c_8$. □

Sage's reduction stops once $|\tau| \ge 0.999$, so w_2 need not be a shortest period; this is why M_σ has the third half-period, which may be dropped when $|\tau| \ge 1$. The point P need not be integral, and no lower bound on $|x(P)|$ is assumed. A reader who needs only the constant can stop here.

The method rests on Zagier's reduction of linear forms in elliptic logarithms [17]; Stroeker and Tzanakis [13] and, independently, Gebel, Pethő and Zimmer [4] made it an algorithm over $\mathbb{Z}$, which Smart extended to S-integral points [11] and Smart and Stephens to number fields [12]. The bounds of the form (1.1) in [4, Lemma 1], [10, §3.3], [16, Lemma 11.2(i)] and in the code proposed in [14], which follows Thongjunthug's Magma code [15, Appendix A.2], assume that $|x(P)|$ is bounded below; over $\mathbb{Q}$ the other points lie in a bounded interval and are found by a search.

In the normalization of Section 2, $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ with $\tau = a + ib$, $0 \le a \le \frac12$ and $|\tau| \ge 1$; we write $M(\tau)$ for the maximum of $|F|$ on V.

Theorem 1.3. *For every such* τ, $M(\tau) \le U(\tau)$, *where* U *is the elementary function of* a *and* b *displayed in* (3.1). *As* $b \to \infty$, *uniformly in* a, *both* $M(\tau)$ *and* $U(\tau)$ *are* $\frac{\pi^2}{3}R^2(1 + O(e^{-\pi b}))$, *with* $R = |\tau|\,|\tau - 1|/(2b)$.

So no bound uniform in the lattice exists: $M(\tau)$ grows like $\pi^2 b^2/12$. In Corollary 1.2, when $|\tau| \ge 1$, $M_\sigma = M(|\operatorname{Re}\tau| + i\operatorname{Im}\tau) \le U(|\operatorname{Re}\tau| + i\operatorname{Im}\tau)$, and with this in place of M_σ the bound (1.3) needs no value of $\wp$.

In November 2026 Cremona added that bounds for real lattices are enough for $\mathbb{Q}$ and for totally real fields, but that arbitrary number fields need the general case [5]. At a real place Sage over $\mathbb{Q}$ uses the real elliptic logarithm alone, so the bound needed is on a segment. A

real lattice is $\omega_1(\mathbb{Z}+\mathbb{Z}\tau)$, with $\omega_1>0$ generating $\Lambda\cap\mathbb{R}$ and $\tau=iy$ (rectangular) or $\frac12+iy$ (rhombic), $y>0$; for rhombic lattices with $y<\frac12$ the segment $(0,\omega_1/2]$ leaves V.

Theorem 1.4. *Let $\Lambda=\omega_1(\mathbb{Z}+\mathbb{Z}\tau)$ with $\omega_1>0$ and $y>0$, and let $e_1=\wp(\omega_1/2)$.*

(a) *If $\tau=iy$, then $t^2\wp(t)$ is strictly increasing on $(0,\omega_1/2]$, so $1<t^2\wp(t)\le\omega_1^2e_1/4$ there.*

(b) *If $\tau=\frac12+iy$, then $t^2\wp(t)$ is increasing on $(0,\omega_1/2]$ if and only if $y\ge\sqrt{3}/2$, that is, if and only if ω_1 is a shortest nonzero vector of Λ. For every $y>0$,*

$$\sup_{0<t\le\omega_1/2}|t^2\wp(t)|=\max\{1,\ \omega_1^2|e_1|/4\}.$$

The proofs go in four steps.

(1) By the maximum principle $|F|$ is largest on ∂V. As $F(-z)=F(z)$, and $|F(\omega-z)|=|F(z)|$ on the edge that bisects 0 and ω, it suffices to treat the half-edges from $\frac12$, $\frac{\tau}{2}$ and $\frac{\tau-1}{2}$ to a vertex.

(2) Dividing by ω carries the edge of ω into the line $\operatorname{Re}z=\frac12$ for $\omega^{-1}\Lambda$, where the q-expansion of $\wp$ converges; bounding it term by term gives Theorem 1.3.

(3) For Theorem 1.1, on each half-edge $|F|$ is at most the largest of the three values. For $b\ge\frac52$ the leading terms of the expansion dominate, and this is proved by hand. For $b\le\frac52$, enclosures of $|F|^2$ and its derivatives, valid for all shapes in a box, pass on each piece of each half-edge a test on signs of derivatives or a comparison of values.

(4) For real lattices, apart from the sign of g_2 and the sign and size of e_1, we use only $\wp'^2=4\wp^3-g_2\wp-g_3$ on $(0,\omega_1/2]$. When $g_2\ge0$ and $e_1>0$ it makes $\wp^{-1/2}$ concave; for rhombic lattices it also gives $t^2F''-4tF'=6F(F-1)-g_2t^4/2$, which excludes an interior maximum of F above $\max\{1,\omega_1^2|e_1|/4\}$.

Appendix A lists what was computed, the software and versions, and where the programs are deposited [6].

2. The cell and the q-expansion on its edges

Let Λ, ω_1, ω_2 and v be as in Theorem 1.1. Then ω_1,ω_2 is a basis of Λ, and $\tau_0=\omega_2/\omega_1$ has $|\tau_0|\ge1$ and $0\le\operatorname{Re}\tau_0\le\frac12$, the upper bound as $|\omega_2-\omega_1|\ge|\omega_2|$. For $c\in\mathbb{C}^\times$, $\wp_{c\Lambda}(cz)=c^{-2}\wp_\Lambda(z)$, so $F_{c\Lambda}(cz)=F_\Lambda(z)$ and $V_{c\Lambda}=cV_\Lambda$; and complex conjugation, with $\wp_{\bar\Lambda}(\bar z)=\overline{\wp_\Lambda(z)}$, conjugates V and F. Dividing by ω_1, and conjugating if $\operatorname{Im}\tau_0<0$, we may therefore assume that

$$\Lambda=\Lambda_\tau=\mathbb{Z}+\mathbb{Z}\tau,\qquad \tau=a+ib,\quad 0\le a\le\tfrac12,\quad |\tau|\ge1,\quad b>0, \tag{2.1}$$

with $\omega_1=1$ and $\omega_2=\tau$. Then $b\ge\sqrt{3}/2$, and v is the circumcenter of 0, 1, τ:

$$v=\tfrac12+i\eta,\qquad \eta=\frac{b^2-a+a^2}{2b},\qquad R=|v|=|\tau-v|=\frac{|\tau|\,|\tau-1|}{2b}. \tag{2.2}$$

