ORBIT CLOSURES OF SCALENE RATIONAL TRIANGLES: RANK ONE AND PRIME DENOMINATORS

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Abstract. The unfolding of a rational triangle is a translation surface, and the closure of its $\mathrm{GL}(2,\mathbb{R})$-orbit has a rank, which is one for lattice surfaces. We determine the scalene rational triangles of rank one: besides the known lattice triangles, which form three infinite families and four further triangles, they are the two triangles found by Delecroix and Rüth's computer search, with angles $(\pi/10, 3\pi/10, 3\pi/5)$ and $(\pi/12, \pi/4, 2\pi/3)$, whose unfoldings are Prym eigenforms. We also prove that for a scalene triangle with angles $p_i\pi/N$, N prime, the orbit closure of the unfolding is a connected component of its stratum. Delecroix and Rüth conjectured that the scalene triangles whose unfoldings do not have dense orbits in their ambient strata form an explicit list. The second theorem proves this for prime denominators, and by the first, every scalene triangle of rank one whose unfolding does not have a dense orbit in its ambient stratum is on their list. The proofs differentiate the holomorphic eigenforms of the unfolding's rotational symmetry along every direction of the orbit closure; at a prime denominator this leaves a statement about subgroups of the units modulo N, proved with character sums, a count of lattice points and a computation.

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This paper is Part I of a series on orbit closures of unfoldings of rational polygons: Parts I and II [20] treat triangles, Part III [19] quadrilaterals and Part IV [18] polygons with at least five sides.

1. Introduction

A rational polygon unfolds to a translation surface, glued from the copies of the polygon under the group generated by the linear parts of the reflections in its sides [42]. The group $\mathrm{GL}(2,\mathbb{R})$ acts on each stratum of translation surfaces, and by Eskin, Mirzakhani and Mohammadi the closure $\mathcal{M}$ of an orbit is an affine invariant submanifold [10, 11]: near a point (X,ω) it is, in period coordinates, a complex linear subspace V of $H^1(X,\Sigma;\mathbb{C})$, where Σ is the set of zeros of ω. The *rank* of $\mathcal{M}$ is half the dimension of the image $p(V)$ of V in absolute cohomology. Lattice surfaces, whose orbits are closed, have rank one, the least possible value. At the other end, Mirzakhani and Wright proved that an orbit closure of full rank, $p(V)=H^1(X;\mathbb{C})$, is a connected component of a stratum or the hyperelliptic locus in one [31, Theorem 1.1].

For triangles, McMullen lists the known lattice triangles and asks whether the list is complete [30, Theorem 3.9 and Question 3.11]. Mirzakhani and Wright showed that infinitely many triangles, built from primitive roots modulo primes, unfold to surfaces with dense orbits in components of their strata [31, Theorem 1.4]. We write $(p_1,p_2,p_3)/N$ for the triangle with angles $p_1\pi/N$, $p_2\pi/N$, $p_3\pi/N$, in any order, where the p_i are positive integers with $\gcd(p_1,p_2,p_3)=1$ and $N=p_1+p_2+p_3$; it is scalene if the p_i are distinct. Its unfolding (X,ω) is taken with the points of cone angle 2π unmarked, and the rank of the triangle is the rank of the orbit closure of (X,ω). A lattice triangle is one whose unfolding is a lattice surface.

Delecroix and Rüth proved with the flatsurf software that the unfolding of every scalene triangle with $N\leq 58$ has a dense orbit in a connected component of its ambient stratum, apart from an explicit list [8, Theorem 4.1]; the ambient stratum is the stratum of (X,ω) for odd N, and for even N, when X has the rotation by π, the locus of holonomy double covers in a stratum of quadratic differentials. The exceptions are lattice triangles from three infinite families and four single ones; $(1,3,6)/10$ and $(1,3,8)/12$, which they found in this search and list as of rank one; and $(1,4,11)/16$, $(1,4,15)/20$ and $(3,4,13)/20$, which they list as of rank two. They conjecture that this list of exceptions is complete [8, §4]. We prove the conjecture for prime denominators, and that every scalene exception of rank one is on their list.

Theorem 1.1. *A scalene rational triangle has rank one if and only if it is* $(1,k,k+1)/(2k+2)$ *with* $k\geq 2$, $(2,2k+1,2k+3)/(4k+6)$ *with* $k\geq 1$, $(1,2,2k-3)/2k$ *with* $k\geq 4$, *or one of* $(1,4,7)/12$, $(1,3,6)/10$, $(1,3,8)/12$, $(2,3,4)/9$, $(3,4,5)/12$, $(3,5,7)/15$. *All of these except* $(1,3,6)/10$ *and* $(1,3,8)/12$ *are lattice triangles.*

Theorem 1.2. *Let N be prime. For every scalene rational triangle $(p_1,p_2,p_3)/N$, the $\mathrm{GL}(2,\mathbb{R})$-orbit closure of the unfolding is the connected component of the stratum $\mathcal{H}(p_1-1,p_2-1,p_3-1)$ containing it (an entry 0 is omitted: that point is not marked). No such triangle is a lattice triangle.*

No triangle on Delecroix and Rüth's list has a prime denominator, so Theorem 1.2 is their conjecture for triangles of prime denominator; their computation covers the primes up to 53. By Theorem 1.1, the one scalene triangle of rank one that is not on their list is $(1,2,3)/6$, whose unfolding is a torus; the orbit of a torus is its whole stratum.

Both proofs differentiate eigenforms in every direction of the orbit closure. Mirzakhani and Wright follow the holomorphic eigenforms of the unfolding's rotation by $2\pi/N$ along the Teichmüller disc. Kahn and Wright's formula, used as they suggest [22, §1], gives the derivative of such a form along any direction v of the orbit closure: paired with a second

eigenform, it is v paired with the class of a meromorphic eigenform on the complement of the zeros of ω. As the unfolding is a cyclic cover of the sphere branched over three points, this class is not zero, and for an eigenvector v with non-zero absolute part the pairing is non-zero whenever the characters match. So each eigenspace in $p(V)$ forces others (Theorem 1.5). At a prime denominator the eigenspaces in $p(V)$ form a subgroup of the units modulo N, and Theorem 1.6, proved with character sums, a count of lattice points and a computation below $N = 2000$, shows that no proper subgroup satisfies Theorem 1.5, apart from $\{\pm 1\}$ for two kinds of triangle. For Theorem 1.1, the directions with zero absolute part give a criterion by residues.

The lattice triangles of Theorem 1.1 are those of Veech [35], Vorobets [36, 37], Ward [38], Kenyon and Smillie [24] and Hooper [16], so the scalene lattice triangles are exactly the scalene triangles of McMullen's list; with the right, isosceles and obtuse ones this answers his question in the convention of our unfolding (Corollary 1.3). The acute scalene ones are $(2, 3, 4)/9$, $(3, 4, 5)/12$ and $(3, 5, 7)/15$. Kenyon and Smillie proved this for denominators up to 10,000 and conjectured it in general [24, Theorem 1 and §1]; it was proved by Puchta under a coprimality assumption [32] and treated in general in his later preprint [33]. The obtuse lattice triangles were determined in [17, Theorem 1.1], and for the obtuse angles in $(\pi/2, 2\pi/3]$ Angdinata, Chen, Ono, Thorner, Zhang and Zhang had excluded all but a proportion $N^{-1+o(1)}$ of the triangles of denominator N [1]. Apisa computed the orbit closures of all right and all isosceles triangles [2, Theorems 7.6 and 7.7], so with Theorem 1.1 every rational triangle of rank one is known, and with Theorem 1.2 the orbit closure of every rational triangle of prime denominator.

Corollary 1.3. *With the unfolding of Section 2, which is that of Kenyon and Smillie, a rational triangle is a lattice triangle if and only if it is of one of the types of McMullen's Theorem 3.9 [30]. So in this convention the answer to his Question 3.11 is yes.*

Remark 1.4. Corollary 1.3, proved at the end of Section 10.3, depends on the convention: in McMullen's, the acute isosceles triangles with apex angle $2\pi/m$, $m \geq 5$ odd, are lattice triangles as well (Section 2.1).

Delecroix and Rüth describe the orbit closures of $(1, 3, 6)/10$ and $(1, 3, 8)/12$ as eigenform loci in a Prym subvariety, referring to work in preparation of Aulicino, Avila and Delecroix [8, Theorem 4.1]. We prove that both have rank one; for $(1, 3, 6)/10$ the orbit closure is then a component of the eigenform locus $\Omega E_5(2, 2, 2)$ by Lanneau and Nguyen's theorem [26, Theorem 1.1], and for $(1, 3, 8)/12$ it has dimension four and is an irreducible component of $\Omega E_{12}(1, 1, 1, 1)$ (Proposition 10.3). By Wright's theorem that orbit closures of rank one consist of completely periodic surfaces [40, Theorem 1.5], billiards in both triangles share a property of the lattice triangles of Theorem 1.1: if one trajectory that starts in a direction is periodic, so is every trajectory that starts in that direction and misses the vertices (Corollary 11.1). The unfoldings of $(1, 4, 11)/16$ and $(1, 4, 15)/20$ are also unfoldings of quadrilaterals, in invariant subvarieties of rank two and dimension four found by Eskin, McMullen, Mukamel and Wright [9, Theorem 1.1 and §5]; by our Theorem 1.1 their rank is not one, so these are their orbit closures. For $(3, 4, 13)/20$, rank at least two also follows from Delecroix and Rüth's lower bound for the dimension [8, Theorem 1.2]. Delecroix announces that this orbit closure has dimension 4, from joint work with Kai Fu [4, pp. 48–49]; his slides of June 2025, on joint work with K. Fu and B. Zykoski, list it with dimension 4 and rank 2 [5, slide 12].

1.1. **Eigenforms along the orbit closure.** The unfolding of $(p_1, p_2, p_3)/N$ is the normalization X of the curve $y^N = x^{p_1}(x-1)^{p_2}$ with $\omega = y\,dx/(x(x-1))$ [31, §6], and $T(x,y) = (x, \zeta y)$, $\zeta = e^{2\pi i/N}$, generates the deck group of x, with $T^*\omega = \zeta\omega$. Let H^1_a be the ζ^a-eigenspace of T^* on $H^1(X;\mathbb{C})$, let $[m] \in \{0, \dots, N-1\}$ be the residue of an integer m modulo N, and

$$\Phi = \{a \in \mathbb{Z}/N : \ ap_1, ap_2, ap_3 \not\equiv 0 \text{ and } [ap_1] + [ap_2] + [ap_3] = N\}.$$

From the dimension formulas of Mirzakhani and Wright [31, Lemma 6.1 and Corollary 6.2], H^1_a is a line if $a \in \Phi \cup -\Phi$ and is zero otherwise, $\Phi \cap -\Phi = \emptyset$, and for $a \in \Phi$ the line is spanned by the holomorphic form $\omega_a = x^{[ap_1]/N-1}(x-1)^{[ap_2]/N-1}dx$, so that $\omega_1 = \omega$. Rotation by $2\pi/N$ and the deck transformation T take (X, ω) to the same point, so $T^*V = V$ (Lemma 2.4; Mirzakhani and Wright prove $T^*p(V) = p(V)$ in this way [31, proof of Lemma 7.7]). Hence

$$p(V) = \bigoplus_{a \in S} H^1_a, \qquad \operatorname{rank} \mathcal{M} = \tfrac{1}{2}|S|, \tag{1.1}$$

for a set $S \subset \Phi \cup -\Phi$ with $S = -S$ and $\pm 1 \in S$, and the problem is to determine S. Here $S = -S$ because $p(V)$ is the complexification of a real subspace and complex conjugation maps H^1_a onto H^1_{-a}; the units in S form a subgroup of $(\mathbb{Z}/N)^\times$ containing -1 [31, Lemma 7.6].

For a unit u modulo N, the Galois conjugate of $p(V)$ under $\zeta \mapsto \zeta^u$ is $\bigoplus_{a \in uS} H^1_a$. By Wright's theorem on the field of definition [39, Theorem 1.5], it is the fiber of a flat summand of the cohomology over $\mathcal{M}$, and by Filip [12] each summand is a variation of Hodge structure. So for $a \in uS \cap \Phi$ the form ω_a extends to a family α of holomorphic forms on the surfaces of $\mathcal{M}$ near (X, ω) that stays in the summand of uS, and the derivative of α along any $v \in V$ is orthogonal to ω_b for every $b \in \Phi$ outside uS. Mirzakhani and Wright, after McMullen [28, Theorem 7.5], use the direction $v = [\bar{\omega}]$ of the Teichmüller disc. There the pairing is the second fundamental form $\frac{i}{2}\int_X \omega_a\omega_b\,\bar{\omega}/\omega$ of the Hodge bundle [14, 15], which is not zero when $a + b \equiv 2$ [31, Proposition 7.3], and this gives their criterion for rank at least two [31, Theorem 7.5].

We use every direction of V. By T-invariance V is the sum of its intersections V_c with the eigenspaces of T^* on $H^1(X, \Sigma; \mathbb{C})$. For $v \in V_c$, Kahn and Wright's formula for the derivative of the projection to moduli space [22, Theorem 1.1] takes the form

$$\int_X \partial_v \alpha \wedge \omega_b = \langle v, [\omega_a\omega_b/\omega] \rangle, \tag{1.2}$$

where on the right $H^1(X, \Sigma; \mathbb{C})$ is paired with $H^1(X \setminus \Sigma; \mathbb{C})$. The meromorphic form $\omega_a\omega_b/\omega$ has character $a + b - 1$, so the right side vanishes unless $c \equiv 1 - a - b$. What makes triangles special is the converse. With three branch points, the eigenspace of $H^1(X \setminus \Sigma; \mathbb{C})$ of character $-c$, for $c \in \Phi \cup -\Phi$, is a line, and $\omega_a\omega_b/\omega$, which is y^{-c} times a rational function of x times dx, has a non-zero class in it. Hence, for such c, the right side of (1.2) is non-zero for every non-zero $v \in V_c$ with $c \equiv 1 - a - b$. The directions $v \in V_c$ with $c \in S$ give the following rule. The directions in $V \cap \ker p$, where the right side is a sum of residues of $\omega_a\omega_b/\omega$ at the zeros of ω, give a criterion for rank at least two by residues (Section 3).

Theorem 1.5. *Let $(p_1, p_2, p_3)/N$ be a rational triangle whose unfolding has genus at least two. If u is a unit modulo N, $a \in uS \cap \Phi$, $a' \in S$ and $b = 1 - a - a' \in \Phi$, then $b \in uS$.*

The case $a' = -1$, $b = 2 - a$, is always available, and with $u = a$ it is Mirzakhani and Wright's criterion. *We call* a unit a modulo N with $2a \not\equiv 2$ and $a, 2 - a \in \Phi$ a *witness*; for obtuse triangles these are the witnesses of [17, §2.3], by Lemma 2.7. For a witness, Theorem 1.5 puts $(2 - a)/a$ in S, and $(2 - a)/a \not\equiv \pm 1$, so the rank is at least two. The

other cases are needed. For $(1,3,9)/13$, $\Phi = \{1,2,3,5,6,9\}$; the witnesses 6 and 9 give $S \supseteq \{\pm 1, \pm 5\}$, and $S = \{\pm 1, \pm 5\}$ satisfies every case of Theorem 1.5 with $a' = -1$. The case $u = a = 3$, $a' = 5$, $b = 6$ puts 2 in S. As 2 is a primitive root modulo 13, the subgroup S is all of $(\mathbb{Z}/13)^\times$, and the orbit closure has full rank.

1.2. **Prime denominators.** Let N be prime; a scalene triangle then has $N \geq 7$. Every non-zero character is a unit, $H^1(X;\mathbb{C})$ is the sum of the $N-1$ lines H^1_a, and S is a subgroup of $G = (\mathbb{Z}/N)^\times$ containing -1. In Theorem 1.5 take $u = x \in \Phi$, so that $x \in uS$, and $a' = h \in S$: whenever $b = 1-h-x$ lies in Φ, the quotient b/x lies in S.

Theorem 1.6. *Let $N \geq 7$ be prime and p_1, p_2, p_3 distinct positive integers with $p_1+p_2+p_3 = N$, and let Φ be as above. Let H be a subgroup of $(\mathbb{Z}/N)^\times$ with $-1 \in H$ and $|H| < N-1$. Then there are $h \in H$ and $x \in \Phi$ such that $b = 1-h-x \in \Phi$ and $b/x \notin H$, unless $|H| = 2$ and $\{p_1,p_2,p_3\}$ is $\{1,(N-3)/2,(N+1)/2\}$ or $\{2,3,6\}$; in these two cases there are none.*

Theorem 1.2 follows from Theorem 1.6 in three steps.

(1) By Theorem 1.6 with $H = S$, either $S = G$, or $|S| = 2$ and the triangle is $(1,(N-3)/2,(N+1)/2)/N$ or $(2,3,6)/11$. For $H = \{\pm 1\}$ the pairs (h,x) are $h = -1$ and the witnesses x, so these are the scalene triangles of prime denominator without a witness.

(2) These have rank at least two, so $S = G$ and $\mathcal{M}$ has rank $(N-1)/2$. For $N \geq 13$ the first is not a lattice triangle [17, Corollary 1.5] and the criterion by residues applies. For $(1,2,4)/7$, $(1,4,6)/11$ and $(2,3,6)/11$ we find completely periodic directions whose cylinder deformations [40], with $[\omega]$ and $[\bar{\omega}]$, span a subspace of $p(V)$ of dimension at least four.

(3) By Mirzakhani and Wright's theorem [31, Theorem 1.1], $\mathcal{M}$ is a connected component of its stratum or the hyperelliptic locus in one, and their Lemma 6.7 excludes the second, as N is odd and the angles are distinct. A lattice triangle has rank one, and $(N-1)/2 \geq 3$.

The proof of Theorem 1.6 (Section 5) uses the condition in the form: $x, y \in \Phi$ and $x+y \in 1+H$ imply $y/x \in H$. Counting the pairs that violate it with the characters of G trivial on H, a bilinear bound for character sums gives a violation when the index of H is less than $(N-5)/(4\sqrt{N})$, and Gauss sums give one when $2\sqrt{N} < |H| \leq (N-1)(N-3)/(8N)$; for $N \geq 2000$ the two ranges overlap. For $|H| \leq 2\sqrt{N}$ the condition allows at most $|H|-1$ elements $x \in \Phi$ with $c - x \in \Phi$, for each non-zero $c \in 1-H$. We count these x exactly as points in triangles of the lattice Λ of vectors $s \in \mathbb{Z}^3$ with $s_1+s_2+s_3 = 0$ congruent modulo N to a multiple of (p_1,p_2,p_3). The count exceeds $2\sqrt{N}$ unless c is represented by a vector of Λ of norm less than $N/4$, and Λ has too few short vectors to represent all of $1-H$ unless its shortest vector is $(-3,1,2)$ up to order and sign; for $|H| = 2$ that case leads to $(1,(N-3)/2,(N+1)/2)/N$. For $7 \leq N < 2000$ we check Theorem 1.6 directly.

1.3. **Rank one.** For Theorem 1.1 we first locate the triangles without a witness. Among obtuse triangles, Larsen, Norton and Zykoski found them when the obtuse angle exceeds $2\pi/3$ [27, Theorem 4.1], and the author when it is at most $2\pi/3$ [17, Theorem 4.1(a)]: finitely many families and finitely many single triangles. For acute triangles we prove the following.

Theorem 1.7. *An acute rational triangle has no witness if and only if it is* $(1,k,k)/(2k+1)$ *with* $k \geq 1$, $(2,m-1,m-1)/2m$ *with* m *even,* $m \geq 4$, $(4,m-2,m-2)/2m$ *with* m *odd,* $m \geq 5$, $(4,m-3,m-1)/2m$ *with* $m \equiv 2 \pmod 4$, $m \geq 6$, *or one of the nine triangles* $(2,3,4)/9$, $(3,5,6)/14$, $(3,5,7)/15$, $(5,6,7)/18$, $(3,8,9)/20$, $(5,8,11)/24$, $(7,9,14)/30$, $(8,9,13)/30$, $(5,17,20)/42$.

Its proof bounds the denominator of an acute triangle without a witness with Jacobsthal's function, as in Puchta's work [32], Kanold's bound for it [23] and an argument of Larsen, Norton and Zykoski [27, Proposition 7.5]; below the bound we find them by enumeration.

The proof of Theorem 1.1 has six steps.

(1) *The listed triangles have rank one.* Lattice triangles do: their orbits are closed [34, Theorem 8.1]. For $(1,3,6)/10$ and $(1,3,8)/12$, not lattice triangles [17, Theorem 1.1], the involution $T^{N/2}$ and the self-adjoint operator $T^* + (T^*)^{-1}$ make the unfolding an eigenform for real multiplication by the orders of discriminant 5 and 12 on a Prym variety of dimension two. McMullen proved the loci of such eigenforms closed and invariant [29, Theorem 3.2], so by Baire's theorem $p(V)$ lies in one two-dimensional eigenspace.

(2) *Witnesses.* A triangle with a witness has rank at least two. A right triangle is $(a,n-a,n)/2n$ with a odd and prime to $2n$; for $a \notin \{1,n-1,n-2\}$ an explicit unit is a witness, as in Apisa's proof of his Lemma 7.4 [2], and $a \in \{1,n-1,n-2\}$ gives the two right families of the list. The acute and obtuse triangles without a witness are those of Theorem 1.7, [27, Theorem 4.1] and [17, Theorem 4.1(a)]. Steps (3)–(6) show that those of them that are scalene and not on the list have rank at least two.

(3) *The involution.* For even N and a scalene triangle other than $(1,2,3)/6$, the powers of T are the only automorphisms of X multiplying ω by a constant, so $\iota = T^{N/2}$ is the only one with $\iota^*\omega = -\omega$ and V lies in the sum of the eigenspaces of odd character. This bounds the rel of $\mathcal{M}$, the dimension of $V \cap \ker p$, by the number of pairs of zeros of ω that ι exchanges. If the rel is zero and the rank one, $\mathcal{M}$ has dimension two and the triangle is a lattice triangle. This settles families of obtuse triangles, not lattice triangles by [17], and $(4,m-3,m-1)/2m$ for $m \geq 14$, not a lattice triangle by [24, Theorem 24].

(4) *Residues.* On three families the criterion by residues, with step (3) for even N, applies: at rank one, the residues of $\omega_a\omega_b/\omega$ at the zeros of ω, for a unit $a \in \Phi$ and $b \in \Phi \setminus \{a\}$, leave no non-zero direction of V in $\ker p$, so $\mathcal{M}$ would have dimension two.

(5) *Second order.* On two families of obtuse triangles with no witness, with $s = 2p_3 - N$ dividing N and N/s odd, the first only for $s \geq 3$, we go to second order along the Teichmüller disc. At rank one, the planes $H^1_u \oplus H^1_{-u}$ for the units u, and the sum of the H^1_a over the non-units, stay of Hodge type along the disc. Along it, the component of ω_a in the summand containing H^1_{-b}, for suitable $b \in \Phi$ with $a+b \equiv 4$, begins in order two and must be holomorphic, so an integral $J(a,b)$ vanishes. $J(a,b)$ is a Bezoutian of two polynomials of degree at most two at $p_1/(p_1+p_2)$, and apart from twelve of these triangles there is a pair on which it is non-zero.

(6) *Cylinders.* Wright's cylinder deformations [40] give rank at least three on $(1,d-2,d+1)/2d$ for odd $d \geq 7$, from an explicit cylinder decomposition. On each of the 62 single triangles we find completely periodic directions whose cylinder deformations, with $[\omega]$ and $[\bar{\omega}]$, span a subspace of $p(V)$ of dimension at least four.

1.4. **Plan.** Section 2 sets up the unfolding, its eigenspaces, the orbit closure, the witness criterion and Wright's cylinder deformations, and Section 3 proves (1.2), Theorem 1.5

and the criterion by residues. Section 4 deduces Theorem 1.2 from Theorem 1.6, treating $(1,(N-3)/2,(N+1)/2)/N$ by residues for every odd $N \geq 13$ other than 15, and Section 5, which is elementary, proves Theorem 1.6. For Theorem 1.1, Section 6 shows that the involution is the only automorphism to consider and draws its consequences for V, and Section 7 proves the second-order criterion; Sections 8, 9 and 10 treat obtuse, acute and right triangles, Section 9 proves Theorem 1.7, and Section 10 treats the two Prym eigenforms and completes the proof of Theorem 1.1, from which Section 11 draws the consequence for billiards.

1.5. **Computations.** Five computations enter the proofs, all in exact arithmetic: Theorem 1.6 for $7 \leq N < 2000$; thirteen constants for the small subgroups in its proof; the enumeration behind Theorem 1.7; a finite search in the second-order criterion; and the cylinder deformations of the 62 single triangles. The first four are programs in C and in Python, on integers and rationals. For the last we follow Delecroix and Rüth's procedure for bounding orbit closures from below [8, §3] (all 62 have $N \leq 48$, so their Theorem 4.1 covers them, with density for all but $(1,4,11)/16$, $(1,4,15)/20$ and $(3,4,13)/20$), with the flatsurf software [6, 7] in Sage, and compute in the number field of each surface. The programs, their output and instructions for rerunning them are deposited [21].

2. The unfolding and its orbit closure

This section fixes the objects of Section 1.1 for an arbitrary denominator $N \geq 3$: the unfolding as a cyclic cover of the sphere, the eigenspaces of its deck group in absolute and relative cohomology, the tangent space V and the set S of (1.1), the Galois conjugates of $p(V)$, the witness criterion of Mirzakhani and Wright, and Wright's cylinder deformations, which give the lower bounds for the rank that we compute. Much of it is in [31, §§6–7].

2.1. **The surface.** Let $\Delta = (p_1,p_2,p_3)/N$, with angle $p_i\pi/N$ at the vertex we number i. It is acute, right or obtuse as $\max_i p_i$ is less than, equal to or greater than $N/2$. Put $g_i = \gcd(p_i,N)$. A prime dividing g_i and g_j for $i \neq j$ divides p_i, p_j and N, hence all three numerators; so the g_i are pairwise coprime. For an integer m, $[m] \in \{0,\dots,N-1\}$ is its residue modulo N, and $\{t\}$ is the fractional part of a real number t.

The unfolding is glued from $2N$ copies of Δ, one for each element of the dihedral group generated by the linear parts of the reflections in the sides [42]. Above vertex i it has g_i points, each of cone angle $2\pi p_i/g_i$. We do not mark the points of cone angle 2π, which lie above the vertices with $p_i \mid N$, and write Σ for the set of the other cone points, $I = \{i : \ p_i \nmid N\}$ for the set of vertices below them, and Σ_i, $i \in I$, for the g_i points above vertex i. The $2N$ triangles have $3N$ edges and $g_1+g_2+g_3$ vertices, so

$$2\,\mathrm{genus} - 2 = N - g_1 - g_2 - g_3. \tag{2.1}$$

As $g_i \leq p_i$, the genus is one exactly when $p_i \mid N$ for every i; then $\Sigma = \emptyset$, and the triangle is $(1,1,1)/3$, $(1,1,2)/4$ or $(1,2,3)/6$. From now on $\Sigma \neq \emptyset$.

As a Riemann surface the unfolding is the normalization X of the curve $y^N = x^{p_1}(x-1)^{p_2}$, with

$$\omega = \frac{y\,dx}{x(x-1)} = x^{p_1/N-1}(x-1)^{p_2/N-1}\,dx \tag{2.2}$$

[31, §6], [27, §2]: on the upper half-plane a branch of $\int\omega$ is the Schwarz–Christoffel map onto a triangle with angles $p_1\pi/N$, $p_2\pi/N$, $p_3\pi/N$ at the images of 0, 1, ∞. So the points of X above 0, 1, ∞ are those above vertices 1, 2, 3, ω vanishes to order $p_i/g_i - 1$ at each point above vertex i, and Σ is the zero set of ω. The deck group of $x\colon X \to \mathbb{P}^1$ is generated

by $T(x,y)=(x,\zeta y)$, $\zeta=e^{2\pi i/N}$, which permutes the points above each vertex cyclically and satisfies $T^*\omega=\zeta\omega$.

In Mirzakhani and Wright's notation, X is the curve X_z with $k=N$ and $q_i=p_i$, after a Möbius map takes the three branch points to 0, 1, ∞, and their set Σ_z [31, §6] is our Σ. Their $\mathcal{M}(\theta)$, the smallest affine invariant submanifold containing the surfaces (X_z,ω_z), is the orbit closure $\mathcal{M}$ of (X,ω), because with three branch points each (X_z,ω_z) is $(X,c\,\omega)$ for a constant $c\in\mathbb{C}^*$: the eigenspace of T^* containing ω is a line (Lemma 2.1).

These conventions are Kenyon and Smillie's [24, §§2.1–2.2]. They ask for a lattice in $\mathrm{SL}^{\pm}(2,\mathbb{R})$ [24, §3], whose orientation-preserving part has index at most two, so their lattice triangles are ours. McMullen takes one copy for each coset of the subgroup that preserves the polygon up to translation [30, (3.1)]. For a triangle that subgroup is non-trivial only for the equilateral triangle, which still unfolds to a torus, and for isosceles triangles of even denominator, which then unfold to our surface of the right triangle obtained by halving them. So in his convention the acute isosceles triangles with apex angle $2\pi/m$, $m\geq 5$ odd, are lattice triangles as well. For instance, $(3,3,4)/10$ then unfolds to our surface of $(2,3,5)/10$, a lattice surface, while in ours it is not a lattice triangle [24, Theorem 2]. Every other triangle is a lattice triangle in one convention exactly when it is in the other.

2.2. **Eigenspaces.** For $\ell\in\mathbb{Z}/N$ let $H^{1,0}_\ell$, H^1_ℓ and $H^1(X,\Sigma)_\ell$ be the ζ^ℓ-eigenspaces of T^* on $H^{1,0}(X)$, $H^1(X;\mathbb{C})$ and $H^1(X,\Sigma;\mathbb{C})$, and put $t(\ell)=\sum_{i=1}^3\{\ell p_i/N\}$. For $\ell\not\equiv 0$, $\dim H^{1,0}_\ell=t(-\ell)-1$ [31, Lemma 6.1], and as $H^1_\ell=H^{1,0}_\ell\oplus\overline{H^{1,0}_{-\ell}}$,

$$\dim H^1_\ell=t(\ell)+t(-\ell)-2\qquad(\ell\not\equiv 0)\tag{2.3}$$

[31, Corollary 6.2].

Lemma 2.1. *Let $a\in\mathbb{Z}/N$.*

(a) *If $ap_i\not\equiv 0$ for $i=1,2,3$, then H^1_a is a line and exactly one of a, $-a$ lies in Φ; if $a\in\Phi$, then $H^1_a=H^{1,0}_a$.*

(b) *Otherwise $H^1_a=0$, and if $a\not\equiv 0$, exactly one of ap_1, ap_2, ap_3 is divisible by N.*

(c) *If $a\in\Phi$, then $H^{1,0}_a$ is spanned by*

$$\omega_a=x^{[ap_1]/N-1}(x-1)^{[ap_2]/N-1}\,dx=y^a\,x^{-i_a}(x-1)^{-j_a}\,dx,\tag{2.4}$$

where, for an integer representative a, $i_a=1+(ap_1-[ap_1])/N$ and $j_a=1+(ap_2-[ap_2])/N$ are integers. The form ω_a vanishes to order $[ap_i]/g_i-1\geq 0$ at each point above vertex i, and $\omega_1=\omega$.

In particular $\Phi\cup-\Phi$ is the set of a with $ap_1,ap_2,ap_3\not\equiv 0$, and it contains every unit.

Proof. (a) Here $\{-ap_i/N\}=1-\{ap_i/N\}$ with $\{ap_i/N\}=[ap_i]/N>0$, so $t(a)+t(-a)=3$, and $t(a)$, $t(-a)$ are positive integers, as $\sum_i ap_i/N=a$ [27, §2]. So one of them is 1 and the other 2, $\dim H^1_a=1$ by (2.3), and $\dim H^{1,0}_a=t(-a)-1$ is 1 exactly when $t(a)=1$, that is, when $a\in\Phi$.

(b) $H^1_0=0$, as the T-invariant classes are pulled back from $X/\langle T\rangle=\mathbb{P}^1$. For $a\not\equiv 0$, N does not divide two of the ap_i, since it would then divide the third, hence $a\gcd(p_1,p_2,p_3)=a$. If it divides one, the other two fractional parts are positive with integer sum less than 2. So $t(a)=t(-a)=1$, and $H^1_a=0$ by (2.3).

(c) Mirzakhani and Wright's form spanning $H^{1,0}_a$ [31, Lemma 6.1], moved by the Möbius map of Section 2.1, is a constant multiple of ω_a. What we add are the orders of vanishing. Near a point above $x=0$ there is a coordinate w with $x=w^{N/g_1}$ and $y=w^{p_1/g_1}h$, h

holomorphic and non-zero, so ω_a has order $ap_1/g_1-(i_a-1)N/g_1-1=[ap_1]/g_1-1$ there. The same computation applies above $x=1$. Above $x=\infty$, ω_a behaves like $x^{([ap_1]+[ap_2])/N-2}dx = x^{-[ap_3]/N-1}dx$, because $a\in\Phi$, and in the coordinate w with $1/x=w^{N/g_3}$ its order is $[ap_3]/g_3-1$. These orders are non-negative, as $[ap_i]$ is a positive multiple of g_i. □

The relative classes in $H^1(X,\Sigma;\mathbb{C})$ are the coboundaries of functions on Σ, and their eigenspaces are read off from the action of T on Σ.

Lemma 2.2. *Let $Z=\ker(p\colon H^1(X,\Sigma;\mathbb{C})\to H^1(X;\mathbb{C}))$ and $Z_c=Z\cap H^1(X,\Sigma)_c$. Then $Z=\delta(\mathbb{C}^\Sigma)\cong\mathbb{C}^\Sigma/\mathbb{C}$, where $(\delta f)(\gamma)=f(\gamma(1))-f(\gamma(0))$ for a path γ with endpoints in Σ, and $T^*f=f\circ T$. For $i\in I$ fix $z_i\in\Sigma_i$, and for $c\in\mathbb{Z}/N$ with $(N/g_i)\mid c$ let $e_{i,c}$ be the function equal to ζ^{ck} at T^kz_i and to 0 off Σ_i.*

(a) *Let $c\not\equiv 0$. If $(N/g_i)\mid c$ for some $i\in I$, this i is unique and Z_c is the line spanned by $\delta e_{i,c}$; otherwise $Z_c=0$.*

(b) *Z_0 has dimension $|I|-1$; it is spanned by the classes $\delta 1_{\Sigma_i}$, $i\in I$, whose sum is the only relation.*

(c) *If $a\in\Phi\cup-\Phi$, then $Z_a=0$, and p maps $H^1(X,\Sigma)_a$ isomorphically onto the line H^1_a.*

Proof. As $\Sigma\neq\emptyset$, the sequence $0\to\mathbb{C}\to\mathbb{C}^\Sigma\xrightarrow{\delta}H^1(X,\Sigma;\mathbb{C})\xrightarrow{p}H^1(X;\mathbb{C})\to 0$ is exact and T-equivariant. Since T permutes Σ_i as a cycle of length g_i, the function $e_{i,c}$ is well defined exactly when $(N/g_i)\mid c$, it satisfies $T^*e_{i,c}=\zeta^c e_{i,c}$, and the g_i functions $e_{i,c}$ form a basis of the functions on Σ_i. If $(N/g_i)\mid c$ and $(N/g_j)\mid c$ with $i\neq j$, then c is divisible by $\operatorname{lcm}(N/g_i,N/g_j)=N$, as g_i and g_j are coprime. This gives (a) and (b). If $a=kN/g_i$, then $ap_i=k(p_i/g_i)N\equiv 0$; so for $a\in\Phi\cup-\Phi$ no N/g_i divides a, $a\not\equiv 0$, $Z_a=0$, and (c) follows from the exact sequence and Lemma 2.1. □

2.3. **The orbit closure.** Let $\mathcal{M}$ be the $\mathrm{GL}(2,\mathbb{R})$-orbit closure of (X,ω) in its stratum, in which only the zeros of ω are marked. Filip writes $\mathcal{M}=\mathcal{M}_0\sqcup\mathcal{M}_1$, with $\mathcal{M}_0$ smooth and open and $\mathcal{M}_1$ closed, $\mathrm{GL}^+(2,\mathbb{R})$-invariant and of lower dimension [13, Remark 1.7], so the dense orbit of (X,ω) lies in $\mathcal{M}_0$, as in the proof of [31, Lemma 7.7]. As $\mathcal{M}$ is linear in period coordinates $H^1(X,\Sigma;\mathbb{C})$ [10, 11], there are then a ball U about $[\omega]$ and a single complex linear subspace

$$V=T_{(X,\omega)}\mathcal{M}\subset H^1(X,\Sigma;\mathbb{C}),\qquad V=\overline{V},$$

with $\mathcal{M}\cap U=V\cap U$ in period coordinates. The same holds for the orbit closure of any translation surface, with any finite set of marked points.