Lemma 2.1. *For τ as in (2.1), V is the convex hull of v, $\tau-v$, $v-1$, $-v$, $v-\tau$, $1-v$, a hexagon inscribed in the circle $|z|=R$, or a rectangle when $a=0$, where $\tau-v=v-1$. Its edges join consecutive vertices and lie on the perpendicular bisectors of 0 and τ, $\tau-1$, -1, $-\tau$, $1-\tau$, 1, with the half-periods as midpoints.*

Proof. The open disc of radius R about v contains no point $m+n\tau$ of Λ: for $n=0,1$ as $|m-\frac12|\ge\frac12$ and $|m+a-\frac12|\ge\frac12-a$, while $|v|^2=\frac14+\eta^2$ and $|\tau-v|^2=(\frac12-a)^2+(b-\eta)^2$ equal R^2; for $n\ge2$ and $n\le-1$ as the imaginary parts differ by at least $2b-\eta$ and $b+\eta$,

and $(2b-\eta)^2-\eta^2 = 2b^2+2a(1-a)$ and $(b+\eta)^2-\eta^2 = b^2+2b\eta$ exceed $\frac14$. So $v\in V$, and as $z\mapsto\tau-z$, $z\mapsto z-1$, $z\mapsto -z$ preserve Λ and $|\tau-v| = |v-1| = R$, the other five points are in V. They lie on $|z|=R$ in the order given, as $|\mathrm{Re}(\tau-v)|\le\frac12$, and consecutive ones are equidistant from 0 and from τ, $\tau-1$, -1, $-\tau$, $1-\tau$, 1 in turn. So their convex hull, which lies in V, is cut out by these six bisectors, and V lies on the side of 0 of each. □

Every vertex is congruent to $\pm v$ and has absolute value R, so $|F| = R^2|\wp(v)|$ at all six. On the bisector of 0 and ω, $|\omega-z| = |z|$ and $\wp(\omega-z) = \wp(z)$, so $z\mapsto\omega-z$, which reverses that edge, preserves $|F|$; and $F(-z) = F(z)$. By the maximum principle,

$$M(\tau) = \max_V |F| = \max_{\omega\in\{1,\tau,\tau-1\}}\ \max_{z\in H_\omega} |F(z)|, \tag{2.3}$$

where $H_1 = [\frac12, v]$, $H_\tau = [\frac\tau2, v]$ and $H_{\tau-1} = [\frac{\tau-1}{2}, v-1]$ are half-edges of V (Figure 1). For $a=0$, $H_{\tau-1}$ is the point $v-1$.

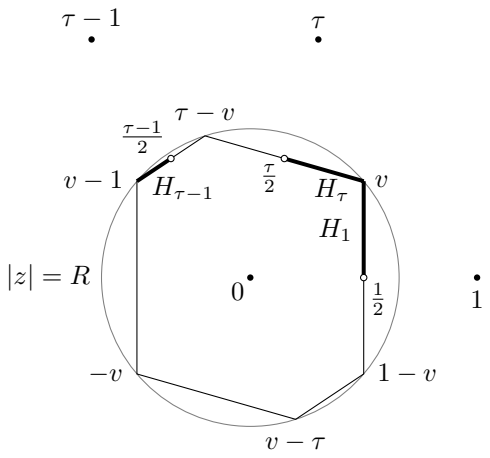


FIGURE 1. The cell V for $\tau = 0.3+1.05i$: its vertices on $|z|=R$, three half-periods at midpoints of edges, and the half-edges H_1, H_τ, $H_{\tau-1}$ (thick).

For $\omega\in\{1,\tau,\tau-1\}$ the lattice $\omega^{-1}\Lambda$ is Λ_{τ_ω}, with $\tau_1=\tau$, $\tau_\tau = -1/\tau$, $\tau_{\tau-1} = 1/(1-\tau)$ and $T_\omega = \mathrm{Im}\,\tau_\omega = b/|\omega|^2$. The map $z\mapsto z/\omega$ takes the bisector of 0 and ω to the line $\mathrm{Re}\,z=\frac12$ and $\omega/2$ to $\frac12$, and (2.2) gives $\mathrm{Im}(v/\tau) = -(1-a)/(2b)$ and $\mathrm{Im}((v-1)/(\tau-1)) = a/(2b)$. Write $\wp_{\tau'}$ for $\wp_{\Lambda_{\tau'}}$ and let $S_1=\eta$, $S_\tau = (1-a)/(2b)$, $S_{\tau-1} = a/(2b)$. Since $\wp_{\tau'}(\frac12 - is) = \wp_{\tau'}(\frac12+is)$, on H_ω

$$|F(z)| = f_\omega(s) = \big(\tfrac14+s^2\big)\big|\wp_{\tau_\omega}(\tfrac12+is)\big|, \qquad 0\le s\le S_\omega, \tag{2.4}$$

with $f_\omega(0) = |F(\omega/2)|$ and $f_\omega(S_\omega) = |F(v)|$. Here $S_\omega < T_\omega$: for $\omega=1$ as $b-\eta = (b^2+a-a^2)/(2b) > 0$, and for the other two as $b^2\ge 1-a^2$ gives $2b|\tau|^2(T_\tau - S_\tau)\ge(1-a)(1+2a)$ and $2b|\tau-1|^2(T_{\tau-1}-S_{\tau-1})\ge 2(1-a)$.