At an orbifold point of a stratum, a surface with an automorphism other than the identity that fixes the form, we read period coordinates, U and V at a lift to a finite regular cover of the stratum that is a manifold, as Eskin and Mirzakhani [10, §1] and Filip [13, Remark 1.7 and the paragraph after it] do, and all later uses of U and V are in this chart. By Theorem 6.2 the unfolding of a scalene triangle is not such a point, while for isosceles triangles with N even and $p_1=p_2$, $(x,y)\mapsto(1-x,-y)$ is such an automorphism.

The *rank* and the *rel* of $\mathcal{M}$ are

$$\operatorname{rank}\mathcal{M}=\tfrac{1}{2}\dim_{\mathbb{C}}p(V),\qquad \operatorname{rel}\mathcal{M}=\dim_{\mathbb{C}}(V\cap Z),$$

so $\dim_{\mathbb{C}}\mathcal{M}=2\operatorname{rank}\mathcal{M}+\operatorname{rel}\mathcal{M}$; $p(V)$ is symplectic [3, Theorem 1.4], so the rank is an integer. Complex conjugation $(x,y)\mapsto(\bar{x},\bar{y})$ preserves the curve and pulls ω back to $\bar{\omega}$, so it is an isomorphism from the image of (X,ω) under $\operatorname{diag}(1,-1)$ to (X,ω), and the $\mathrm{GL}(2,\mathbb{R})$-orbit of (X,ω) is its $\mathrm{GL}^+(2,\mathbb{R})$-orbit. A *lattice surface* is one whose stabilizer in $\mathrm{SL}(2,\mathbb{R})$ is a lattice.

Lemma 2.3. *If (X,ω) is a lattice surface, then $\mathcal{M}$ is the $\mathrm{GL}(2,\mathbb{R})$-orbit of (X,ω), $\dim_{\mathbb{C}}\mathcal{M}=2$, $\operatorname{rank}\mathcal{M}=1$ and $\operatorname{rel}\mathcal{M}=0$. Conversely, if $\dim_{\mathbb{C}}\mathcal{M}=2$, then (X,ω) is a lattice surface.*

Proof. A translation surface is a lattice surface if and only if its $\mathrm{SL}(2,\mathbb{R})$-orbit is closed [34, Theorem 8.1]. Smillie and Weiss work in the locus of area one modulo $-\mathrm{Id}$. As $-\mathrm{Id}\in\mathrm{SL}(2,\mathbb{R})$ and the locus of area one is closed in our stratum, an $\mathrm{SL}(2,\mathbb{R})$-orbit is closed there if and only if it is closed in ours. As the scalars $\lambda>0$ only change the area, this holds if and only if the $\mathrm{GL}^+(2,\mathbb{R})$-orbit is closed. For (X,ω) the latter is the $\mathrm{GL}(2,\mathbb{R})$-orbit.

If (X,ω) is a lattice surface, its orbit is therefore $\mathcal{M}$. It is the image of a group of real dimension four, so $\dim_{\mathbb{C}}\mathcal{M}\leq 2$. As V contains $[\omega]$ and $[\bar{\omega}]$, whose images under p are independent, $\dim_{\mathbb{C}}\mathcal{M}=\dim p(V)=2$ and $V\cap Z=0$. Conversely, if $\dim_{\mathbb{C}}\mathcal{M}=2$, every $\mathrm{GL}^+(2,\mathbb{R})$-orbit in $\mathcal{M}$ is open in $\mathcal{M}$, since the group and $\mathcal{M}$ have the same real dimension. As $\mathcal{M}$ is connected, it is a single orbit [41, §3.2]. So the orbit of (X,ω) is closed, and (X,ω) is a lattice surface. □

The next lemma is the source of every eigenspace argument in the paper.

Lemma 2.4. *$T^*V=V$, so $V=\bigoplus_{\ell\in\mathbb{Z}/N}V_\ell$ with $V_\ell=V\cap H^1(X,\Sigma)_\ell$. Let S be the set of $a\in\mathbb{Z}/N$ with $0\neq H^1_a\subset p(V)$. Then $S\subset\Phi\cup-\Phi$, $\pm1\in S$,*

$$p(V)=\bigoplus_{a\in S}H^1_a,$$

and $V_a=H^1(X,\Sigma)_a$, a line, for every $a\in S$. In particular $\operatorname{rank}\mathcal{M}=|S|/2$.

Proof. Mirzakhani and Wright prove $T^*p(V)=p(V)$ with the path $(X,e^{i\phi}\omega)$, $0\leq\phi\leq 2\pi/N$, in the orbit of (X,ω), whose monodromy is T^* [31, proof of Lemma 7.7]. The same path gives $T^*V=V$ on relative cohomology, as it stays in $\mathcal{M}_0$, where $\mathcal{M}$ is a single subspace near each point. At an orbifold point it runs at the lift of Section 2.3, where T acts by a deck transformation of the cover. A subspace invariant under a linear map of finite order is the sum of its intersections with the eigenspaces.

That $p(V)$ is the sum of the lines H^1_a it contains, so that $S\subset\Phi\cup-\Phi$, is Mirzakhani and Wright's Corollary 7.9 [31] with Lemma 2.1. For $a\in S$, $p(V_a)=H^1_a\neq 0$, as p is equivariant, and $H^1(X,\Sigma)_a$ is a line by Lemma 2.2(c). So $V_a=H^1(X,\Sigma)_a$. Finally $[\omega]\in V_1$ and $[\bar{\omega}]\in V_{-1}$ have non-zero images, so $\pm1\in S$. □

2.4. **Galois conjugates.** Let H^1 be the flat bundle over $\mathcal{M}$ with fiber $H^1(Y;\mathbb{C})$ at (Y,η), and $k(\mathcal{M})$ the field of definition of $\mathcal{M}$. Wright proves that $H^1=\bigoplus_\rho V_\rho\oplus W$, where ρ runs over the field embeddings $k(\mathcal{M})\to\mathbb{C}$, V_{Id} has fiber $p(V)$ at (X,ω), each V_ρ is a simple flat bundle Galois conjugate to V_{Id}, and W is a semisimple flat bundle with no subbundle isomorphic to any V_ρ [39, Theorem 1.5]. Here $V_\rho=\sigma(V_{\mathrm{Id}})$ for any automorphism σ of $\mathbb{C}$ extending ρ, acting on $H^1(Y;\mathbb{Q})\otimes\mathbb{C}$ through the coefficients: Wright defines V_ρ in this way [39, proof of Lemma 6.5], and the choice of σ does not matter, as V_{Id} is defined over $k(\mathcal{M})$ [39, Theorem 1.5]. The V_ρ are pairwise non-isomorphic [39, Lemma 6.5]. Filip proves that each such summand underlies a variation of Hodge structure [12, proof of Theorem 7.3, with Theorem 7.2]; in the form used by Mirzakhani and Wright [31, §7.1], at each point the fiber of each summand is the sum of its intersections with $H^{1,0}$ and $H^{0,1}$.

For a unit u modulo N, let σ_u be an automorphism of $\mathbb{C}$ with $\sigma_u(\zeta)=\zeta^u$.

Lemma 2.5. *Let u be a unit modulo N. Then $\sigma_u(p(V)) = \bigoplus_{a\in uS} H^1_a$, and this is the fiber at (X,ω) of one of Wright's summands V_ρ. At every point of $\mathcal{M}\cap U$ the fiber of V_ρ is the sum of its intersections with $H^{1,0}$ and $H^{0,1}$. Two such conjugates of $p(V)$ are equal or meet only in 0.*

Proof. T^* is defined over $\mathbb{Q}$, so $\sigma_u(H^1_a) = H^1_{ua}$, and the first statement follows from Lemma 2.4. With $\rho = \sigma_u|_{k(\mathcal{M})}$, $\sigma_u(p(V))$ is the fiber of $V_\rho = \sigma_u(V_{\mathrm{Id}})$, and the second statement is Filip's. If two conjugates V_ρ and $V_{\rho'}$ meet in a non-zero vector at one point, their intersection is a non-zero flat subbundle of both; as both are simple, both equal it, so they are isomorphic and $\rho = \rho'$. □

2.5. **Witnesses and full rank.** For triangles ($n = 3$, $k = N$), Mirzakhani and Wright's criterion [31, Theorem 7.5] says: if there is a unit a modulo N with $2a \not\equiv 2$, $t(-a) > 1$ and $t(-(2-a)) > 1$, then $\mathcal{M}(\theta)$ has rank greater than $n-2 = 1$. For every b, $t(\pm b)$ is an integer, as $\sum_i bp_i/N = b$. If $b \in \Phi \cup -\Phi$, then $t(b)$ and $t(-b)$ are positive with sum 3, so $t(-b) > 1$ means $t(b) = 1$, that is, $b \in \Phi$; otherwise some $\{-bp_i/N\}$ is zero, the other two are less than 1, and $t(-b) \le 1$. So, with witnesses as defined in Section 1.1:

Theorem 2.6 (Mirzakhani–Wright [31, Theorem 7.5])**.** *If Δ has a witness, then* $\operatorname{rank}\mathcal{M} \ge 2$.

In terms of the residues $r_i = [ap_i]$, a witness is found by comparing r_i with $[2p_i]$, which is $2p_i$ at an angle less than $\pi/2$ and $2p_i - N$ at an obtuse angle. For obtuse triangles the following is the form of the criterion used by Larsen, Norton and Zykoski [27, Proposition 2.1].

Lemma 2.7. *Let a be a unit modulo N with $2a \not\equiv 2$, and $r_i = [ap_i]$.*

(a) *If Δ is acute, a is a witness if and only if $r_i < 2p_i$ for $i = 1, 2, 3$.*
(b) *If Δ is obtuse with $p_3 > N/2$, a is a witness if and only if at least two of $r_1 < 2p_1$, $r_2 < 2p_2$, $r_3 < 2p_3 - N$ hold.*

Proof. Put $\hat{p}_i = [2p_i]$. As a is a unit, each r_i lies in $[1, N-1]$, and $r_1 + r_2 + r_3$ is N or $2N$, with N exactly when $a \in \Phi$. As in Larsen, Norton and Zykoski's proof of their Proposition 2.1 [27], $[(2-a)p_i]$ is 0 if $r_i = \hat{p}_i$, $\hat{p}_i - r_i$ if $r_i < \hat{p}_i$, and $\hat{p}_i - r_i + N$ if $r_i > \hat{p}_i$. Let k be the number of i with $r_i < \hat{p}_i$. If no r_i equals $\hat{p}_i$, then $2 - a \in \Phi \cup -\Phi$ and

$$\textstyle\sum_i [(2-a)p_i] = \sum_i \hat{p}_i - \sum_i r_i + (3-k)N. \tag{2.5}$$

For $a \in \Phi$, (b) is their computation: (2.5) equals N exactly when $k = 2$. We add (a), and in (b) that two of the inequalities force $a \in \Phi$.

(a) Here $\sum_i \hat{p}_i = 2N$. If a is a witness, $\sum_i r_i = N$ and (2.5) equals N, so $k = 3$. Conversely, if $r_i < 2p_i$ for all i, then $\sum_i r_i < 2N$, so $\sum_i r_i = N$. No r_i equals $\hat{p}_i$, $k = 3$, and (2.5) equals N.

(b) Here $\sum_i \hat{p}_i = N$. Let $k \ge 2$. If the remaining index j had $r_j \le \hat{p}_j$, then $\sum_i r_i < \sum_i \hat{p}_i = N$, which is impossible. So $r_j > \hat{p}_j$, $k = 2$, and (2.5) equals $2N - \sum_i r_i$, which is positive and a multiple of N only if $\sum_i r_i = N$. So a and $2 - a$ lie in Φ. □

Full rank turns into density by Mirzakhani and Wright's theorem [31, Theorem 1.1]: an affine invariant submanifold of full rank, $\operatorname{rank}\mathcal{M} = \text{genus}$, is a connected component of a stratum or the locus of hyperelliptic surfaces in one. If N is odd and the three angles are distinct, $\mathcal{M}$ is not contained in the hyperelliptic locus [31, Lemma 6.7].

2.6. **Cylinder deformations.** Our lower bounds for the rank of single triangles, and of one family, come from Wright's cylinder deformations [40]. We allow some of the points of cone angle 2π above the vertices to be marked: let $\Sigma' \supseteq \Sigma$ be Σ together with some of them, $\mathcal{M}'$ the orbit closure of (X,ω) with the points of Σ' marked, and $V' \subset H^1(X,\Sigma';\mathbb{C})$ its tangent space, a single subspace by Section 2.3. Forgetting the added points is $\mathrm{GL}(2,\mathbb{R})$-equivariant and linear in period coordinates and commutes with p, so it maps $\mathcal{M}'$ into $\mathcal{M}$ and $p(V') \subset p(V)$.

Let (Y,η) be horizontally periodic, with maximal horizontal cylinders $C_1,\dots,C_r$ (bounded by saddle connections between points of Σ') of heights h_i, circumferences c_i and moduli $m_i = h_i/c_i$. Let $u^{\mathcal{C}}_{t_1,\dots,t_r}$ apply the shear $\left(\begin{smallmatrix}1 & t_i\\ 0 & 1\end{smallmatrix}\right)$ to C_i and nothing to the rest of the surface, so that $u_t = u^{\mathcal{C}}_{t,\dots,t}$ is the horocycle flow [40, proof of Lemma 3.1]. Let $\Xi_i \in H^1(Y,\Sigma';\mathbb{Z})$ be the class that vanishes on relative cycles disjoint from the interior of C_i and is one on a path crossing C_i from a point of Σ' on its bottom to one on its top. Wright proves that $u^{\mathcal{C}}_{t_1,\dots,t_r}(Y)$ has period coordinates $[\eta] + \sum_i t_i h_i \Xi_i$ for small t [40, Lemmas 2.3 and 2.4], and that the closure of the horocycle orbit of (Y,η) is the set of the $u^{\mathcal{C}}_{t_1/m_1,\dots,t_r/m_r}(Y)$ with $(t_1,\dots,t_r)$ in the space $\mathcal{T} \subset \mathbb{R}^r$ of solutions of every homogeneous linear relation with rational coefficients satisfied by $(m_1,\dots,m_r)$ [40, Corollary 3.4].

Lemma 2.8. *Let θ be a direction in which the straight-line flow on (X,ω) is completely periodic, let C_i, c_i, Ξ_i and $\mathcal{T}$ be as above for $(Y,\eta) = e^{-i\theta}(X,\omega)$ with the points of Σ' marked, and $L = \{\sum_i t_i c_i \Xi_i : t \in \mathcal{T}\}$. Then $L \subset V'$, and if $\Sigma' = \Sigma$, then $\operatorname{rank}\mathcal{M} \geq \dim_{\mathbb{R}} p(L)$. If $\sum_i b_i \Xi_i$ lies in the kernel of p on $H^1(X,\Sigma';\mathbb{C})$, there is a function λ on Σ', real if the b_i are real, with $b_i = \lambda(z^+) - \lambda(z^-)$ for every i and all points z^- of Σ' on the bottom and z^+ on the top of C_i.*

Proof. The rotation $e^{-i\theta}$ is a complex scalar on period coordinates, which preserves the orbit closure and V'; so we may take $\theta = 0$ and $(Y,\eta) = (X,\omega)$. The orbit closure $\mathcal{M}'$ is closed and invariant under u_t, so it contains $u^{\mathcal{C}}_{t_1/m_1,\dots,t_r/m_r}(X)$ for every $t \in \mathcal{T}$. For small t this surface has period coordinates $[\omega] + \sum_i (t_i/m_i) h_i \Xi_i = [\omega] + \sum_i t_i c_i \Xi_i$, and since $\mathcal{M}'$ is a single subspace near (X,ω), $L \subset V'$.

For $\Sigma' = \Sigma$, L lies in Wright's twist space [40, Definition 8.1], whose image under p is isotropic [40, Lemma 8.10]. As $p(V) \cap H^1(X;\mathbb{R})$ is symplectic of dimension $2\operatorname{rank}\mathcal{M}$ [3, Theorem 1.4], this gives $\dim_{\mathbb{R}} p(L) \leq \operatorname{rank}\mathcal{M}$, as in the proof of [40, Corollary 8.11].

Finally, the kernel of p on $H^1(X,\Sigma';\mathbb{C})$ is $\delta(\mathbb{C}^{\Sigma'})$, as in Lemma 2.2. If $\sum_j b_j \Xi_j = \delta\lambda$, evaluate on a path inside C_i from z^- to z^+: it gives $b_i = \lambda(z^+) - \lambda(z^-)$. If the b_j are real, $\delta(\operatorname{Re}\lambda)$ is the same class. □

For single triangles we use the lemma as Delecroix and Rüth do in their procedure for bounding orbit closures from below [8, §3].

Lemma 2.9. *Let Σ' be the set of all points above the vertices. Suppose that there are directions $\theta_1,\dots,\theta_n$ in which the straight-line flow on (X,ω) is completely periodic such that the spaces L of Lemma 2.8 for these directions, together with $\operatorname{Re}[\omega]$ and $\operatorname{Im}[\omega]$, span a subspace of $H^1(X,\Sigma';\mathbb{R})$ whose image in $H^1(X;\mathbb{R})$ has dimension at least four. Then $\operatorname{rank}\mathcal{M} \geq 2$.*

Proof. By Lemma 2.8 each L lies in V', and so do $\operatorname{Re}[\omega]$ and $\operatorname{Im}[\omega]$, as V' is complex and contains $[\omega]$ and $[\bar{\omega}]$. So $p(V')$, which is defined over $\mathbb{R}$, has complex dimension at least four, and $p(V') \subset p(V)$. □

3. Derivatives of eigenforms

This section proves (1.2), Theorem 1.5 and the criterion by residues. A holomorphic eigenform that stays in a Galois conjugate of $p(V)$ has a derivative, along every $v \in V$, orthogonal to the eigenforms outside that conjugate (Lemma 3.5); Kahn and Wright's formula turns this orthogonality into the vanishing of $\langle v, [\omega_a\omega_b/\omega]\rangle$ (Lemma 3.1); and with three branch points the class $[\omega_a\omega_b/\omega]$ is never zero (Lemma 3.2). We keep the notation of Section 2, with $\Sigma \neq \emptyset$.

3.1. **The pairing and the derivative.** Classes in $H^1(X,\Sigma;\mathbb{C}) = H^1_c(X\setminus\Sigma;\mathbb{C})$ are represented by closed smooth 1-forms with compact support in $X\setminus\Sigma$, and classes in $H^1(X\setminus\Sigma;\mathbb{C})$ by closed 1-forms on $X\setminus\Sigma$. By Poincaré–Lefschetz duality the pairing

$$\langle[\nu],[\theta]\rangle = \int_X \nu\wedge\theta$$

between the two spaces is perfect. It is T-invariant, so it pairs $H^1(X,\Sigma)_c$ with $H^1(X\setminus\Sigma)_{c'}$ to zero unless $c+c'\equiv 0$. If θ is closed on all of X, then $\langle w,[\theta]\rangle = \int_X p(w)\wedge\theta$. If θ is a meromorphic 1-form on X, holomorphic on $X\setminus\Sigma$, and $f\in\mathbb{C}^\Sigma$, then

$$\langle \delta f,[\theta]\rangle = -2\pi i\sum_{z\in\Sigma} f(z)\operatorname{Res}_z\theta\,: \tag{3.1}$$

δf is represented by $d(\sum_z f(z)\kappa_z)$, where κ_z equals 1 near z and is supported in a small disc D_z, and Stokes' theorem on D_z minus a smaller disc about z gives the residue.

Let (X_ϖ,ω_ϖ), $\varpi\in U$, be the marked surfaces with period coordinates ϖ, and $[\alpha]_{\mathrm{rel}}\in H^1(X,\Sigma;\mathbb{C})$ the relative class of a holomorphic form α on one of them. For holomorphic θ, the next lemma is Kahn and Wright's formula for the derivative of the projection from a stratum to moduli space [22, Theorem 1.1, Corollary 2.3 and Remark 2.4], written as a pairing. We give the short proof, along their path [22, Lemma 2.2].

Lemma 3.1. *Let $v\in H^1(X,\Sigma;\mathbb{C})$, and let $\varpi\mapsto\alpha_\varpi$ be a family of holomorphic 1-forms on the surfaces X_ϖ, of class C^1 in t along the path $[\omega]+tv$, as forms on the fixed surface in the identification of the proof, with $\alpha_{[\omega]}=\alpha$. Let θ be a meromorphic 1-form on X, holomorphic on $X\setminus\Sigma$, with at most simple poles. Then*

$$\langle \partial_v[\alpha_\varpi]_{\mathrm{rel}},[\theta]\rangle = \langle v,[\alpha\theta/\omega]\rangle.$$

The form $\alpha\theta/\omega$ on the right is meromorphic on X and holomorphic on $X\setminus\Sigma$. For $\theta=\omega_b$ the left side is $\int_X \partial_v\alpha\wedge\omega_b$, which gives (1.2).

Proof. Represent v by a closed form ν with compact support in $X\setminus\Sigma$. For small t the form $\omega_t=\omega+t\nu$ is closed, equals ω near Σ, and satisfies $i\,\omega_t\wedge\bar\omega_t>0$ off Σ, so there is a unique complex structure Y_t on the surface, equal to that of X near Σ, for which ω_t is of type $(1,0)$, hence holomorphic. With the identity marking, (Y_t,ω_t) has period coordinates $[\omega]+tv$, so by the local injectivity of period coordinates it is the path of the statement. Pulled back to the fixed surface, $\alpha_t=\alpha_{[\omega]+tv}$ is a C^1 family of closed forms of type $(1,0)$ for Y_t; so $\alpha_t\wedge\omega_t=0$, and near Σ the α_t are holomorphic in a fixed coordinate. Differentiating at $t=0$,

$$\dot\alpha\wedge\omega+\alpha\wedge\nu=0,$$

and since $\omega\neq 0$ off Σ this says that the $(0,1)$-part of $\dot\alpha$ for X is $(\alpha/\omega)\,\nu^{0,1}$. The form $\dot\alpha$ is closed and holomorphic near Σ. Its relative class $\partial_v[\alpha_\varpi]_{\mathrm{rel}}$ is represented by $\dot\alpha - d(\kappa\dot A)$,

where κ is a cut-off function equal to 1 near Σ and $\dot{A}$ is the primitive of $\dot{\alpha}$ near each $z \in \Sigma$ with $\dot{A}(z) = 0$. Hence

$$\langle [\dot{\alpha}]_{\mathrm{rel}}, [\theta] \rangle = \int_X \dot{\alpha} \wedge \theta + \sum_{z \in \Sigma} \lim_{\varepsilon \to 0} \oint_{|w|=\varepsilon} \dot{A}\, \theta,$$

and the last terms vanish because $\dot{A}(z) = 0$ and θ has at most a simple pole at z. Finally $\dot{\alpha} \wedge \theta = \dot{\alpha}^{0,1} \wedge \theta = (\alpha/\omega)\, \nu^{0,1} \wedge \theta = \nu \wedge (\alpha\theta/\omega)$, which is supported where ν is; integrating gives $\langle v, [\alpha\theta/\omega] \rangle$. □

The Beltrami differential of the path (Y_t, ω_t) is $\nu^{0,1}/\omega$, and for holomorphic θ the right side is, up to a constant, its pairing with the quadratic differential $\alpha\theta$. For $v = [\bar{\omega}]$ and $\theta = \beta$ holomorphic, it is a multiple of $\int_X \alpha\beta\,\bar{\omega}/\omega$, which with the factor $i/2$ is the form $B_\omega(\alpha, \beta)$ that Forni, Matheus and Zorich define following Forni [15, (2.11)], [14], the second fundamental form of the Hodge bundle along the Teichmüller flow [15, Lemma 2.1], as Mirzakhani and Wright use it [31, §7.1, Lemma 7.2]. For $v = \delta f$ it is a sum of residues, by (3.1).

3.2. **Three branch points.** Let $\hat{\Sigma} \supseteq \Sigma$ be the set of all points of X above the three vertices. For four or more branch points the eigenspaces are larger and a single class of this kind can pair to zero with a tangent direction; for three it cannot.

Lemma 3.2. *Let $c \in \Phi \cup -\Phi$ and $k, l \in \mathbb{Z}$. The meromorphic form $y^{-c}x^k(x-1)^l\,dx$ on X, which is holomorphic on $X \setminus \hat{\Sigma}$, has non-zero class in $H^1(X \setminus \hat{\Sigma}; \mathbb{C})$, hence in $H^1(X \setminus \Sigma'; \mathbb{C})$ for every $\Sigma' \subseteq \hat{\Sigma}$ off which it is holomorphic.*

Proof. Call the form $\xi_{k,l}$; on a branch it is $x^{s_1}(x-1)^{s_2}\,dx$ with $s_1 = k - cp_1/N$, $s_2 = l - cp_2/N$ for a fixed representative c. Over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ the equation $y^N = x^{p_1}(x-1)^{p_2}$ defines a connected unramified cyclic cover, as $\gcd(p_1, p_2, N) = 1$; so $X \setminus \hat{\Sigma}$ is the smooth affine curve it defines, y is a unit on it, and the regular functions and 1-forms of character $-c$ are $y^{-c}R$ and $y^{-c}R\,dx$, with $R = \mathbb{C}[x, 1/x, 1/(x-1)]$. By the algebraic de Rham theorem, $H^1(X \setminus \hat{\Sigma}; \mathbb{C})_{-c}$ is the quotient of $y^{-c}R\,dx$, which the $\xi_{k,l}$ span, by the differentials of $y^{-c}R$. The cokernel of $H^1(X; \mathbb{C}) \to H^1(X \setminus \hat{\Sigma}; \mathbb{C})$ is spanned by classes dual to small loops about the points of $\hat{\Sigma}$, whose characters are multiples of N/g_i, and no multiple ℓ of N/g_i is $-c$, as $\ell p_i \equiv 0$ while $cp_i \not\equiv 0$. So $H^1(X \setminus \hat{\Sigma}; \mathbb{C})_{-c} \cong H^1_{-c}$, a line by Lemma 2.1. From

$$d(y^{-c}x^{k+1}(x-1)^{l+1}) = (s_1+1)\xi_{k,l+1} + (s_2+1)\xi_{k+1,l}, \qquad \xi_{k+1,l} - \xi_{k,l+1} = \xi_{k,l},$$

we get, in this line,

$$[\xi_{k+1,l}] = \frac{s_1+1}{s_1+s_2+2}\,[\xi_{k,l}], \qquad [\xi_{k,l+1}] = -\frac{s_2+1}{s_1+s_2+2}\,[\xi_{k,l}].$$

Modulo $\mathbb{Z}$, $s_1 + 1 \equiv -cp_1/N$, $s_2 + 1 \equiv -cp_2/N$ and $s_1 + s_2 + 2 \equiv cp_3/N$, and none of these is an integer since $cp_i \not\equiv 0$. So all the classes $[\xi_{k,l}]$ are non-zero multiples of one another, and as they span a line, none is zero. □

Lemma 3.3. *Let $c \in \Phi \cup -\Phi$ and $a, b \in \Phi$ with $a + b \equiv 1 - c$. Then $\langle v, [\omega_a \omega_b/\omega] \rangle \neq 0$ for every non-zero $v \in V_c$.*

Proof. By (2.4), $\omega_a\omega_b/\omega = y^{a+b-1}x^k(x-1)^l\,dx$ with integers k, l, and replacing y^{a+b-1} by y^{-c} changes only k and l, by a multiple of $y^N = x^{p_1}(x-1)^{p_2}$. The form is holomorphic off the zeros Σ of ω, so its class in $H^1(X \setminus \Sigma; \mathbb{C})_{-c}$ is not zero by Lemma 3.2; as in the proof of Lemma 3.2, with Σ in place of $\hat{\Sigma}$, this eigenspace is the line H^1_{-c}. Since $c \in \Phi \cup -\Phi$, $H^1(X, \Sigma; \mathbb{C})_c \cong H^1_c$ is a line (Lemma 2.2(c)), and the pairing, being perfect and T-invariant, restricts to a perfect pairing between it and the line $H^1(X \setminus \Sigma; \mathbb{C})_{-c}$; so $\langle v, [\omega_a\omega_b/\omega] \rangle \neq 0$ for every non-zero $v \in V_c$. □

3.3. **The rule.** The real structure of V and the Galois conjugates of $p(V)$ constrain S as follows, at every N.

Lemma 3.4. *$S = -S$. If u is a unit modulo N and $S \cap uS \neq \emptyset$, then $uS = S$. If N is prime, S is a subgroup of $(\mathbb{Z}/N)^\times$ containing -1.*

Proof. Since $V = \overline{V}$ and p is defined over $\mathbb{R}$, $p(V)$ is the complexification of a real subspace, and complex conjugation maps H^1_a onto H^1_{-a} because T^* is real; so $S = -S$. If $a \in S \cap uS$, then $p(V)$ and $\sigma_u(p(V))$ both contain H^1_a, so they are equal by Lemma 2.5, which rests on Wright's theorem [39, Theorem 1.5], and $uS = S$. If N is prime, every non-zero character is a unit; for $a \in S$ we have $a = a \cdot 1 \in S \cap aS$, so $aS = S$. Hence S is closed under multiplication, and $-1 \in S$. □

For the set of units in S in place of S this lemma is Mirzakhani and Wright's Lemma 7.6 [31]: the units in S form a subgroup of $(\mathbb{Z}/N)^\times$ containing -1; at prime N every non-zero character is a unit, so its last clause is theirs.

The derivative of an eigenform vanishes against the eigenforms outside its Galois conjugate, because that conjugate is a variation of Hodge structure.

Lemma 3.5. *Let u be a unit modulo N, $a \in uS \cap \Phi$, and $b \in \Phi \setminus uS$. There is a holomorphic family α_ϖ of holomorphic 1-forms on the surfaces of $\mathcal{M} \cap U$ with $\alpha_{[\omega]} = \omega_a$ such that*

$$\langle \partial_v [\alpha_\varpi]_{\mathrm{rel}}, [\omega_b] \rangle = 0 \qquad \text{for every } v \in V.$$

Proof. Let V_ρ be the summand of Lemma 2.5 with fiber $\bigoplus_{c \in uS} H^1_c$ at (X, ω). Over $\mathcal{M} \cap U$ the bundle H^1 is the constant bundle $H^1(X; \mathbb{C})$ and V_ρ is a constant subspace of it, closed under complex conjugation because $uS = -uS$ (Lemma 3.4). At $\varpi \in \mathcal{M} \cap U$, with $H^{1,0}(X_\varpi) \subset H^1(X; \mathbb{C})$ through the marking, V_ρ is the sum of its intersections with $H^{1,0}(X_\varpi)$ and its conjugate (Lemma 2.5); conjugation exchanges the two, so $L_\varpi = V_\rho \cap H^{1,0}(X_\varpi)$ has dimension $\frac{1}{2}\dim V_\rho$ at every point. Being the kernel of the holomorphic family of maps $V_\rho \to H^1(X; \mathbb{C})/H^{1,0}(X_\varpi)$ of constant rank, L is a holomorphic bundle, and $L_{[\omega]} = \bigoplus_{c \in uS \cap \Phi} H^{1,0}_c$ contains ω_a. Let α_ϖ be a holomorphic section of L through ω_a.

Let $v \in V$. For small t the path $[\omega] + tv$ of the proof of Lemma 3.1 lies in $V \cap U = \mathcal{M} \cap U$. Its Beltrami coefficient $t\nu^{0,1}/(\omega + t\nu^{1,0})$ is holomorphic in t, and so are the classes $p[\alpha_{[\omega]+tv}]$, as α_ϖ is a holomorphic section and the path is affine; as a holomorphic form is the harmonic representative of its class, and the harmonic representative of a fixed class depends smoothly on the complex structure, the forms $\alpha_{[\omega]+tv}$, pulled back to the fixed surface, form a C^1 family. Their classes lie in the fixed space V_ρ, so $p(\partial_v[\alpha_\varpi]_{\mathrm{rel}}) = \partial_v p[\alpha_\varpi] \in V_\rho$, and as ω_b is closed on X,

$$\langle \partial_v [\alpha_\varpi]_{\mathrm{rel}}, [\omega_b] \rangle = \int_X \partial_v p[\alpha_\varpi] \wedge \omega_b.$$

The cup product pairs H^1_c with H^1_b only if $c \equiv -b$, and $-b \notin uS$ because $uS = -uS$ and $b \notin uS$. So the integral is zero. □

Proof of Theorem 1.5. Let u, a, a' and b be as in the theorem, and suppose that $b \notin uS$. Take the family α_ϖ of Lemma 3.5 and a non-zero $v \in V_{a'}$, which exists because $V_{a'} = H^1(X, \Sigma)_{a'}$ is a line (Lemma 2.4). By Lemma 3.5 and Lemma 3.1 with $\theta = \omega_b$,

$$0 = \langle \partial_v [\alpha_\varpi]_{\mathrm{rel}}, [\omega_b] \rangle = \langle v, [\omega_a \omega_b / \omega] \rangle.$$

But $a' \in S \subset \Phi \cup -\Phi$ and $a + b \equiv 1 - a'$, so the right side is not zero by Lemma 3.3. Hence $b \in uS$. □

For $a' = -1$ the direction is $[\bar\omega]$, and the non-vanishing is Mirzakhani and Wright's Proposition 7.3(2) [31].

3.4. **Residues.** The directions of V in Z are not reached by Theorem 1.5. For them the right side of (1.2) is a sum of residues. Let $\mathcal{P}$ be the set of pairs (a, b) with a a unit, $a, b \in \Phi$ and $b \not\equiv a$. For $(a, b) \in \mathcal{P}$ let $F_{ab} = \omega_a \omega_b / \omega$, a meromorphic form holomorphic on $X \setminus \Sigma$, and

$$\lambda_{ab}(f) = \sum_{z \in \Sigma} f(z) \operatorname{Res}_z F_{ab} \qquad (f \in \mathbb{C}^\Sigma),$$

a linear form on $\mathbb{C}^\Sigma / \mathbb{C} \cong Z$ by the residue theorem.

Lemma 3.6. *Let $(a, b) \in \mathcal{P}$.*

(a) *At a point z above vertex i, $\operatorname{Res}_z F_{ab} \neq 0$ if and only if $[ap_i] + [bp_i] = p_i$.*
(b) *λ_{ab} vanishes on $(\mathbb{C}^\Sigma)_{c'}$ for every $c' \not\equiv 1 - a - b$.*
(c) *If $c \equiv 1 - a - b$, $i \in I$ and $(N/g_i) \mid c$, then $\lambda_{ab}(e_{i,c}) = g_i \operatorname{Res}_{z_i} F_{ab}$. If $a + b \equiv 1$, then $\lambda_{ab}(1_{\Sigma_i}) = g_i \operatorname{Res}_{z_i} F_{ab}$ for $i \in I$, and these numbers sum to zero.*

Proof. (a) As in the proof of Lemma 2.1(c), in a coordinate w at z with w^{N/g_i} equal to x, $x - 1$ or $1/x$, we have $\omega_a = w^{[ap_i]/g_i - 1} h_a(w^{N/g_i})\, dw$ with h_a holomorphic and $h_a(0) \neq 0$, and likewise for ω_b and $\omega = \omega_1$. So $F_{ab} = w^{m-1} H(w^{N/g_i})\, dw$ with $m = ([ap_i] + [bp_i] - p_i)/g_i$ and $H(0) \neq 0$, and the residue is the coefficient of w^{-m} in $H(w^{N/g_i})$. It is $H(0)$ if $m = 0$ and zero if $m > 0$. If $m < 0$ it could be non-zero only if $p_i - [ap_i] - [bp_i] = -mg_i$, which lies in $(0, p_i)$, were a multiple of N.