Lemma 2.2. *Let* $\mathrm{Im}\,\tau' = T>0$, $q = e^{2\pi i\tau'}$ *and* $a_k = kq^k/(1-q^k)$. *For* $|\mathrm{Im}\,z|<T$,

$$\wp_{\tau'}(z) = \frac{\pi^2}{\sin^2\pi z} - \frac{\pi^2}{3} + 16\pi^2\sum_{k\ge1} a_k\sin^2\pi kz,$$

and, with $r = e^{-2\pi T}$ *and* $G(x) = x/(1-x)^2 = \sum_{k\ge1} kx^k$,

$$\Big|\wp_{\tau'}(z) - \frac{\pi^2}{\sin^2 \pi z} + \frac{\pi^2}{3}\Big| \le 4\pi^2 D(T, \operatorname{Im} z), \qquad D(T,s) = \frac{G(re^{2\pi s}) + 2G(r) + G(re^{-2\pi s})}{1-r}. \tag{2.5}$$

The function $D(T,\cdot)$ *is even, and increasing in* $|s|$ *on* $(-T, T)$.

Proof. The expansion is [3, 23.8.1] for the lattice with $2\omega_1 = 1$ and $2\omega_3 = \tau'$, rewritten with $\cos 2\pi kz = 1 - 2\sin^2 \pi kz$; the series converges, and the identity holds by evenness, for $|\operatorname{Im}(z/\omega_1)| < 2\operatorname{Im}(\omega_3/\omega_1)$, that is, $|\operatorname{Im} z| < T$. The constant term follows from $\wp(z) - z^{-2} \to 0$ [3, 23.9.2], as $\pi^2/\sin^2 \pi z = z^{-2} + \pi^2/3 + O(z^2)$. For the bound, $|a_k| \le kr^k/(1-r)$ and $|\sin \pi kz|^2 \le \cosh^2(\pi k \operatorname{Im} z) = \frac14(e^{2\pi k \operatorname{Im} z} + 2 + e^{-2\pi k \operatorname{Im} z})$; summing over k gives $4\pi^2 D(T, \operatorname{Im} z)$, the series converging as $re^{2\pi|\operatorname{Im} z|} < 1$. Finally $G(re^{2\pi s}) + G(re^{-2\pi s}) = 2\sum_k kr^k \cosh 2\pi ks$. □

3. The explicit bound

Each half-edge is bounded by Lemma 2.2 in its own frame (2.4). For the edges of τ and $\tau - 1$ the frame of 1 gives a second bound, and we take the smaller. With D as in (2.5) and η, R as in (2.2), set

$$\begin{aligned} B(T,S) &= \pi^2 \max\Big\{\frac16,\ \Big(\frac14 + S^2\Big)\Big|\operatorname{sech}^2 \pi S - \frac13\Big|\Big\} + 4\pi^2\Big(\frac14 + S^2\Big)D(T,S) \qquad (0 \le S < T),\\ C(\tau) &= \pi^2 R^2\Big(\frac13 + \frac{1}{\sinh^2 \pi\eta} + 4D(b, b-\eta)\Big),\\ U(\tau) &= \max\Big\{B(b,\eta),\ \min\Big\{B\Big(\frac{b}{|\tau|^2}, \frac{1-a}{2b}\Big), C(\tau)\Big\},\ \min\Big\{B\Big(\frac{b}{|\tau-1|^2}, \frac{a}{2b}\Big), C(\tau)\Big\}\Big\}, \end{aligned} \tag{3.1}$$

with the last entry of U omitted when $a = 0$.

Lemma 3.1. *Let* $\varphi(s) = (\frac14 + s^2)(\operatorname{sech}^2 \pi s - \frac13)$. *Then* $\max_{0\le s\le S} |\varphi(s)| = \max\{\frac16, |\varphi(S)|\}$ *for every* $S \ge 0$.

Proof. Let $s_0 > 0$ be the zero of φ, so $\cosh^2 \pi s_0 = 3$ and $s_0 = 0.3648\ldots$. On $[s_0, \infty)$, $-\varphi$ is a product of two nonnegative increasing functions. On $(0, s_0]$, using $\operatorname{sech}^2 \le 1$,

$$\varphi'(s) = 2s\Big(\operatorname{sech}^2 \pi s - \frac13\Big) - 2\pi\Big(\frac14 + s^2\Big)\operatorname{sech}^2 \pi s \tanh \pi s \le s\operatorname{sech}^2 \pi s\Big(\frac43 - \frac{\pi}{2}\frac{\tanh \pi s}{s}\Big) < 0,$$

because $\tanh(\pi s)/s \ge \tanh(\pi s_0)/s_0 = \sqrt{2/3}/s_0 > 2.2$ by concavity of tanh. So φ decreases from $\varphi(0) = \frac16$ to 0 on $[0, s_0]$. □

Proof of Theorem 1.3. By (2.3) it suffices to bound $|F|$ on the three half-edges. On H_ω, (2.4) and (2.5) at $z = \frac12 + is$, where $\pi^2/\sin^2 \pi z = \pi^2 \operatorname{sech}^2 \pi s$, give

$$f_\omega(s) \le \pi^2|\varphi(s)| + 4\pi^2(\tfrac14 + s^2)D(T_\omega, s) \le B(T_\omega, S_\omega) \qquad (0 \le s \le S_\omega),$$

by Lemma 3.1 and the monotonicity of D. This is the first entry of U for $\omega = 1$, and the first term of the minimum for $\omega = \tau$ and $\omega = \tau - 1$. The half-edges H_τ and $H_{\tau-1}$ lie on the edges that join v, $\tau - v$ and $v - 1$, so their points satisfy $|z| \le R$ and $\eta \le \operatorname{Im} z \le b - \eta$, where $\eta \le b/2$. Lemma 2.2 for $\tau' = \tau$, with $|\sin \pi z| \ge \sinh(\pi \operatorname{Im} z)$, gives $|\wp(z)| \le \pi^2(\frac13 + \sinh^{-2} \pi\eta + 4D(b, b-\eta))$ there, so $|F(z)| \le C(\tau)$. For $a = 0$ the half-edge $H_{\tau-1}$ is the point $v - 1$, where $|F| = |F(v)| \le B(b,\eta)$. Hence $M(\tau) \le U(\tau)$.

For the asymptotics, $a(1-a) \le \frac14$ gives $b/2 - 1/(8b) \le \eta \le b/2$. So $D(b,\eta)$, $D(b,b-\eta)$, $\sinh^{-2}\pi\eta$ and $\operatorname{sech}^2\pi\eta$ are $O(e^{-\pi b})$ uniformly in a; for instance $e^{-2\pi b}e^{2\pi\eta} \le e^{-\pi b}$. For large b the maximum in $B(b,\eta)$ is its second entry, so $B(b,\eta)$ and $C(\tau)$ are at most $\frac{\pi^2}{3}R^2(1+O(e^{-\pi b}))$, and so is $U(\tau) \le \max\{B(b,\eta), C(\tau)\}$. From below, (2.5) at $z = v$ gives $|\wp(v)| \ge \frac{\pi^2}{3} - O(e^{-\pi b})$, and $U(\tau) \ge M(\tau) \ge |F(v)| = R^2|\wp(v)|$. □

Numerically, with M from Theorem 1.1, $U/M \le 2.09$ on every shape we tried (PARI/GP; the shapes are listed in [6]). The largest value, 2.0875, is at $a \approx 6.9 \cdot 10^{-4}$ on $|\tau| = 1$, where the two terms of the third entry of U are equal, and for $b \ge 8$ the ratio was within 10^{-9} of 1.

4. The maximum

By (2.3) and (2.4), Theorem 1.1 for τ as in (2.1) says that each f_ω is at most $\max\{f_1(0), f_\tau(0), |F(v)|\}$ on $[0, S_\omega]$. We prove this by hand for $b \ge \frac52$, where $|F(v)|$ alone suffices, and by a computation for $b \le \frac52$.

Proposition 4.1. *Let τ be as in (2.1) with $b \ge \frac52$. Then $|F| \le |F(v)|$ on V, so $M(\tau) = |F(v)|$.*

Proof. Let $r_0 = e^{-5\pi}$, $x_1 = e^{-\pi b}$, $x_2 = e^{-2\pi b}$ and $\eta_0 = b/2 - 1/(8b)$, so that $\eta_0 \le \eta \le b/2$. We work in the frame of 1, where $|a_k| \le k x_2^k/(1-r_0)$. On the strip $\eta \le y \le b/2$, $y = \operatorname{Im} z$, which contains H_τ, $H_{\tau-1}$ and v, the bounds $|\sin\pi z| \ge \sinh\pi\eta_0$, $|\sin\pi kz|^2 \le \cosh^2\pi ky$, $|\cos 2\pi kz| \le \cosh 2\pi ky$ and $x_2^k e^{2\pi ky} \le x_1^k$ turn Lemma 2.2 into $|\wp(z) + \pi^2/3| \le \varepsilon_0$ and $|\wp''(z)| \le \varepsilon_2$, where

$$\begin{aligned} \varepsilon_0 &= \frac{\pi^2}{\sinh^2\pi\eta_0} + \kappa, \qquad \kappa = \frac{4\pi^2}{1-r_0}\bigl(G(x_1) + 3G(x_2)\bigr), \\ \varepsilon_2 &= 2\pi^4\Bigl(\frac{3}{\sinh^4\pi\eta_0} + \frac{2}{\sinh^2\pi\eta_0}\Bigr) + \frac{16\pi^4}{1-r_0}\sum_{k\ge1} k^3(x_1^k + x_2^k), \end{aligned} \tag{4.1}$$

using $(\pi^2/\sin^2\pi z)'' = 2\pi^4(3\sin^{-4}\pi z - 2\sin^{-2}\pi z)$ and $(\sin^2\pi kz)'' = 2\pi^2k^2\cos 2\pi kz$. At $b = \frac52$, $\varepsilon_0 \le 0.03637$ and $\varepsilon_2 \le 1.43904$.

Along H_ω, $\omega \in \{\tau, \tau-1\}$, let t be the distance from $\omega/2$. Then $|z|^2 = |\omega|^2/4 + t^2$, $|\wp'(z)| \le t\varepsilon_2$ as $\wp'(\omega/2) = 0$, and $|z|^2 \le R^2 \le (1+b^2)/4$, so $g = |F|^2 = |z|^4|\wp|^2$ satisfies

$$\frac{dg}{dt} \ge 4t|z|^2|\wp|^2 - 2|z|^4|\wp|\,|\wp'| \ge 2t|z|^2|\wp|\Bigl(2\Bigl(\frac{\pi^2}{3} - \varepsilon_0\Bigr) - \frac{1+b^2}{4}\,\varepsilon_2\Bigr),$$

and the bracket is at least 3.8987 at $b = \frac52$. So $|F|$ increases along H_ω towards the vertex.

On H_1 let $z = \frac12 + is$, $0 \le s \le \eta$, $P(s) = \wp(z)$ and $P_0(s) = \pi^2(\operatorname{sech}^2\pi s - \frac13)$. The same estimates, with $|\sin 2\pi kz| = \sinh 2\pi ks$ on $\operatorname{Re} z = \frac12$, give $|P - P_0| \le \kappa$ and $|P' - P_0'| \le \kappa_1 = 8\pi^3(1-r_0)^{-1}\sum_k k^2x_1^k$ for $0 \le s \le b/2$. For $s \le \frac45$, Lemma 3.1 and $|\varphi(\frac45)| \le 0.2737 < \frac{89}{300}$ give

$$|F(z)| \le \pi^2|\varphi(s)| + (\tfrac14 + s^2)\kappa \le \frac{89}{100}\Bigl(\frac{\pi^2}{3} + \kappa\Bigr) \le 2.9417.$$

For $\frac45 \le s \le \eta$, $g = (\frac14 + s^2)^2|P|^2$ has $g'/g = 4s/(\frac14 + s^2) + 2\operatorname{Re}(P'/P)$, so $g' > 0$ where $2|P| > (s + \frac{1}{4s})|P'|$. Here $|P| \ge \frac{\pi^2}{3} - \pi^2\operatorname{sech}^2\frac{4\pi}{5} - \kappa$, and $(s+\frac{1}{4s})|P'| \le 2\pi^3 h(\frac45) + (\frac{b}{2} + \frac{5}{16})\kappa_1$ with $h(s) = (s + \frac{1}{4s})\operatorname{sech}^2\pi s\tanh\pi s$. Indeed h decreases on $[\frac45, \infty)$: its logarithmic derivative is at most $2s/(\frac14 + s^2) - 2\pi\tanh\pi s + 2\pi/\sinh 2\pi s$, which decreases for $s \ge \frac12$ and is -4.32

at $s = \frac{4}{5}$. At $b = \frac{5}{2}$ these bounds give $2|P| - (s + \frac{1}{4s})|P'| \geq 4.1231$, so $|F|$ increases on $[\frac{4}{5}, \eta]$. Finally

$$|F(v)| = R^2|\wp(v)| \geq (\tfrac{1}{4} + \eta_0^2)\Big(\frac{\pi^2}{3} - \varepsilon_0\Big) \geq 5.4984 > 2.9417,$$

so $|F| \leq |F(v)|$ on all three half-edges, and (2.3) gives the claim.