(b), (c) $T^* F_{ab} = \zeta^{a+b-1} F_{ab}$, so $\operatorname{Res}_{T^k z} F_{ab} = \zeta^{k(a+b-1)} \operatorname{Res}_z F_{ab}$ and $\lambda_{ab}(T^* f) = \zeta^{1-a-b} \lambda_{ab}(f)$, which gives (b). For $c \equiv 1 - a - b$ the terms of $\lambda_{ab}(e_{i,c}) = \sum_{k=0}^{g_i - 1} \zeta^{kc} \operatorname{Res}_{T^k z_i} F_{ab}$ all equal $\operatorname{Res}_{z_i} F_{ab}$; the case $c \equiv 0$ is the same, and the sum is zero by the residue theorem, as F_{ab} has no poles off Σ. □

Let $K \subset Z$ be the common kernel of the λ_{ab}, $(a, b) \in \mathcal{P}$, and $\bar K$ its complex conjugate.

Theorem 3.7. *Let Δ be a rational triangle that is not a lattice triangle. If $K \cap \bar K = 0$, then* $\operatorname{rank} \mathcal{M} \geq 2$.

Proof. Suppose $\operatorname{rank} \mathcal{M} = 1$. Then $S = \{\pm 1\}$, $V = \mathbb{C}[\omega] \oplus \mathbb{C}[\bar\omega] \oplus R$ with $R = V \cap Z$, and $R = \bar R$ since V and Z are real. If $R = 0$, then $\dim_{\mathbb{C}} \mathcal{M} = 2$ and Δ is a lattice triangle by Lemma 2.3; so $R \neq 0$. Let $(a, b) \in \mathcal{P}$ and $u = a$, so that $uS = \{\pm a\}$ and $a \in uS \cap \Phi$. Then $b \notin uS$, as $b \not\equiv a$ and $-a \notin \Phi$. For $\delta f \in R$, Lemma 3.5, Lemma 3.1 with $\theta = \omega_b$, and (3.1) give

$$0 = \langle \delta f, [\omega_a \omega_b / \omega] \rangle = -2\pi i\, \lambda_{ab}(f).$$

So $R \subset K$, hence $R = \bar R \subset K \cap \bar K = 0$, a contradiction. □

The criterion is a first-order consequence of Filip's theorem that flat summands are variations of Hodge structure, with Wright's decomposition (Lemma 2.5). It bounds $V \cap Z$ itself, while Filip's theorem on the torsion of relative periods [13, Theorem 1.3] constrains the combinations of zeros that $V \cap Z$ annihilates.

For $c \not\equiv 0$ with $Z_c \neq 0$, the vertex $i \in I$ with $(N/g_i) \mid c$ is unique (Lemma 2.2); we call it the vertex of c and write $\mathcal{P}(c)$ for the set of $(a,b) \in \mathcal{P}$ with $a+b \equiv 1-c$ and $[ap_i]+[bp_i] = p_i$ at that vertex. By Lemma 3.6, $\lambda_{ab}(e_{i,c}) \neq 0$ for $(a,b) \in \mathcal{P}(c)$.

Corollary 3.8. *Let Δ be a rational triangle that is not a lattice triangle, with $|I| \leq 2$. Suppose that*

(i) *for every $c \not\equiv 0$ with $Z_c \neq 0$, $\mathcal{P}(c)$ or $\mathcal{P}(-c)$ is not empty, and*

(ii) *if $|I| = 2$, there is $(a,b) \in \mathcal{P}$ with $a+b \equiv 1$ and $[ap_i]+[bp_i] = p_i$ for some $i \in I$.*

Then $\operatorname{rank} \mathcal{M} \geq 2$.

Proof. We show $K \cap \bar{K} = 0$. Let $\delta f \in K \cap \bar{K}$, with $f = \sum_c f_c$, $f_c \in (\mathbb{C}^\Sigma)_c$. By Lemma 3.6(b), $\lambda_{ab}(f) = \lambda_{ab}(f_{1-a-b})$. Let $c \not\equiv 0$. If $Z_c = 0$, then $f_c = 0$; otherwise f_c is a multiple of $e_{i,c}$ for the vertex i of c. For $(a,b) \in \mathcal{P}(c)$, $\lambda_{ab}(f) = 0$ and $\lambda_{ab}(e_{i,c}) \neq 0$ then give $f_c = 0$. If $\mathcal{P}(-c) \neq \emptyset$, apply the same to $\bar{f}$: $\delta\bar{f} = \overline{\delta f}$ lies in K, and the component of $\bar{f}$ of character $-c$ is $\overline{f_c}$. If $|I| = 2$, Z_0 is the line spanned by $\delta 1_{\Sigma_i}$, and (ii) with Lemma 3.6(c) gives $\delta f_0 = 0$; if $|I| = 1$, $Z_0 = 0$. So $\delta f = 0$, and Theorem 3.7 applies. □

Remark 3.9. The condition that a be a unit cannot be dropped. The triangle $(1,3,6)/10$ is not a lattice triangle and has rank one (Proposition 10.2); here $I = \{2,3\}$ and $Z = Z_0 \oplus Z_5$, with 3 the vertex of 5. The pair $(7,4) \in \mathcal{P}$ has $7+4 \equiv 1$ and $[7 \cdot 3]+[4 \cdot 3] = 3 = p_2$, so condition (ii) holds. The characters $2, 4 \in \Phi$, which are not units, have $2+4 \equiv 1-5$ and $[2 \cdot 6]+[4 \cdot 6] = 6 = p_3$; with such pairs admitted, condition (i) would hold as well.

4. Prime denominators

This section deduces Theorem 1.2 from Theorem 1.6. What remains beyond Theorem 1.5 is to show that the two kinds of triangle excepted in Theorem 1.6 have rank at least two. For $(1,(N-3)/2,(N+1)/2)/N$ we do this with the criterion by residues for every odd $N \geq 13$ other than 15, since Section 8 needs the composite N as well.

4.1. **The family $(1,(N-3)/2,(N+1)/2)/N$.** Let $\Delta = (1,(N-3)/2,(N+1)/2)/N$ with N odd. As $2p_3 \equiv 1$ and $2p_2 \equiv -3$, writing a character as $a \equiv 2\alpha$ with $0 \leq \alpha < N$ gives

$$[ap_1] = [2\alpha], \qquad [ap_2] = [-3\alpha], \qquad [ap_3] = \alpha. \tag{4.1}$$

For $N \geq 13$, $p_2 = (N-3)/2$ does not divide N (it would divide $N - 2p_2 = 3$), and $g_1 = g_3 = 1$, $g_2 = \gcd(3,N)$. So the point above vertex 1 is regular, $I = \{2,3\}$, and by Lemma 2.2 $Z = Z_0$ if $3 \nmid N$, while $Z = Z_0 \oplus Z_{N/3} \oplus Z_{2N/3}$, with 2 the vertex of $\pm N/3$, if $3 \mid N$.

Lemma 4.1. *$2\alpha \in \Phi$ if and only if $0 < \alpha < N/3$ or $N/2 < \alpha < 2N/3$.*

Proof. For $0 < \alpha < N/2$ the three residues in (4.1) sum to $3\alpha + [-3\alpha]$, which is N exactly when $3\alpha \leq N$. For $N/2 < \alpha < N$ they sum to $3\alpha - N + [-3\alpha]$, which is N exactly when $3\alpha \leq 2N$. When $3\alpha \in \{N, 2N\}$ the residue $[ap_2]$ is 0; otherwise none of the three is. □

Lemma 4.2. *Let $N \geq 13$ be odd with $N \neq 15$. There is $(a,b) \in \mathcal{P}$ with $a+b \equiv 1$ and $[ap_3]+[bp_3] = p_3$.*

Proof. Take $a = 2\alpha$, $b = 2\beta$ with $\alpha + \beta = (N+1)/2$. Then $a + b \equiv 1$ and $[ap_3] + [bp_3] = \alpha + \beta = p_3$. By Lemma 4.1, $a, b \in \Phi$ when $0 < \alpha, \beta < N/3$; a is a unit when $\gcd(\alpha, N) = 1$; and $b \not\equiv a$ when $\alpha \neq \beta$. Writing $\alpha = (N+1-j)/4$ and $\beta = (N+1+j)/4$, these conditions are

$$j \equiv N+1 \pmod 4, \qquad 0 < |j| < (N-3)/3, \qquad \gcd(j-1, N) = 1.$$

If $N \equiv 1 \pmod 4$, $j = 2$ satisfies them, as $N \geq 13$. If $N \equiv 3 \pmod 4$, let ℓ be the least odd prime not dividing N, and put $j = \ell + 1$ if $\ell \equiv 3 \pmod 4$ and $j = 1 - \ell$ if $\ell \equiv 1 \pmod 4$. Then $4 \mid j$ and $\gcd(j-1, N) = \gcd(\ell, N) = 1$, and we need $3|j| + 3 < N$. For $\ell = 3$, $|j| = 4$, and $N \geq 19$ since $3 \nmid N$. For $\ell = 5$, $|j| = 4$, and $N \geq 27$ since $3 \mid N$, $5 \nmid N$ and $N \equiv 3 \pmod 4$. For $\ell = 7$, $|j| = 8$, and $N \geq 75$ since $15 \mid N$, $N \neq 15$ and $N \equiv 3 \pmod 4$. For $\ell \geq 11$ every odd prime below ℓ divides N; by Bertrand's postulate one of them, ℓ', lies in $(\ell/2, \ell)$, and $\ell' \geq 7$, so $N \geq 15\ell' > 7\ell > 3(\ell+1) + 3 \geq 3|j| + 3$. □

Lemma 4.3. *Let $3 \mid N$ with $N \geq 21$, and let a be the least odd prime not dividing N. Then $(a, 2N/3 + 1 - a) \in \mathcal{P}(N/3)$.*

Proof. First $a < N/3$: either $a = 5$ and $N \geq 21$, or $a = 7$, $15 \mid N$, $N \neq 15$ and $N \geq 45$, or $a \geq 11$ and $N > 7a$ as in the previous proof. Now $a = 2\beta$ with $\beta = (N+a)/2 \in (N/2, 2N/3)$, and $b = 2N/3 + 1 - a = 2\alpha$ with $\alpha = N/3 - (a-1)/2 \in (0, N/3)$, so $a, b \in \Phi$ by Lemma 4.1; a is a unit, and $a < b < N$. Next, $a + b \equiv 1 - N/3$, and the vertex of $N/3$ is 2. Finally, by (4.1),

$$[ap_2] + [bp_2] = (2N - 3\beta) + (N - 3\alpha) = \frac{N - 3a}{2} + \frac{3(a-1)}{2} = \frac{N-3}{2} = p_2. \qquad \square$$

Proposition 4.4. *For every odd $N \geq 13$ with $N \neq 15$, $(1, (N-3)/2, (N+1)/2)/N$ has rank at least two.*

Proof. The triangle is obtuse, N is odd and $p_2 \nmid N$, so it is not a lattice triangle [17, Theorem 1.2(a)]. We apply Corollary 3.8, with $|I| = 2$. Lemma 4.2 gives condition (ii). If $3 \nmid N$ there is no $c \not\equiv 0$ with $Z_c \neq 0$. If $3 \mid N$, then $N \geq 21$, and Lemma 4.3 gives condition (i) for $c = N/3$ and for $c = 2N/3 \equiv -N/3$. □

4.2. **Proof of Theorem 1.2.** Let N be prime and Δ scalene; then $N \geq 7$. Every p_i is prime to N, so above vertex i there is a single point, of cone angle $2\pi p_i$; by (2.1) the genus is $(N-1)/2$. Every non-zero character is a unit and lies in $\Phi \cup -\Phi$, so $H^1(X; \mathbb{C})$ is the sum of the $N-1$ lines H^1_a, $a \in G = (\mathbb{Z}/N)^\times$.

By Lemma 3.4, S is a subgroup of G containing -1. For $h \in S$ and $x \in \Phi$ with $b = 1 - h - x \in \Phi$, Theorem 1.5 with $u = x$, $a = x \in xS$ and $a' = h$ gives $b \in xS$, that is, $b/x \in S$. So $H = S$ admits no h and x as in Theorem 1.6, and either $S = G$, or $|S| = 2$ and Δ is $(1, (N-3)/2, (N+1)/2)/N$ or $(2, 3, 6)/11$.

These triangles have rank at least two, so $|S| \neq 2$. For $N \geq 13$ this is Proposition 4.4. The ones with $N < 13$ are $(1, 2, 4)/7$, $(1, 4, 6)/11$ and $(2, 3, 6)/11$, and for each of them we find, as in the proof of Proposition 8.9, completely periodic directions for which Lemma 2.9 gives rank at least two [21].

Hence $S = G$, $p(V) = H^1(X; \mathbb{C})$, and $\mathcal{M}$ has full rank $(N-1)/2$. By Mirzakhani and Wright's theorem (Section 2.5), $\mathcal{M}$ is a connected component of its stratum or the hyperelliptic locus in one, and the second is excluded by their Lemma 6.7, as N is odd and the angles are distinct. The stratum is $\mathcal{H}(p_1 - 1, p_2 - 1, p_3 - 1)$, with an entry 0 omitted, since the zeros of ω are the only marked points. Finally, a lattice triangle has rank one by Lemma 2.3, while $(N-1)/2 \geq 3$. □

5. Subgroups of the units modulo a prime

This section proves Theorem 1.6. Apart from the elementary Lemma 4.1 in its last step, it uses nothing from the rest of the paper: $N \geq 7$ is prime, p_1, p_2, p_3 are distinct positive integers with sum N, $G = (\mathbb{Z}/N)^\times$, and Φ is the set of $x \in G$ with $[xp_1] + [xp_2] + [xp_3] = N$. For every $x \in G$ the residue sum is N or $2N$, and it is N for exactly one of $\pm x$; so $|\Phi| = (N-1)/2$. In the terms of Koblitz and Rohrlich, Φ is the set H_{p_1,p_2}, a CM type of a factor of the Jacobian of the Fermat curve of degree N [25, §1]. Large subgroups are handled with character sums (Section 5.2); for small ones we count exactly, with a lattice attached to the triangle (Sections 5.3 and 5.4).

5.1. **Two forms of the condition.** Let $H \subset G$ be a subgroup with $-1 \in H$, of order k and index $m = (N-1)/k$. *We call H stable* if there are no $h \in H$ and $x \in \Phi$ with $1 - h - x \in \Phi$ and $(1-h-x)/x \notin H$; Theorem 1.6 says that no proper H is stable, with the two exceptions. Since $H = -H$, stability can be put in two ways.

(S1) For $x, y \in \Phi$, $x + y \in 1 + H$ implies $y/x \in H$.

(S2) For every $c \in Q_H = (1 - H) \setminus \{0\}$, the set $W(c) = \{x \in \Phi : \ c - x \in \Phi\}$ lies in $\{c/(1+h) : \ h \in H, \ h \neq -1\}$. In particular $|W(c)| \leq k - 1$.

Indeed, $x \in W(c)$ with $(c-x)/x = h$ means $x = c/(1+h)$; and $x + y = 0$ is impossible for $x, y \in \Phi$. Note $|Q_H| = k - 1$ and $2 \in Q_H$. As $1 \in W(2)$, and the other elements of $W(2)$ are the witnesses, $\{\pm 1\}$ is stable if and only if the triangle has no witness.

For example, for $(1, 4, 8)/13$ and $H = \{\pm 1, \pm 5\}$ we have $\Phi = \{1, 2, 4, 5, 7, 10\}$ and $Q_H = \{2, 6, 9\}$, and $W(9) = \{2, 4, 5, 7\}$ has more than $k - 1 = 3$ elements; so H is not stable.

5.2. **Character sums.** Let $\widehat{G/H}$ be the set of the m characters of G trivial on H, extended to $\mathbb{F}_N$ by $\chi(0) = 0$ (also for the trivial character χ_0), so that $\mathbf{1}_H = \frac{1}{m} \sum_{\chi \in \widehat{G/H}} \chi$ on $\mathbb{F}_N$. Let $\mathrm{e}(t) = e^{2\pi i t}$. In this subsection Φ enters only through containing exactly one of $\pm x$ for each $x \in G$. We use the classical bilinear bound in this form.