These values were evaluated at $b = \frac{5}{2}$ in ball arithmetic (Appendix A), and each bound is monotone in b the right way: with r_0 fixed, κ, κ_1 and ε_0 decrease, $\frac{1}{4} + \eta_0^2$ increases, each $(\frac{b}{2} + \frac{5}{16})k^2 e^{-\pi k b}$ decreases for $b \geq 1$, and $(1+b^2)\varepsilon_2$ decreases, since $\eta_0' \geq \frac{1}{2}$ and $2b/(1+b^2) \leq 1$ bound the logarithmic derivatives of its terms by $1 - 2\pi$, $1 - \pi$, $1 - \pi k$ and $1 - 2\pi k$. □

For $b \leq \frac{5}{2}$ we use $g_\omega = f_\omega^2$, which is real-analytic and even in s, so $g_\omega'(0) = g_\omega'''(0) = 0$.

Proposition 4.2. *Let τ be as in (2.1) with $b \leq \frac{5}{2}$. Then $|F| \leq \max\{|F(\frac{1}{2})|, |F(\frac{\tau}{2})|, |F(v)|\}$ on V.*

Proof. The proof is a computation in ball arithmetic. The shapes are covered by finitely many boxes $[a_1, a_2] \times [b_1, b_2]$; each S_ω is monotone in a and in b, so its range over a box is read off at two corners. For each box and ω, a fixed subdivision of $[0, 1]$ cuts $[0, S_\omega]$ into pieces, each tested on an interval containing it for every τ in the box, where Lemma 2.2, with an explicit bound for the tail, encloses g_ω and its first four derivatives in s. Each piece passes one of the tests

(i) $g' > 0$; (ii) $g' < 0$; (iii) $g'' > 0$;
(iv) the piece starts at 0 and $g'' < 0$; (v) the piece starts at 0 and $g'''' > 0$;
(vi) $g < \max\{g_1(0), g_\tau(0)\}$, comparing the upper end of the ball for g with the lower ends of the balls for $g_1(0)$ and $g_\tau(0)$ over the box.

Let $s^* \in (0, S_\omega)$ be a local maximum of $g = g_\omega$. Then $g'(s^*) = 0$ and $g''(s^*) \leq 0$, which excludes (i)–(iii). In a piece of type (iv), $g'(s^*) = \int_0^{s^*} g'' < 0$. In a piece of type (v), $\Psi = sg'' - g'$ vanishes at 0 and $\Psi' = sg''' = s\int_0^s g'''' > 0$, so $s^* g''(s^*) = \Psi(s^*) > 0$. So s^* lies in a piece of type (vi), and $g(s^*) < \max\{g_1(0), g_\tau(0)\}$. Hence, by (2.3), the maximum of $|F|^2$ on ∂V is attained at $s = 0$ or $s = S_\omega$ for some ω, and it remains to show $g_{\tau-1}(0) \leq \max\{g_1(0), g_\tau(0), |F(v)|^2\}$. For each box one of the following holds.

(a) $g_{\tau-1}(0) < g_1(0)$ or $g_{\tau-1}(0) < g_\tau(0)$ on the box, by comparing balls.
(b) Every piece of $[0, S_{\tau-1}]$ is of type (i) or (iii). Then $g = g_{\tau-1}$ increases: the piece at 0 is of type (iii), as $g'(0) = 0$, so $g' > 0$ just after 0, and at a first zero s_1 of g' we would have $g''(s_1) \leq 0$, which neither test allows.
(c) $g_{\tau-1}(0) < |F(v)|^2$ on the box, by comparing balls.
(d) The function $\Phi(\alpha) = |F_{\Lambda_{\alpha+ib}}((\alpha + ib)/2)|^2$ has $\partial\Phi/\partial\alpha < 0$ for $a_1 \leq \alpha \leq 1 - a_1$ and $b_1 \leq b \leq b_2$. As $\Lambda_{-\alpha+ib} = \overline{\Lambda_{\alpha+ib}}$, Φ is even in α, and as $\Lambda_{\tau-1} = \Lambda_\tau$, $g_{\tau-1}(0) = \Phi(a-1) = \Phi(1-a) \leq \Phi(a) = g_\tau(0)$.

The computation verifies them on a cover of $\{0 \leq a \leq \frac{1}{2},\ |\tau| \geq 1,\ b \leq \frac{5}{2}\}$ (Appendix A). □

Remark 4.3. At $\tau = i$ the vertex $\frac{1+i}{2}$ is a half-period where $\wp$ has a double zero, so g_1 and g_τ vanish to order four at $s = S_\omega$, and on a box containing i no test (i)–(v) can hold on the last piece of these half-edges. On the boxes within 10^{-7} of i we use test (vi), a comparison of values, which does not decide whether g has an interior local maximum there. We found no proof by hand for $b < \frac{5}{2}$; along the edge of 1, $|F|$ was not convex for any shape we tried.

Proof of Theorem 1.1. By Section 2 we may assume (2.1), where $|\mathrm{Im}(\omega_2/\omega_1)| = b$. Propositions 4.1 and 4.2 give the formula, and Proposition 4.1 gives $M(\tau) = |F(v)|$ for $b \geq \frac{5}{2}$. For $\tau = i$, $iv \equiv v$ modulo $\Lambda = i\Lambda$, so $\wp(v) = \wp(iv) = -\wp(v) = 0$, and $|F(\frac{1}{2})| = |F(\frac{i}{2})| = e_1/4 = \Gamma(\frac{1}{4})^4/(32\pi)$ by [3, 23.5.3] with $\omega_1 = \frac{1}{2}$. For $\tau = \rho = e^{\pi i/3}$, $\rho^2 v \equiv v$ modulo $\Lambda = \rho^2\Lambda$, so $\wp(v) = \rho^{-4}\wp(v) = 0$, and the half-periods $\frac{1}{2}$, $\frac{\rho}{2}$, $\frac{\rho-1}{2}$ have absolute value $\frac{1}{2}$ and $|e_j| = 4\Gamma(\frac{1}{3})^6/(2^{14/3}\pi^2)$ by [3, 23.5.7] with $\omega_1 = \frac{1}{2}$. □

5. Real lattices

In Theorem 1.4 we may take $\omega_1 = 1$, as $F_{c\Lambda}(ct) = F_\Lambda(t)$ for $c > 0$; then $F(t) = t^2\wp(t)$ with $\wp = \wp_\tau$, real on $\mathbb{R}$ as $\bar{\Lambda} = \Lambda$, and $e_1 = \wp(\frac{1}{2})$. The zeros of $\wp'$ are the half-periods modulo Λ [3, 23.2.10], and $\frac{\tau}{2}$, $\frac{1+\tau}{2}$ are not congruent to real numbers. Hence $\wp$ decreases from $+\infty$ to e_1 on $(0, \frac{1}{2}]$, and $F(t) \to 1$ as $t \to 0^+$.

Lemma 5.1. *If $g_2 \geq 0$ and $e_1 > 0$, then $t^2\wp(t)$ is strictly increasing on $(0, \frac{1}{2}]$.*

Proof. As $\wp \geq e_1 > 0$ on $(0, \frac{1}{2}]$, $u = \wp^{-1/2}$ is defined there, and $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, $\wp'' = 6\wp^2 - \frac{1}{2}g_2$ [3, 23.3.10, 23.3.12] give

$$u'' = \frac{3\wp'^2 - 2\wp\wp''}{4\wp^{5/2}} = -\frac{2g_2\wp + 3g_3}{4\wp^{5/2}}.$$

Since $g_2 \geq 0$, $\wp \geq e_1$ and $4e_1^3 = g_2e_1 + g_3$, we get $2g_2\wp + 3g_3 \geq 2g_2e_1 + 3g_3 = 2e_1\wp''(\frac{1}{2})$. As $\wp(1-t) = \wp(t)$, $\wp$ has a local minimum on $\mathbb{R}$ at $\frac{1}{2}$, and $\wp''(\frac{1}{2}) = 2(e_1 - e_2)(e_1 - e_3) \neq 0$, from [3, 23.3.11], as the e_j are distinct. So $\wp''(\frac{1}{2}) > 0$ and $u'' < 0$ on $(0, \frac{1}{2}]$. Thus u is strictly concave with $u(0^+) = 0$, $u(t)/t$ strictly decreases, and $t^2\wp(t) = (t/u(t))^2$ strictly increases. □

Lemma 5.2. *Let $\tau = \frac{1}{2} + iy$ with $y > 0$.*

(a) *$g_2 > 0$ for $y > \sqrt{3}/2$, $g_2 = 0$ for $y = \sqrt{3}/2$, and $g_2 < 0$ for $1/(2\sqrt{3}) < y < \sqrt{3}/2$.*
(b) *If $y \geq \sqrt{3}/2$, then $e_1 \geq 5.8953$. If $y \leq 1/(2\sqrt{3})$, then $e_1 < 0$ and $|e_1|/4 \geq 4.4214$.*

Proof. (a) Since $\bar{\Lambda} = \Lambda$, g_2 is real. Suppose $g_2 = 0$, and let $\zeta = e^{2\pi i/3}$. Then $\zeta\Lambda$ has invariants $\zeta^{-4}g_2 = 0$ and $\zeta^{-6}g_3 = g_3$ [3, 23.3.1, 23.3.2]. The Laurent coefficients of $\wp$ at 0 are polynomials in g_2, g_3 [3, 23.9.2, 23.9.4, 23.9.5], so $\wp_{\zeta\Lambda} = \wp_\Lambda$ and, comparing poles, $\zeta\Lambda = \Lambda$. So for a shortest vector λ of Λ, $\pm\lambda$, $\pm\zeta\lambda$, $\pm\zeta^2\lambda$ are six shortest vectors. But $|m + n\tau|^2 = (m + \frac{n}{2})^2 + n^2y^2$ is at least $\frac{1}{4} + y^2$ for $|n| = 1$, with equality only at $\pm\tau$, $\pm(\tau - 1)$, and at least $4y^2$ for $|n| \geq 2$. So the shortest vectors are ± 1 alone for $y > \sqrt{3}/2$, and $\pm\tau$, $\pm(\tau - 1)$ alone for $1/(2\sqrt{3}) < y < \sqrt{3}/2$, and g_2 has no zero on either interval. As $y \to \infty$, $g_2 \to 60\sum_{m\neq 0} m^{-4} = 4\pi^4/3$, termwise in [3, 23.3.1]. For $y = \frac{1}{2}$, $\Lambda = \frac{1+i}{2}\mathbb{Z}[i]$, so $g_2 = (\frac{1+i}{2})^{-4}g_2(\mathbb{Z}[i]) = -4g_2(\mathbb{Z}[i]) < 0$, as $g_2(\mathbb{Z}[i]) > 0$ by the proof of Theorem 1.4(a), which does not use this lemma. For $y = \sqrt{3}/2$, $\zeta\Lambda = \Lambda$, so $g_2 = \zeta^{-4}g_2$, which forces $g_2 = 0$. Continuity gives (a).