Lemma 5.1. *For $\chi \neq \chi_0$, $a \in \mathbb{F}_N$ and complex weights α, β on $\mathbb{F}_N$,*

$$\Big| \sum_{x,y} \alpha_x \beta_y \chi(x + y + a) \Big| \leq \sqrt{N}\, \|\alpha\|_2 \|\beta\|_2.$$

Proof. By Cauchy–Schwarz the square of the left side is at most $\|\alpha\|^2 \sum_{y,y'} \beta_y \bar{\beta}_{y'} \Gamma(y' - y)$, where $\Gamma(\nu) = \sum_z \chi(z) \bar{\chi}(z + \nu)$. Now $\Gamma(0) = N - 1$, and for $\nu \neq 0$ the substitution $u = z/(z+\nu)$, a bijection from $\mathbb{F}_N \setminus \{0, -\nu\}$ onto $\mathbb{F}_N \setminus \{0, 1\}$, gives $\Gamma(\nu) = \sum_{u \neq 0,1} \chi(u) = -1$. So the double sum is $N\|\beta\|^2 - |\sum_y \beta_y|^2 \leq N\|\beta\|^2$. □

Proposition 5.2. *If $H \neq G$ and $m < (N-5)/(4\sqrt{N})$, then H is not stable.*

Proof. By (S1), H is stable if and only if $\mathcal{N} = \#\{(x, y) \in \Phi^2 : \ x + y - 1 \in H, \ y/x \notin H\}$ is zero. Write $\mathcal{N} = \mathcal{S}_1 - \mathcal{S}_2$ with

$$\mathcal{S}_1 = \sum_{x,y \in \Phi} \mathbf{1}_H(x + y - 1) = \frac{1}{m} \sum_{\chi} A(\chi),$$

$$\mathcal{S}_2 = \sum_{x,y \in \Phi} \mathbf{1}_H(x + y - 1)\, \mathbf{1}_H(y/x) = \frac{1}{m^2} \sum_{\chi,\psi} B(\chi, \psi),$$

where $A(\chi)=\sum_{x,y\in\Phi}\chi(x+y-1)$, $B(\chi,\psi)=\sum_{x,y\in\Phi}\chi(x+y-1)\psi(y)\bar{\psi}(x)$, and χ,ψ run over $\widehat{G/H}$. Let $W_1=|W(1)|\le|\Phi|$. Then $A(\chi_0)=B(\chi_0,\chi_0)=|\Phi|^2-W_1$. For $\psi\neq\chi_0$ we have $\psi(-1)=1$, so $\sum_\Phi\psi=\frac12\sum_G\psi=0$, and $B(\chi_0,\psi)=|\sum_\Phi\psi|^2-\sum_{x\in W(1)}\psi((1-x)/x)$ has modulus at most W_1. For $\chi\neq\chi_0$, Lemma 5.1 with $\alpha=\mathbf{1}_\Phi\bar{\psi}$, $\beta=\mathbf{1}_\Phi\psi$ gives $|A(\chi)|,|B(\chi,\psi)|\le\sqrt{N}|\Phi|$. Collecting terms,

$$\mathcal{N}\ge\frac{m-1}{m^2}\Big(|\Phi|^2-2W_1-2m\sqrt{N}|\Phi|\Big)\ge\frac{m-1}{m^2}|\Phi|\Big(\frac{N-1}{2}-2-2m\sqrt{N}\Big),$$

which is positive under the hypothesis. □

Proposition 5.3. *Let $k,m\ge2$ and $\eta^*=((m-1)\sqrt{N}+1)/m$. If $(N-1)^2k-(N^2-1)\eta^*>4N(k-1)^2$, then H is not stable.*

Proof. The number $\mathcal{S}_1$ of pairs $(x,y)\in\Phi^2$ with $x+y\in1+H$ is $\sum_{h\in H}|W(1-h)|$. If H is stable, then $|W(1-h)|\le k-1$ for $h\neq1$ by (S2), and $W(0)=\emptyset$, so $\mathcal{S}_1\le(k-1)^2$. On the other hand, with $F(t)=\sum_{x\in\Phi}\mathrm{e}(tx/N)$ and $\eta(u)=\sum_{h\in H}\mathrm{e}(uh/N)$,

$$\mathcal{S}_1=\frac{1}{N}\sum_{t\in\mathbb{F}_N}F(t)^2\,\mathrm{e}(-t/N)\,\eta(-t).$$

The term $t=0$ is $|\Phi|^2k/N$. For $u\neq0$, $\eta(u)=\frac1m\sum_\chi\bar{\chi}(u)\,g(\chi)$ with the Gauss sums $g(\chi)=\sum_z\chi(z)\mathrm{e}(z/N)$, and $g(\chi_0)=-1$, $|g(\chi)|=\sqrt{N}$ otherwise, so $|\eta(u)|\le\eta^*$. By Parseval, $\sum_{t\neq0}|F(t)|^2=N|\Phi|-|\Phi|^2$. With $|\Phi|=(N-1)/2$,

$$\mathcal{S}_1\ge\frac{|\Phi|^2k}{N}-\eta^*\Big(|\Phi|-\frac{|\Phi|^2}{N}\Big)=\frac{(N-1)^2k-(N^2-1)\eta^*}{4N}>(k-1)^2.$$ □

5.3. **The lattice of the triangle.** For $k<\sqrt{N}$ these bounds are empty, and we count $W(c)$ exactly. Let

$$\Lambda_0=\{s\in\mathbb{Z}^3:\ s_1+s_2+s_3=0\},\qquad\Lambda=\{s\in\Lambda_0:\ s\equiv y(p_1,p_2,p_3)\bmod N,\ y\in\mathbb{Z}/N\}.$$

For $s\in\Lambda$ the class $y(s)$ is determined by s, as p_1 is a unit. Let $\|s\|=\max_l|s_l|$ on $\Lambda_0\otimes\mathbb{R}$; areas are taken in the coordinates (s_a,s_b) for any $a\neq b$, which identify Λ_0 with $\mathbb{Z}^2$. For $y\in G$ let $\mu(y)$ be the least norm of a vector $s\in\Lambda$ with $y(s)=y$; so $\mu(y)/N$ is the distance from 0 of the class of $y(p_1,p_2,p_3)/N$ in $\mathbb{R}^3/\mathbb{Z}^3$, which lies in the subtorus $(\Lambda_0\otimes\mathbb{R})/\Lambda_0$. For $r\ge0$ let $\mathcal{Y}(r)=\{y\in G:\ \mu(y)\le r\}$ and $\mathcal{Y}_\Phi(r)=\mathcal{Y}(r)\cap\Phi$. Let $\lambda=\min_{y\in G}\mu(y)$, attained at $w\in\Lambda$, and $x_0=y(w)$. Lattices of this kind, for the divisors of the denominator and with their duals, are used by Angdinata, Chen, Ono, Thorner, Zhang and Zhang to count witnesses [1, §§4.2–4.3].

Lemma 5.4. (a) *Let $y\in\Phi$, $d_l=[yp_l]$, and i the index of the largest d_l. Then $\mu(y)=N-d_i$, attained by $d-N\mathbf{e}_i$, whose only negative coordinate is at i. Also $\mu(-y)=\mu(y)$, and if $\mu(y)<N/2$ the vector attaining $\mu(y)$ is unique.*

(b) *Λ has index N in Λ_0.*

(c) *For $a,b\in\Lambda_0\otimes\mathbb{R}$, $|a_1b_2-a_2b_1|\le\|a\|\,\|b\|$; for $a,b\in\Lambda$, $a_1b_2-a_2b_1\in N\mathbb{Z}$.*

(d) *λ is the least norm of a non-zero vector of Λ, and $3\le\lambda\le2\sqrt{N/3}$. Every $s\in\Lambda$ with $0<\|s\|<N$ has non-zero, pairwise distinct coordinates; $\lambda=3$ if and only if w is an ordering of $\pm(-3,1,2)$; and w is primitive in Λ and in $\mathbb{Z}^3$.*

(e) *For $r<N/2$, $s\mapsto y(s)$ is a bijection from $\{s\in\Lambda:\ 0<\|s\|\le r\}$ onto $\mathcal{Y}(r)$, and $|\mathcal{Y}_\Phi(r)|=|\mathcal{Y}(r)|/2$.*

Proof. The d_l are distinct, as the p_l are and y is a unit, and they sum to N.

(a) The vectors of Λ_0 congruent to $y(p_1,p_2,p_3)$ are $d+Nu$ with $u\in\mathbb{Z}^3$, $u_1+u_2+u_3=-1$. A coordinate with $u_l\geq 1$ or $u_l\leq -2$ has absolute value at least N, and one with $u_l=-1$ has absolute value $N-d_l\geq N-d_i$; as some $u_l\neq 0$, every such vector has norm at least $N-d_i$. The vector $d-N\mathbf{e}_i$ has norm $\max(N-d_i,d_{i'},d_{i''})=N-d_i$, where $\{i,i',i''\}=\{1,2,3\}$, since $d_{i'}+d_{i''}=N-d_i$. Negation gives $\mu(-y)=\mu(y)$, and two vectors with the same y differ by a non-zero vector of $N\Lambda_0$, of norm at least N.

(b) $s\mapsto y(s)$ maps Λ onto $\mathbb{Z}/N$ with kernel $N\Lambda_0$, and $[\Lambda_0:N\Lambda_0]=N^2$.

(c) In the coordinates (s_1,s_2) the unit ball of $\|\cdot\|$ is the hexagon with vertices $\pm(1,0)$, $\pm(0,1)$, $\pm(1,-1)$, on whose pairs of vertices $a_1b_2-a_2b_1$ lies in $[-1,1]$; a bilinear form takes its extreme values on a product of polygons at pairs of vertices. If $a\equiv y(p_1,p_2,p_3)$ and $b\equiv y'(p_1,p_2,p_3)$, then $a_1b_2-a_2b_1\equiv yy'(p_1p_2-p_2p_1)=0$.

(d) Let λ' be the least norm of a non-zero vector of Λ. The open hexagon $\{\|s\|<\lambda'\}$ has area $3\lambda'^2$ and no non-zero point of Λ, so $3\lambda'^2\leq 4N$ by Minkowski's theorem and (b). Non-zero vectors of $N\Lambda_0$ have norm at least $N>2\sqrt{N/3}$, so λ' is attained at some s with $y(s)\neq 0$, and $\lambda'=\lambda$. Let $s\in\Lambda$ with $0<\|s\|<N$; then $y(s)\neq 0$. If $s_l=0$, then $y(s)p_l\equiv 0$, which is impossible; if $s_a=s_b$ for $a\neq b$, then $y(s)(p_a-p_b)\equiv 0$, so $p_a=p_b$, which is excluded. The vectors of Λ_0 with non-zero, pairwise distinct coordinates and norm at most 3 are the orderings of $\pm(-3,1,2)$; so $\lambda\geq 3$. Finally, if $w=nw'$ with an integer $n\geq 2$ and $w'\in\mathbb{Z}^3$, then $w'\in\Lambda_0$, and $n\leq\lambda<N$ is prime to N, so $w'\equiv n^{-1}x_0(p_1,p_2,p_3)$ lies in Λ and is shorter than w.

(e) The map lands in $\mathcal{Y}(r)$ and is onto it by the definition of μ; it is injective because two vectors with the same y differ by a non-zero vector of $N\Lambda_0$, of norm at least $N>2r$. Finally $\mathcal{Y}(r)=-\mathcal{Y}(r)$, and exactly one of $\pm y$ lies in Φ. □

The exact count is in terms of the following numbers. For $l\in\{1,2,3\}$, with $\{l,l',l''\}=\{1,2,3\}$, and an integer $1\leq D\leq N$, let

$$n_l(D)=\#\{y\in G:\ [yp_{l'}]+[yp_{l''}]<D\}.$$

Such a y lies in Φ and has $[yp_l]>N-D$, since a residue sum $2N$ would need $[yp_l]>N$.

Lemma 5.5. *Let $c\in G$, c' the one of $\pm c$ in Φ, $d_l=[c'p_l]$, i the index of the largest d_l, and $\{i,i',i''\}=\{1,2,3\}$. Let ε_c be 1 if $c\in\Phi$ and 0 if not. Then $\mu(c)=N-d_i=d_{i'}+d_{i''}$, and*

$$|W(c)|=\tfrac{N-1}{2}-\varepsilon_c-n_1(d_1)-n_2(d_2)-n_3(d_3), \tag{5.1}$$

$$|W(c)|=\mu(c)-\varepsilon_c-n_i(\mu(c)+1)-n_{i'}(d_{i'})-n_{i''}(d_{i''}). \tag{5.2}$$

Proof. The value of $\mu(c)$ is Lemma 5.4(a). For a unit $x\neq c$ write $a_l=[xp_l]$. We first show that

$$\#\{y\in\Phi:\ [yp_l]<N-d_l\text{ for all }l\}=\tfrac{N-1}{2}-n_1(d_1)-n_2(d_2)-n_3(d_3). \tag{5.3}$$

An element $y\in\Phi$ cannot have $[yp_l]>N-d_l$ for two indices, since $2N-d_l-d_{l'}>N$, nor $[yp_l]=N-d_l$, which would mean $y=-c'\notin\Phi$. And $y\in\Phi$ has $[yp_l]>N-d_l$ exactly when $[yp_{l'}]+[yp_{l''}]<d_l$, which by the remark above counts $n_l(d_l)$ elements.

If $c\in\Phi$, then $c'=c$ and $a_l\neq d_l$. Let $E_l=\{x\in G:\ a_l<d_l\}$, of d_l-1 elements. For $x\neq c$, $[(c-x)p_l]=d_l-a_l+N\cdot\mathbf{1}\{a_l>d_l\}$, so $c-x$ has residue sum $N-\sum_l a_l+N\nu_x$, with ν_x the number of l with $x\notin E_l$. So $x,c-x\in\Phi$ exactly when $\nu_x=1$ (then $\sum_l a_l=2N$ would give $c-x$ residue sum 0): $W(c)$ is the set of x in exactly two of the E_l, and no x is in all three, as $\nu_x=0$ would give $c-x$ residue sum $N-\sum_l a_l\leq 0$. If $\nu^{(t)}$ counts the $x\in G$ in exactly t of the E_l, then $\sum_t\nu^{(t)}=N-1$ and $\nu^{(1)}+2\nu^{(2)}=N-3$, so $|W(c)|=\nu^{(2)}=\nu^{(0)}-2$.

The x in none of the E_l are c and the x with $a_l > d_l$ for all l, which have $\sum_l a_l > N$, so $\sum_l a_l = 2N$, and are the $-y$ with y as in (5.3). This gives (5.1).

If $c \notin \Phi$, then $[cp_l] = N - d_l$, and $c - x$ has residue sum $2N - \sum_l a_l + N \cdot \#\{l : \ a_l > N - d_l\}$. So $x, c - x \in \Phi$ exactly when $a_l < N - d_l$ for all l, which forces $\sum_l a_l < 2N$; and $W(c)$ is the set in (5.3).

For (5.2), write $\mu = \mu(c) \le 2N/3$. Of the μ units y with $[yp_i] \le \mu$, those outside Φ are the $-z$ with $z \in \Phi$ and $[zp_{i'}] + [zp_{i''}] < \mu + 1$; so $\#\{y \in \Phi : \ [yp_i] \le \mu\} = \mu - n_i(\mu + 1)$, and $n_i(d_i) = \frac{N-1}{2} - \mu + n_i(\mu + 1)$. Substitute in (5.1). □

For $(1, 4, 8)/13$, μ takes the values $5, 5, 7, 6, 6, 3$ at $y = 1, 2, 4, 5, 7, 10$, and $w = (-3, 1, 2)$, $x_0 = 10$. For $c = 9$: $c' = 4$, $d = (4, 3, 6)$, $\mu(9) = 7$, $n_3(8) = 2$, $n_1(4) = 1$, $n_2(3) = 0$, and (5.2) gives $|W(9)| = 4$.

Corollary 5.6. *The three sets counted in* (5.2) *are pairwise disjoint subsets of* $\mathcal{Y}_\Phi(\mu(c))$. *Hence* $|W(c)| \ge \mu(c) - 1 - |\mathcal{Y}_\Phi(\mu(c))|$.

Proof. A y counted by $n_l(D)$ lies in Φ and has $\mu(y) \le N - [yp_l] < D$ by Lemma 5.4(a); the thresholds are $\mu(c) + 1$, $d_{i'} < \mu(c)$ and $d_{i''} < \mu(c)$. A y in the first two sets would have $[yp_i] \ge d_i$ and $[yp_{i'}] > N - d_{i'}$, of sum more than N; one in the last two would have $[yp_{i'}] + [yp_{i''}] > 2N - \mu(c) > N$; and the first and third are like the first two. □

We bound the sets $\mathcal{Y}_\Phi$ by counting lattice points.

Lemma 5.7. *Let* $\mathcal{K}$ *be a compact convex subset of* $\Lambda_0 \otimes \mathbb{R}$. *Then*

$$\#(\mathcal{K} \cap \Lambda) \le \frac{\operatorname{area} \mathcal{K}}{N} + \frac{\operatorname{diam} \mathcal{K}}{\lambda} + \frac{\lambda \operatorname{diam} \mathcal{K}}{N} + 1,$$

with the diameter for $\|\cdot\|$. *Consequently:*

(a) *for* $r < N/2$, $|\mathcal{Y}(r)| \le 3r^2/N + 2r/\lambda + 2\lambda r/N$, *and* $|\mathcal{Y}_\Phi(r)|$ *is half of that at most;*
(b) *if* $\lambda r < N$, *then* $\mathcal{Y}(r) = \{jx_0 : \ 0 < |j| \le r/\lambda\}$ *and* $|\mathcal{Y}_\Phi(r)| = \lfloor r/\lambda \rfloor$;
(c) *for every integer* $3 \le D \le N$, $n_l(D) \le (D-3)^2/(2N) + (D-3)/\lambda + \lambda(D-3)/N + 1$, *and* $n_l(D) = 0$ *for the integers* $D \le 2$.

Proof. By Lemma 5.4(d) we can complete w to a basis (w, w_2) of Λ, and $\Pi(t, u) = tw + uw_2$ carries $\mathbb{Z}^2$ onto Λ with Jacobian N. So $\mathcal{K}' = \Pi^{-1}\mathcal{K}$ is convex of area $\operatorname{area} \mathcal{K}/N$. Its chord at height u has a length $L(u)$, concave where it is defined, and $L(u) \le \operatorname{diam} \mathcal{K}/\lambda$, since two points of $\mathcal{K}$ on a line parallel to w differ by tw with $|t|\lambda \le \operatorname{diam} \mathcal{K}$. Such a chord contains at most $L(u) + 1$ integer points. For a concave $L \ge 0$ on an interval, $\sum_{u \in \mathbb{Z}} L(u) \le \int L + \max L$: an integer at distance at least 1 to the left of a maximum point of L is paired with the unit interval to its right, one at distance at least 1 to the right with the unit interval to its left, and the at most two remaining integers $u_- < u_+$ have $L(u_-) + L(u_+) \le 2\int_{u_-}^{u_+} L \le \int_{u_-}^{u_+} L + \max L$. Points of $\mathcal{K}$ at heights u and u' differ by s with $|\det(w, s)| = N|u - u'|$, so by Lemma 5.4(c) the heights met form an interval of length at most $\lambda \operatorname{diam} \mathcal{K}/N$, containing at most $\lambda \operatorname{diam} \mathcal{K}/N + 1$ integers. Summing gives the bound.