(b) By Lemma 2.2 at $z = \frac{1}{2}$, $e_1 = \frac{2\pi^2}{3} + 16\pi^2\sum_{k \text{ odd}} a_k$, where now $q = -x$ with $x = e^{-2\pi y}$. For odd k, $a_k = -kx^k/(1 + x^k) \geq -kx^k$, so

$$e_1 \geq \frac{2\pi^2}{3} - 16\pi^2\sum_{k \text{ odd}} kx^k = \frac{2\pi^2}{3} - \frac{16\pi^2 x(1 + x^2)}{(1 - x^2)^2}.$$

The right side decreases in x and is $5.8953\ldots$ at $x = e^{-\pi\sqrt{3}}$, which gives the first claim. For the second, let $y' = 1/(4y) \geq \sqrt{3}/2$ and $\tau' = \frac{1}{2} + iy'$. The vectors $2iy = 2\tau - 1$ and $2iy\tau' = \tau - 1$ form a basis of Λ, so $\Lambda = 2iy\Lambda_{\tau'}$; and $\frac{1}{2} = 2iy \cdot (-iy')$ with $-iy' = \frac{1}{2} - \tau'$. Hence $e_1 = (2iy)^{-2}e_1(\tau') = -4y'^2 e_1(\tau') < 0$, and $|e_1|/4 \geq \frac{3}{4} \cdot 5.8953 > 4.4214$. □

Proof of Theorem 1.4. (a) For $\tau = iy$, $q = e^{-2\pi y} > 0$, so every a_k is positive and $e_1 > 2\pi^2/3$ by Lemma 2.2 at $z = \frac{1}{2}$. Each half-period is congruent to its complex conjugate, so the e_j are real and $g_2 = 2(e_1^2 + e_2^2 + e_3^2) > 0$ [3, 23.3.6]. Now apply Lemma 5.1.

(b) If $y \geq \sqrt{3}/2$, then $g_2 \geq 0$ and $e_1 > 0$ by Lemma 5.2, and Lemma 5.1 applies. If $1/(2\sqrt{3}) < y < \sqrt{3}/2$, then $g_2 < 0$, and $F(t) = 1 + \frac{g_2}{20}t^4 + O(t^6)$ [3, 23.9.2, 23.9.4] is less than 1 for small $t > 0$. If $y \leq 1/(2\sqrt{3})$, then $F(\frac{1}{2}) = e_1/4 < 0$. In neither case is F increasing. As $|m + n\tau| \geq \min\{|\tau|, 2y\}$ for $n \neq 0$, 1 is a shortest vector if and only if $|\tau|^2 = \frac{1}{4} + y^2 \geq 1$.

Let $\mu = |e_1|/4 = |F(\frac{1}{2})|$. On $(0, \frac{1}{2}]$, $F(t) \geq t^2 e_1 \geq -\mu$. Suppose that $F > \max\{1, \mu\}$ somewhere. As F extends continuously to $[0, \frac{1}{2}]$ with $F(0) = 1$ and $F(\frac{1}{2}) \leq \mu$, it has a maximum at some $t^* \in (0, \frac{1}{2})$, with $F(t^*) > \max\{1, \mu\}$, $F'(t^*) = 0$ and $F''(t^*) \leq 0$. From $\wp'' = 6\wp^2 - \frac{1}{2}g_2$,

$$t^2F'' - 4tF' = 6F(F-1) - \tfrac{1}{2}g_2t^4, \tag{5.1}$$

so $6F(F-1) \leq \frac{1}{2}g_2t^{*4}$ at t^*, which is impossible if $g_2 \leq 0$. If $g_2 > 0$, then $y > \sqrt{3}/2$ or $y \leq 1/(2\sqrt{3})$ by Lemma 5.2(a), and $\mu \geq \frac{4}{3}$ by Lemma 5.2(b). As $\overline{\tau/2} \equiv \frac{1+\tau}{2}$, e_2 and e_3 are complex conjugates, and with $e_1 + e_2 + e_3 = 0$ this gives $g_2 = 3e_1^2 - 4(\operatorname{Im} e_2)^2 \leq 48\mu^2$. With $t^* \leq \frac{1}{2}$ we get $F(F-1) \leq g_2/192 \leq \mu^2/4$ at t^*, so $F(t^*) \leq \frac{1}{2}(1 + \sqrt{1+\mu^2}) \leq \mu$, as $\mu \geq \frac{4}{3}$. This contradicts $F(t^*) > \mu$. So $|F| \leq \max\{1, \mu\}$ on $(0, \frac{1}{2}]$, and both values are approached, at 0 and at $\frac{1}{2}$. □

6. Integral points

In Sage, with `L = E.period_lattice(`σ`)` and `(w1, w2) = L.normalised_basis()`, $\wp(z)$ is `L.elliptic_exponential(z, to_curve=False)[0]` (with the default `to_curve=True` the first coordinate is $\wp(z) - b_2/12$). For $w = w_2$, w_1, $w_1 - \varepsilon w_2$, $|F(\frac{w}{2})| = |w|^2|x(\frac{w}{2}) + b_2/12|/4$, where $x(\frac{w}{2})$ is the root in `L.ei()` of the 2-division polynomial nearest to the first coordinate of `L.elliptic_exponential(w/2)`. So M_σ needs one further value of $\wp$, at v, to be enclosed rigorously, for instance in ball arithmetic as $F(v) = (v/w_2)^2\wp_\tau(v/w_2)$, with $\wp_\tau$ from `ComplexBall.elliptic_p`, which takes the periods 1 and τ.

For the representative ψ nearest to 0, write the value of `elliptic_logarithm()` in the basis w_1, w_2 (at a real place `L.coordinates()` uses `basis()`, not this basis), reduce both coordinates into $[0, 1)$, and subtract the nearest of the four corners 0, w_1, w_2, $w_1 + w_2$. That corner is a nearest point of Λ. Indeed, the diagonal from w_1 to w_2 if $\varepsilon = 1$, or from 0 to $w_1 + w_2$ if $\varepsilon = -1$, cuts the parallelogram into lattice translates of the triangle 0, w_2, εw_1 and of its negative. This triangle is a translate of the triangle 0, ω_1, ω_2 in the proof of Corollary 1.2, so it is covered by the Voronoi cells of its vertices: each cell is convex and contains its vertex, the midpoints of the two sides through it and the circumcenter (Lemma 2.1), and the circumcenter lies in the triangle, as no angle is obtuse. Rounding the coordinates to the nearest integers instead gives the representative in the parallelogram centered at 0, discussed below.

In the code proposed in [14] the constant for this step at σ is $c_7 = 8\delta^2 + c_8^2|b_2|/12$, with $\delta = 1$ at a real and $\delta = 1 + \pi$ at a complex place and c_8 as in Corollary 1.2. The last term

of (1.3) is at most that of c_7, but M_σ is not always below $8\delta^2$: by Theorem 1.3 it grows like $\pi^2(\operatorname{Im}\tau)^2/12$, with $\tau = w_1/w_2$.

The choice of representative matters in practice. In Sage 10.9 we compared the largest value of $|x(P)|\,|\psi|^2$ over a set of points, for ψ the value of `elliptic_logarithm()` and for the representative nearest to 0. For 37a1 at its real place and kP, $1 \le k \le 59$, $P = (0,0)$, it was 955 against 1.77; for $y^2 = x^3 + ix + 1$ over $\mathbb{Q}(i)$ and kP, $1 \le k \le 25$, $P = (0,1)$, it was 419 against 1.35; for $y^2 = x^3 - 112x + 400$ over $\mathbb{Q}(\sqrt{-5})$, the curve of the examples in [14], and the 80 nonzero combinations $\sum_j c_j P_j$ with $c_j \in \{-1, 0, 1\}$ of $P_1 = (4, -4)$, $P_2 = (0, -20)$, $P_3 = (-4, -28)$ and $P_4 = (-\frac{89}{5}, -\frac{637}{25}\sqrt{-5})$, it was 135 against 1.85. The bound (1.3) is 1.88, 1.60 and 2.10 in the three cases.

If a real place is treated with the real elliptic logarithm alone, a point on the identity component of $E(\mathbb{R})$ has a logarithm $\psi \in (-\omega_1/2, \omega_1/2]$, where $\omega_1 > 0$ generates $\Lambda \cap \mathbb{R}$. As F is even, Theorem 1.4 gives $|x(P)|\psi^2 \le \max\{1, \omega_1^2|e_1|/4\} + \omega_1^2|b_2|/48$ for such a point. Over $\mathbb{Q}$, Sage's `integral_points()` finds the points with $x(P) + b_2/12 < 2\max_j |e_j|$ by a search and uses $c_3 = 8 + \omega_1^2|b_2|/48$ for the others; the bound above has $\max\{1, \omega_1^2|e_1|/4\}$ in place of the 8, for every point of the identity component. A point on the bounded real component, which exists only for rectangular lattices, has a representative $\omega_2/2 + t$, where $\omega_2 = iy\omega_1$ generates $\Lambda \cap i\mathbb{R}$, with $|t| \le \omega_1/2$ on the boundary of V, to which Corollary 1.2 applies.

Bosser and Surroca use, in the proof of [1, Prop. 4.1], that $|z^2\wp(z)|$ is bounded on the fundamental parallelogram centered at 0, without an explicit constant. That region can carry a larger maximum than V: for the basis 1, ρ of Λ_ρ, the corner $\frac{1+\rho}{2}$ of $\{s + t\rho : |s|, |t| \le \frac{1}{2}\}$ is a half-period with $|\frac{1+\rho}{2}|^2 = \frac{3}{4} = 3|\frac{1}{2}|^2$, and $|\wp|$ takes the same value at the three half-periods [3, 23.5.7], so $|F(\frac{1+\rho}{2})| = 3M(\rho)$.

For $\wp'$, a result of Masser [9], as quoted in [8, Fact 4.2], gives $|\wp'(z)|\,d(z,\Lambda)^3 \le \Gamma(\frac{1}{3})^9/((2\pi)^3 3^{3/2})$ for every lattice Λ and every $z \in \mathbb{C} \smallsetminus \Lambda$; the constant is the value of $|z^3\wp'(z)|$ at a vertex of the Voronoi cell of Λ_ρ, where $\wp = 0$ and $\wp'^2 = -g_3$ [3, 23.5.8]. We have not had access to [9] itself, which may contain bounds of the same kind for $\wp$.

Appendix A. Computations

The proof of Proposition 4.2 reads the following from the computation. The set $\{0 \le a \le \frac{1}{2},\ \frac{27}{32} \le b \le \frac{5}{2},\ |\tau| \ge 1\}$ is covered by 13,107 boxes, of side $\frac{1}{64}$ at first and bisected down to 2^{-13}, or to 2^{-30} within $\frac{1}{1024}$ of i. For each box, each of the three half-edges is cut into pieces that pass one of the tests (i)–(vi), and one of (a)–(d) holds for the point $\frac{\tau-1}{2}$. Test (vi) is used only on boxes within $8.3 \cdot 10^{-8}$ of i, and there only within 10^{-5} of the vertex along the edge.

The enclosures follow Lemma 2.2, as in [15, Lemma 3.2.1 and §3.2.2]: the series is summed until the bound for its tail is below 2^{-65}, derivatives come from Leibniz's rule, and Taylor forms in s narrow the enclosures over each piece. In (d), $\Phi(\alpha) = |e_1(\tau')|^2/16$ with $\tau' = -1/(\alpha + ib)$ and $e_1 = \frac{2\pi^2}{3} + 16\pi^2 \sum_{k \text{ odd}} a_k$, and $\partial\Phi/\partial\alpha$ is enclosed from this series. This is done in ball arithmetic with Arb [7], through python-flint 0.8.0 (FLINT 3.3.1) in Python 3.12, at 128 bits. The constants of Proposition 4.1 at $b = \frac{5}{2}$ are evaluated in the same arithmetic. A second program, written separately, checks the cover of Proposition 4.2 and the constants of Proposition 4.1. The numerical experiments of Sections 3 and 6 used PARI/GP 2.17.3 and Sage 10.9. The programs, their outputs and instructions for rerunning them are deposited at Zenodo [6].

Use of AI. This paper and its results were produced by an AI research system that the author designed, built and directs, running on Anthropic's Claude Fable 5.2 and Claude Opus 5.5 in Claude Code. The author chooses the problems and prioritizes the work. Separate AI agents checked each result, and a second program confirmed the computations the proofs use, with any exceptions listed in the public deposit. The system also searched the literature and checked the citations against reliable sources. The author read the paper, reran the computations from the deposit and takes full responsibility for all of the paper's contents.

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