(a) Take the ball of radius r, of area $3r^2$ and diameter $2r$, leave out the origin, and apply Lemma 5.4(e).

(b) If $\|s\| \le r$ and $s \in \Lambda$, then $|\det(w, s)| \le \lambda r < N$ and $\det(w, s) \in N\mathbb{Z}$, so $s \in \mathbb{R}w \cap \Lambda = \mathbb{Z}w$. Here $r < N/3$, so Lemma 5.4(e) applies, and $y(jw) = jx_0$.

(c) By the remark before Lemma 5.5, $y \mapsto s$ with $s_{l'} = [yp_{l'}]$, $s_{l''} = [yp_{l''}]$, $s_l = -s_{l'} - s_{l''}$ is a bijection from the set counted by $n_l(D)$ onto the points of Λ with $s_{l'}, s_{l''} \ge 1$ and

$s_{l'} + s_{l''} \le D-1$. For $D \ge 3$ these lie in a triangle of area $(D-3)^2/2$ and diameter $D-3$; for $D \le 2$ there are none. □

Lemma 5.8. *For every $c \in G$, $|W(c)| \ge \mu(c)(N-\mu(c))/N - N/\lambda - \lambda - 9/2$.*

Proof. By Lemma 5.7(c), $n_l(d) \le d^2/(2N) + d/\lambda + \lambda d/N + 1$ for every integer $1 \le d \le N$. Insert this in (5.1), with $\varepsilon_c \le 1$ and $\sum_l d_l = N$:

$$|W(c)| \ge \frac{N-3}{2} - \sum_l \Big(\frac{d_l^2}{2N} + \frac{d_l}{\lambda} + \frac{\lambda d_l}{N} + 1 \Big) = \frac{N^2 - \sum_l d_l^2}{2N} - \frac{N}{\lambda} - \lambda - \frac{9}{2}.$$

As $d_i = N - \mu(c)$ and $d_{i'} + d_{i''} = \mu(c)$, $\sum_l d_l^2 \le (N-\mu(c))^2 + \mu(c)^2$. □

When λ is small, Lemma 5.8 is weak, and we count along the line through w instead. For an integer vector v with $v_1 + v_2 + v_3 = 0$ and non-zero coordinates, and $\tau \in \mathbb{R}/\mathbb{Z}$, let $S_v(\tau) = \sum_l \{\tau v_l\}$, and let B_v be the finite set of τ at which some τv_l is an integer. Off B_v, S_v takes the values 1 and 2. Let

$$J_v = \{\tau \notin B_v : \ S_v(\tau) = 1\}, \qquad \ell_v(\gamma) = \big|J_v \cap (\gamma - J_v)\big|, \qquad \rho_v(\gamma) = \min_{n \in \Lambda_0} \|\gamma v - n\|,$$

where $|\cdot|$ is Lebesgue measure on $\mathbb{R}/\mathbb{Z}$.

Lemma 5.9. *J_v is a union of at most $\|v\|$ open arcs. For $c \in G$ let $\gamma = u/N$ with $u \equiv c/x_0$. Then:*

(a) *$x \in \Phi$ if and only if $[x/x_0]/N \in J_w$;*
(b) *$\mu(c) = N\rho_w(\gamma)$;*
(c) *$|W(c)| \ge N\ell_w(\gamma) - 2\lambda$.*

Proof. As τ increases through a point of B_v, each coordinate with $v_l > 0$ for which τv_l passes an integer lowers S_v by one, and each with $v_l < 0$ raises it by one; if two of the τv_l are integers, so is the third. So an arc of J_v can begin only where some τv_l with $v_l > 0$ passes an integer, at one of at most $\sum_{v_l>0} v_l$ points; for a vector with sum zero and three non-zero coordinates this sum is $\|v\|$.

(a) With $t = [x/x_0] \in [1, N-1]$ we have $x(p_1,p_2,p_3) \equiv tw$, so $[xp_l] = N\{tw_l/N\}$ and the residue sum of x is $NS_w(t/N)$; and $t/N \notin B_w$, as $0 < |w_l| \le \lambda < N$.

(b) The vectors of Λ_0 congruent to $c(p_1,p_2,p_3)$ are $uw + Nn$ with $n \in \Lambda_0$.

(c) The map $x \mapsto [x/x_0]$ is a bijection from G onto $[1, N-1]$ taking $c - x$ to $u - t$. By (a), $|W(c)|$ is the number of $t \in \mathbb{Z}/N$ with $t/N \in J_w \cap (\gamma - J_w)$, as $t \equiv 0$ gives $t/N = 0 \notin J_w$ and $t \equiv u$ gives $\gamma - t/N = 0 \notin J_w$. Each arc of this intersection begins where an arc of J_w or of $\gamma - J_w$ begins, so it has at most 2λ arcs, and an open arc of length L contains at least $NL - 1$ points of $N^{-1}\mathbb{Z}$. □

For short w the measure ℓ_w is bounded below by a finite computation.

Proposition 5.10. *Let v be a primitive integer vector with sum zero, non-zero pairwise distinct coordinates and $\|v\| \le 9$, and $\kappa_v = \min\{\ell_v(\gamma) : \ \rho_v(\gamma) \ge 1/4\}$. Then $\kappa_v \ge 3/20$. Consequently, if $\lambda \le 9$ and $\mu(c) \ge N/4$, then $|W(c)| \ge 3N/20 - 18$.*

Proof. Permuting the coordinates changes neither ℓ_v nor ρ_v, and $J_{-v} = -J_v$ gives $\ell_{-v}(\gamma) = \ell_v(-\gamma)$ and $\rho_{-v}(\gamma) = \rho_v(\gamma) = \rho_v(-\gamma)$; so κ_v depends only on the class of v under these operations. Each class contains exactly one $(-(a+b), a, b)$ with $1 \le a < b$, $\gcd(a,b) = 1$ and $a + b \le 9$; there are thirteen. Let $M = 4\,\mathrm{lcm}_l |v_l|$. Up to finitely many points, J_v is a union of intervals between consecutive points of $M^{-1}\mathbb{Z}$, so ℓ_v is linear between such points and

$M\ell_v(j/M)$ is an integer. The set $\{\rho_v \geq 1/4\}$ is a finite union of closed arcs whose endpoints, where some $|\gamma v_l - n_l|$ equals $1/4$, lie in $M^{-1}\mathbb{Z}$. So κ_v is the least value of $\ell_v(j/M)$ over the j with $\rho_v(j/M) \geq 1/4$, and we find [21]:

v	$(-3,1,2)$	$(-4,1,3)$	$(-5,1,4)$	$(-6,1,5)$	$(-7,1,6)$	$(-8,1,7)$	$(-9,1,8)$
κ_v	$1/6$	$1/6$	$3/20$	$1/6$	$5/28$	$5/28$	$1/6$
v	$(-5,2,3)$	$(-7,2,5)$	$(-9,2,7)$	$(-7,3,4)$	$(-8,3,5)$	$(-9,4,5)$	
κ_v	$1/5$	$5/28$	$4/21$	$5/28$	$3/16$	$11/60$	

The least value, $3/20$, is attained for $v = (-5,1,4)$ at $\gamma = 1/4$ and $\gamma = 3/4$.

For the consequence, w has these properties by Lemma 5.4(d); $\mu(c) \geq N/4$ means $\rho_w(\gamma) \geq 1/4$ by Lemma 5.9(b), and Lemma 5.9(c) gives $|W(c)| \geq 3N/20 - 2\lambda$. □

5.4. **Small subgroups.**

Proposition 5.11. *Let $N \geq 2000$, and let H have order k with $2 \leq k \leq 2\sqrt{N}$. Then H is not stable, unless $k = 2$ and $\{p_1, p_2, p_3\} = \{1, (N-3)/2, (N+1)/2\}$.*

Proof. Suppose H is stable. Then $|W(c)| \leq k-1$ for each of the $k-1$ elements c of Q_H. Put $\sigma = \sqrt{N} \geq \sqrt{2000} > 44.72$. The idea is that each $c \in Q_H$ must have $\mu(c)$ small, and Λ has too few short vectors to supply $k-1$ such c, except when $\lambda = 3$.

Step 1: $\mu(c) < N/4$ for every $c \in Q_H$. By Lemma 5.4(a), $\mu(c) \leq 2N/3$. Suppose $\mu(c) \geq N/4$. If $\lambda \leq 9$, Proposition 5.10 gives $|W(c)| \geq 3N/20 - 18$, and $3\sigma^2/20 - 2\sigma - 18 > 0$ for $\sigma \geq 44$. If $\lambda \geq 10$, then $\mu(c)(N-\mu(c))/N \geq 3N/16$, and Lemma 5.8 with $N/\lambda \leq N/10$ and $\lambda \leq 2\sqrt{N/3}$ gives $|W(c)| \geq 7N/80 - (2/\sqrt{3})\sigma - 9/2$, which exceeds 2σ because $7\sigma^2/80 - (16/5)\sigma - 9/2 > 0$ for $\sigma \geq 44$. Either way $|W(c)| > 2\sqrt{N} \geq k$.

Step 2: $\lambda = 3$. Replacing (w, x_0) by $(-w, -x_0)$ if needed, w is an ordering of $(-3,1,2)$; let i be the position of -3. Let $c \in Q_H$. As $3\mu(c) < N$, Lemma 5.7(b) gives $c = j_c x_0$ with $3|j_c| \leq \mu(c)$, and since $\|j_c w\| = 3|j_c|$, the vector $j_c w$ attains $\mu(c) = 3|j_c|$. By Lemma 5.4(a) the vector attaining $\mu(y)$ has one negative coordinate if $y \in \Phi$ and two if not; so $c \in \Phi$ if and only if $j_c > 0$, and $\mathcal{Y}_\Phi(\mu(c)) = \{jx_0 : \ 1 \leq j \leq |j_c|\}$. All these elements, c' among them, have their largest residue at i, as their vectors jw are negative at i; so i is the index of Lemma 5.5 for c. A y counted in (5.2) by some n_l has $[yp_l] > N - \mu(c) - 1 > N/2$, so its largest residue is at l, and it lies in $\mathcal{Y}_\Phi(\mu(c))$ by Corollary 5.6; hence $l = i$, and $n_{i'}(d_{i'}) = n_{i''}(d_{i''}) = 0$. Every $y \in \mathcal{Y}_\Phi(\mu(c))$ has $[yp_{i'}] + [yp_{i''}] = \mu(y) \leq \mu(c)$, so $n_i(\mu(c)+1) = |j_c|$. Now (5.2) reads

$$|W(c)| = 2|j_c| - \mathbf{1}\{j_c > 0\}.$$

As k is even, the bound $|W(c)| \leq k-1$ puts j_c in $J^* = \{-(k/2-1), \dots, -1, 1, \dots, k/2\}$. The map $c \mapsto j_c$ is injective and $|J^*| = k - 1 = |Q_H|$, so $Q_H = x_0 J^*$ and

$$H = \{1 - jx_0 : \ -(k/2-1) \leq j \leq k/2\}.$$

The group H is cyclic of order k, so $\sum_{h\in H} h^t = 0$ in $\mathbb{F}_N$ for $1 \leq t \leq k-1$. For $t = 1$ this is $k - x_0 k/2 = 0$, so $x_0 = 2$. If $k = 2$, then $2(p_1, p_2, p_3)$ is congruent to an ordering of $(-3,1,2)$, and $\{p_1, p_2, p_3\} = \{1, (N-3)/2, (N+1)/2\}$. If $k \geq 4$, then $H = \{\pm 1, \pm 3, \dots, \pm(k-1)\}$, and for $t = 2$,

$$\sum_{h\in H} h^2 = 2\sum_{j=1}^{k/2} (2j-1)^2 = \frac{k(k-1)(k+1)}{3},$$

which N does not divide, since $N > 3$ and $1 \leq k-1 < k+1 < N$.

Step 3: $\lambda \geq 4$. Let $\xi = 1 - 1/\lambda - \lambda/N$. The convex function $\lambda \mapsto 1/\lambda + \lambda/N$ is at most $1/4 + 2/\sqrt{3N}$ at both ends of $[4, 2\sqrt{N/3}]$, so $\xi \geq 3/4 - 2/\sqrt{3N} > 0.72418$. Let $c \in Q_H$ and $\mu = \mu(c) < N/4$. By Corollary 5.6 and Lemma 5.7(a),

$$k - 1 \geq |W(c)| \geq \mu - 1 - \Big(\frac{1.5\mu^2}{N} + \frac{\mu}{\lambda} + \frac{\lambda\mu}{N}\Big) = q(\mu) - 1, \qquad q(\mu) = \xi\mu - \frac{1.5\mu^2}{N}.$$

Since $3k/N \leq 6/\sigma < 0.13417$, the concave function q has $q(2k) = 2k(\xi - 3k/N) > 1.18k$ and $q(N/4) = N(\xi/4 - 3/32) > 0.0872N > 2\sigma \geq k$, and $2k \leq 4\sigma < N/4$. So $q > k$ on $[2k, N/4]$, hence $\mu < 2k$, and then $q(\mu) \geq \mu(\xi - 3k/N)$ gives

$$\mu(c) \leq \mu^* = \frac{k}{\xi - 3k/N} < 1.695k \qquad (c \in Q_H).$$

If $k = 2$, then $Q_H = \{2\}$ and $\lambda \leq \mu(2) < 4$, a contradiction; so $k \geq 4$.

(3a) If $\lambda\mu^* < N$, then for $c \in Q_H$, Lemma 5.7(b) gives $|\mathcal{Y}_\Phi(\mu(c))| = \lfloor \mu(c)/\lambda \rfloor \leq \mu(c)/4$, so $\mu(c) \leq 4k/3$ by Corollary 5.6, and $c = j_c x_0$ with $0 < |j_c| \leq 4k/(3\lambda) \leq k/3$. As $c \mapsto j_c$ is injective, $k - 1 \leq 2k/3$, so $k \leq 3$.

(3b) If $\lambda\mu^* \geq N$, then $\mu^*/\lambda \leq (\mu^*)^2/N$ and $\mu^* < N/2$, and $Q_H \subset \mathcal{Y}(\mu^*)$. By Lemma 5.7(a), with $\lambda \leq 2\sqrt{N/3}$ and $k \leq 2\sigma$,

$$k - 1 \leq \frac{5(\mu^*)^2}{N} + \frac{2\lambda\mu^*}{N} \leq \frac{5(1.695k)^2}{N} + \frac{4}{\sqrt{3}} \cdot \frac{1.695k}{\sigma} \leq \frac{32.66k}{\sigma} \leq 0.731k,$$

which is false for $k \geq 4$. □

The case $\lambda = 3$ is where the estimates are tight: it consists of the triangles $(1, 2, N-3)/N$ and their images under multiplication by units, and there the $k - 1$ elements of Q_H can fill all of J^*. For $k \geq 4$ only the power sums of H exclude this; for $k = 2$ nothing does, and the image with $x_0 = 2$ is $(1, (N-3)/2, (N+1)/2)/N$.

5.5. **Proof of Theorem 1.6.** Let H be as in the theorem; k is even and $m \geq 2$. For the primes $7 \leq N < 2000$ we find, for every scalene triangle and every such H, an $h \in H$ and an $x \in \Phi$ as in the theorem, except when $k = 2$ and the triangle is $(1, (N-3)/2, (N+1)/2)/N$ or $(2, 3, 6)/11$ [21]. Let $N \geq 2000$. If $k \leq 2\sqrt{N}$, apply Proposition 5.11. If $2\sqrt{N} < k \leq (N-1)(N-3)/(8N)$, then $\eta^* < \sqrt{N} < k/2$, so

$$(N-1)^2 k - (N^2 - 1)\eta^* > \frac{k(N-1)(N-3)}{2} \geq 4Nk^2 > 4N(k-1)^2,$$

and Proposition 5.3 applies. If $k > (N-1)(N-3)/(8N)$, then $m < 8N/(N-3) < 8.02 < 11.15 < (N-5)/(4\sqrt{N})$, and Proposition 5.2 applies.

It remains to see that in the two exceptional cases $\{\pm 1\}$ is stable, that is, that the triangle has no witness. For $(1, (N-3)/2, (N+1)/2)/N$, write $a = 2\alpha$ and $2 - a = 2\beta$ with $0 \leq \alpha, \beta < N$, so that $\alpha + \beta \in \{1, N+1\}$. The residues of a are $[2\alpha]$, $[-3\alpha]$, α, and by Lemma 4.1, $a, 2 - a \in \Phi$ forces $\alpha, \beta \in (0, N/3) \cup (N/2, 2N/3)$, which leaves only $\alpha = \beta = (N+1)/2$, that is $a = 1$. For $(2, 3, 6)/11$, $\Phi = \{1, 2, 4, 6, 8\}$ and $2 - a \notin \Phi$ for $a = 2, 4, 6, 8$. □

6. Automorphisms and the involution

For even N the unfolding has the involution $\iota = T^{N/2}$, with $\iota^*\omega = -\omega$, and the orbit closure lies in the locus of holonomy double covers over the stratum of $(X/\iota, \omega^2)$. This section shows that for a scalene triangle ι is the only automorphism that matters (Theorem 6.2), and draws the consequences for V: its characters are odd, its rel is bounded by the pairs of

zeros that ι exchanges, and the criterion by residues need only be checked at odd characters (Corollary 6.3).

Let $\mathcal{G}$ be the group of automorphisms τ of the Riemann surface X with $\tau^*\omega \in \mathbb{C}\omega$; as no points of cone angle 2π are marked, τ need not preserve them. The quotient $X/\langle T\rangle$, with the metric induced by $|\omega|$, is the double of Δ, a flat sphere with three cone points; for a larger $\mathcal{G}$ it would cover another flat sphere, and the next lemma lists when that can happen.

Lemma 6.1. *Let D be a flat sphere with exactly three cone points, of angles $2\pi\alpha_1$, $2\pi\alpha_2$, $2\pi\alpha_3$, where $0 < \alpha_i < 1$ and $\alpha_1+\alpha_2+\alpha_3 = 1$. Let D' be a compact Riemann surface with a flat cone metric and $f\colon D \to D'$ a holomorphic map of degree $q \geq 2$ such that*

$$\theta_D(d) = e_d\,\theta_{D'}(f(d)) \qquad (d \in D), \tag{6.1}$$

where θ_D, $\theta_{D'}$ are the cone angles and e_d is the ramification index of f at d. Then either $q = 2$ and two of the α_i are equal, or $(\alpha_1,\alpha_2,\alpha_3)$ is, up to order, $(1,1,4)/6$, $(1,2,3)/6$, $(1,1,2)/4$ or $(1,1,1)/3$.

Proof. D' is a sphere, being the image of one. No point of D has angle above 2π, so by (6.1) no point of D' does. Let $s_1,\dots,s_k$ be the points of D' of angle less than 2π, of angles $2\pi\beta_j$. Over a point of angle 2π, (6.1) forces $e = 1$; so f is unramified off the fibers over the s_j, and each of the three cone points of D lies over some s_j. Let P_j be the set of cone points of D over s_j; a cone point $v \in P_j$ of angle $2\pi\alpha_v$ has $e_v = \alpha_v/\beta_j$, and a point over s_j that is not a cone point has $e = 1/\beta_j =: n_j$, which is then an integer at least 2. Let r_j be the number of such points and $E_j = \sum_{v\in P_j} e_v \geq |P_j|$. Then

$$q = E_j + r_j n_j, \tag{6.2}$$

and the Riemann–Hurwitz formula, $-2 = -2q + \sum_j (q - |P_j| - r_j)$, gives

$$\textstyle\sum_j r_j = (k-2)q - 1. \tag{6.3}$$

If $k \leq 2$ the right side is negative. Since $r_j \leq (q-E_j)/2 \leq (q-|P_j|)/2$ by (6.2), $\sum_j r_j \leq (kq-3)/2$, which with (6.3) gives $(k-4)q \leq -1$. Hence $k = 3$ and $\sum_j r_j = q-1$. Multiplying (6.2) by β_j and summing over j gives $q(\beta_1+\beta_2+\beta_3) = 1 + (q-1)$, so $\beta_1+\beta_2+\beta_3 = 1$.

Let $J = \{j :\ r_j > 0\}$, which is not empty, as $q \geq 2$. If $J = \{j_1\}$, then $q - 1 = r_{j_1} \leq q/2$, so $q = 2$, $r_{j_1} = 1$, $n_{j_1} = 2$ and $E_{j_1} = 0$. The other two fibers have $E_j = 2$ and contain the three cone points, so one consists of a single cone point with $e = 2$ and the other of two cone points with $e = 1$, which then have the same angle $2\pi\beta_j$. If $|J| = 3$, then $\beta_j = 1/n_j$ for all j, and (n_1,n_2,n_3) is $(2,3,6)$, $(2,4,4)$ or $(3,3,3)$ up to order. If $J = \{1,2\}$ with $n_1 \leq n_2$, then $r_1 + r_2 = q - 1$ reads

$$q\Big(1 - \frac{1}{n_1} - \frac{1}{n_2}\Big) = 1 - \frac{E_1}{n_1} - \frac{E_2}{n_2} \leq 1, \tag{6.4}$$

with $1 - 1/n_1 - 1/n_2 = \beta_3 > 0$, and $r_2 \geq 1$ gives $q \geq n_2 + E_2$. If $n_1 \geq 3$, the left side of (6.4) is at least $n_2(2/3 - 1/n_2) \geq 1$, with equality only for $n_1 = n_2 = q = 3$, and then $\beta_3 = 1/3$. If $n_1 = 2$ and $n_2 = n \geq 3$, (6.4) reads $q(n-2) = 2n - nE_1 - 2E_2$, so $E_1 \leq 1$, and $q \geq n$ gives $n \leq 4 - E_1 - 2E_2/n$. If $E_1 = 1$, then $n = 3$, $E_2 = 0$, $q = 3$ and $\beta_3 = 1/6$; if $E_1 = 0$, then $n = 4$, with $E_2 = 0$, $q = 4$ and $\beta_3 = 1/4$, or $n = 3$, with $q = 6 - 2E_2 \geq 3 + E_2$, so $E_2 \leq 1$, and $\beta_3 = 1/6$. So if $|J| \geq 2$, $(\beta_1,\beta_2,\beta_3)$ is $(1/2,1/3,1/6)$, $(1/2,1/4,1/4)$ or $(1/3,1/3,1/3)$ up to order, and each $\alpha_v = e_v\beta_j$ is a multiple of $1/6$, $1/4$ or $1/3$ respectively; three such numbers in $(0,1)$ with sum 1 are those listed. □

Theorem 6.2. *Let Δ be scalene and not $(1,2,3)/6$. Then $\mathcal{G} = \langle T \rangle$. In particular the identity is the only automorphism τ of X with $\tau^*\omega = \omega$; for odd N none has $\tau^*\omega = -\omega$, and for even N only ι does.*

Proof. By (2.1) and $g_i \le p_i$, the genus of X is at least two unless p_1, p_2, p_3 all divide N, which for a scalene triangle happens only for $(1,2,3)/6$. So $\mathcal{G}$ is finite. Writing $\tau^*\omega = c(\tau)\omega$, c is a homomorphism from a finite group to $\mathbb{C}^*$, so $|c(\tau)| = 1$ and $\mathcal{G}$ preserves the flat metric $|\omega|$. If a finite group of automorphisms of a Riemann surface preserves a flat cone metric, the quotient carries a flat cone metric whose angle at the image of z is the angle at z divided by the order of the stabilizer of z, since the stabilizer is cyclic and acts on a punctured neighborhood of z freely, by rotations of the cone. The quotient by $\langle T \rangle$ is $x\colon X \to \mathbb{P}^1$ with the metric $|x^{p_1/N-1}(x-1)^{p_2/N-1}dx|$, the double D of Δ, with three cone points of angles $2\pi p_i/N$. Let $D' = X/\mathcal{G}$ and $f\colon D \to D'$ the induced map, of degree $q = [\mathcal{G} : \langle T \rangle]$. For $z \in X$ over $d \in D$, the ramification index of f at d is $|\mathcal{G}_z|/|\langle T \rangle_z|$, and the angles at d and $f(d)$ are the angle at z divided by $|\langle T \rangle_z|$ and by $|\mathcal{G}_z|$; this is (6.1). If $q \ge 2$, Lemma 6.1 says that two of the p_i are equal or that Δ is $(1,1,4)/6$, $(1,2,3)/6$, $(1,1,2)/4$ or $(1,1,1)/3$, and only $(1,2,3)/6$ of these is scalene. So $q = 1$. Finally $(T^j)^*\omega = \zeta^j\omega$ is ω only for $N \mid j$, and $-\omega$ only for N even and $j \equiv N/2 \pmod N$. □

For even N let

$$E^- = \{u \in H^1(X, \Sigma; \mathbb{C}) : \ \iota^* u = -u\} = \bigoplus_{\ell \text{ odd}} H^1(X,\Sigma)_\ell,$$

since $\iota^* = (T^*)^{N/2}$ acts on $H^1(X,\Sigma)_\ell$ by $(-1)^\ell$, and $Z^- = E^- \cap Z = \bigoplus_{c \text{ odd}} Z_c$.

Corollary 6.3. *Let N be even and Δ scalene, not $(1,2,3)/6$. Then $V \subset E^-$. Consequently:*

(i) *every character in S is odd;*

(ii) *$V \cap Z \subset Z^-$, and $\dim Z^-$ is the sum of $g_i/2$ over the $i \in I$ with N/g_i odd, the number of pairs of zeros of ω that ι exchanges; in particular $\operatorname{rel} \mathcal{M} = 0$ if N/g_i is even for every $i \in I$;*

(iii) *if Δ is not a lattice triangle, and $\mathcal{P}(c)$ or $\mathcal{P}(-c)$ is not empty for every odd c with $Z_c \ne 0$, then $\operatorname{rank} \mathcal{M} \ge 2$.*

Proof. Since ι is an isomorphism from $(X,-\omega)$ to (X,ω), it is one from $(-\mathrm{Id})\, g(X,\omega)$ to $g(X,\omega)$ for every $g \in \mathrm{GL}(2,\mathbb{R})$. So $-\mathrm{Id}$ fixes the orbit of (X,ω) pointwise, hence also $\mathcal{M}$. Mark (X,ω) by $\varphi\colon (S_0,\Sigma_0) \to (X,\Sigma)$. By Theorem 6.2 the identity is the only automorphism fixing ω, so the period map of marked surfaces is a chart of the stratum at (X,ω), which we read in $H^1(X,\Sigma;\mathbb{C})$ through φ on a small neighborhood $U' \subset U$ of $[\omega]$ with $\iota^* U' = -U'$. In this chart $-\mathrm{Id}$ lifts to $(Y,\eta,\psi) \mapsto (Y,-\eta,\psi\varphi^{-1}\iota\varphi)$, which fixes (X,ω,φ), as $\iota^2 = 1$, and acts on periods by the linear map $-\iota^*$. As the chart is injective, $\mathcal{M} \cap U'$ lies in the fixed space of $-\iota^*$, which is E^-; so $V \subset E^-$, as $\mathcal{M} \cap U = V \cap U$ (Section 2.3).

(i) For $a \in S$ the line $V_a = H^1(X,\Sigma)_a$ (Lemma 2.4) lies in E^-, while $E^- \cap H^1(X,\Sigma)_a = 0$ for even a.

(ii) As Z is T^*-stable, $V \cap Z \subset E^- \cap Z = Z^-$. By Lemma 2.2, Z_0 has the even character 0, and Z_c for $c \not\equiv 0$ is a line if c is a multiple of N/g_i for some $i \in I$, and zero otherwise. An odd multiple of N/g_i exists only if N/g_i is odd; then g_i is even, and the odd multiples are kN/g_i with k odd, $0 < k < g_i$, which are $g_i/2$ in number. On the g_i points of Σ_i, which T permutes cyclically, ι is the shift by $N/2$ modulo g_i: trivial if N/g_i is even, and the shift by $g_i/2$, which exchanges them in $g_i/2$ pairs, if N/g_i is odd.

(iii) Suppose the rank is one. Then $S = \{\pm 1\}$, and $R = V \cap Z$ is not zero, since otherwise $\dim_{\mathbb{C}} \mathcal{M} = 2$ and Δ would be a lattice triangle by Lemma 2.3; $R = \bar{R}$, as V and Z are real; and $R \subset K$, since for $(a, b) \in \mathcal{P}$, so that $b \notin \{\pm a\} = aS$, and $\delta f \in R$, Lemma 3.5 with $u = a$, Lemma 3.1 with $\theta = \omega_b$ and (3.1) give $\lambda_{ab}(f) = 0$. Also $R \subset Z^-$ by (ii). Let $\delta f \in R$. As δ is equivariant with the constants as kernel, we may take $f = \sum_{c \text{ odd}} f_c$ with $f_c \in (\mathbb{C}^\Sigma)_c$, a multiple of $e_{i,c}$ for the vertex i of c if $Z_c \neq 0$ and zero otherwise. A pair $(a, b) \in \mathcal{P}(c)$ gives $f_c = 0$, as $0 = \lambda_{ab}(f) = \lambda_{ab}(f_c)$ by Lemma 3.6(b) and $\lambda_{ab}(e_{i,c}) \neq 0$; a pair in $\mathcal{P}(-c)$ does the same through $\bar{f}$, as $\delta \bar{f} \in \bar{R} = R \subset K$ and the component of $\bar{f}$ of character $-c$ is $\overline{f_c}$. So $R = 0$, a contradiction. □

So for even N, part (iii) is Corollary 3.8 with condition (i) asked only at odd c, and without condition (ii) or the bound on $|I|$, because by part (ii) $V \cap Z$ lies in the sum of the Z_c with c odd.

7. A second-order criterion

This section proves a criterion of second order along the Teichmüller disc and applies it to two families of obtuse triangles without a witness from [17, §4], defined in Section 7.4: to $\mathcal{A}$ for $s \geq 3$ and to $\mathcal{B}$. At rank one the cohomology splits along the disc into flat summands that stay of Hodge type (Lemma 7.3). We follow ω_a as a holomorphic form η_t on the deformed surface X_t. For the pairs (a, b) of Section 7.1, the component of $[\eta_t]$ in the summand containing H^1_{-b} begins with its t^2 term, which is then holomorphic and pairs to zero with ω_b; this pairing is a multiple of an integral $J(a, b)$ (Theorem 7.4). Lemma 7.2 solves the $\bar{\partial}$-problem behind it, and Proposition 7.5 evaluates it as a Bezoutian, through a relation between two moments (Lemma 7.1).

Throughout, $\Delta = (p_1, p_2, p_3)/N$ is obtuse with $p_3 > N/2$. For $|t| < 1$ let X_t be the surface with the complex structure in which $\omega + t\bar{\omega}$ is holomorphic, marked by the identity. Its holomorphic charts are $\vartheta + t\bar{\vartheta}$ for a local primitive ϑ of ω; $(X_t)_{|t|<1}$ is the Teichmüller disc of (X, ω) and lies in its $\mathrm{GL}(2, \mathbb{R})$-orbit, and through the marking $H^{1,0}(X_t) \subset H^1(X; \mathbb{C})$. We write d^2x for Lebesgue measure on $\mathbb{C}$ and use the coordinate x of (2.2).

7.1. **The pairs and the integral.** For an integer m let $\langle m \rangle = 1 - \{-m/N\} \in (0, 1]$, which is $\{m/N\}$ unless $N \mid m$, when it is 1. For $k \in \mathbb{Z}/N$ put

$$\mathrm{w}(k) = \langle kp_1 \rangle + \langle kp_2 \rangle + \langle kp_3 \rangle \in \{1, 2, 3\}.$$

By Lemma 2.1 and its proof, $\mathrm{w}(k) = 1$ if and only if $k \in \Phi$; $\mathrm{w}(k) + \mathrm{w}(-k) = 3$ if $k \in \Phi \cup -\Phi$; and if $k \not\equiv 0$ is not in $\Phi \cup -\Phi$, then exactly one $\langle kp_i \rangle$ is 1, so $\mathrm{w}(k) = 2$, and $H^1_k = 0$. For $k \in \Phi$, $\omega_k = \varphi_k(x)\, dx$ on the upper half-plane with $\varphi_k(x) = x^{\langle kp_1 \rangle - 1}(x - 1)^{\langle kp_2 \rangle - 1}$, on one branch of y, the same for every k. For every k the same formula defines a meromorphic form $\varphi_k\, dx$ of character k on X, namely $y^k x^i (x - 1)^j\, dx$ with the integers i, j for which the exponents are $\langle kp_1 \rangle - 1$ and $\langle kp_2 \rangle - 1$. Write $\varphi = \varphi_1$ and

$$\mu = \frac{p_1 + p_2}{N} < \frac{1}{2}, \qquad \xi = \frac{p_1}{p_1 + p_2}, \qquad I_0 = \int_{\mathbb{C}} |\varphi|^2\, d^2x, \qquad I_1 = \int_{\mathbb{C}} |\varphi(x)|^2 x\, d^2x.$$

The map $\int \varphi\, dx$ takes the upper half-plane onto a triangle Δ_0 similar to Δ and the lower half-plane onto its mirror image, so I_0 is twice the area of Δ_0. The integral I_1 converges, as $|\varphi|^2 = O(|x|^{-2-2p_3/N})$ at infinity and $2p_3/N > 1$.

Lemma 7.1. $I_1 = \xi I_0$.

Proof. The function $h(x) = x(x-1)|\varphi(x)|^2 = x^{p_1/N}(x-1)^{p_2/N}\,\overline{\varphi(x)}$ has $\partial_x h = |\varphi|^2(\frac{p_1}{N}(x-1) + \frac{p_2}{N}x) = |\varphi|^2(\mu x - p_1/N)$. By Stokes' formula, $\int \partial_x h\, d^2x$ over the region $\epsilon < |x| < R$, $|x-1| > \epsilon$ is a sum of integrals of $h\, d\bar{x}$ over three circles. On $|x| = R$, $|h| = O(R^{2\mu-2})$, so that integral is $O(R^{2\mu-1})$, which tends to 0 because $\mu < 1/2$; this is where we use that Δ is obtuse. On $|x| = \epsilon$, $|h| = O(\epsilon^{2p_1/N-1})$ and the integral is $O(\epsilon^{2p_1/N})$, and likewise at 1. Hence $\mu I_1 = (p_1/N)I_0$. □

Let $\mathcal{P}_2$ be the set of ordered pairs (a,b) with $a, b \in \Phi$, $a + b \equiv 4$, $a \not\equiv 1$, $\mathrm{w}(c) = 2$ for $c = a - 2$, and either b a unit with $a \not\equiv \pm b$, or b not a unit, a a unit, and c a unit or N dividing one of cp_1, cp_2, cp_3. The conditions on c and on units keep H^1_a and H^1_c out of the summand containing H^1_{-b} at rank one (step (iv) of the proof of Theorem 7.4); $\mathrm{w}(c) = 2$ is also the hypothesis of Lemma 7.2.

For $a, b \in \Phi$ with $a + b \equiv 4$ and $\mathrm{w}(a-2) = 2$, and $c = a - 2$, let $\epsilon_1 = \langle ap_1\rangle - p_1/N$, $\epsilon_2 = \langle ap_2\rangle - p_2/N$,

$$E_1 = \langle bp_1\rangle + \langle cp_1\rangle - \tfrac{2p_1}{N}, \qquad E_2 = \langle bp_2\rangle + \langle cp_2\rangle - \tfrac{2p_2}{N},$$
$$E_1' = \langle ap_1\rangle - \langle cp_1\rangle - \tfrac{2p_1}{N} + 1, \qquad E_2' = \langle ap_2\rangle - \langle cp_2\rangle - \tfrac{2p_2}{N} + 1,$$
$$P(x) = x^{E_1}(x-1)^{E_2}, \qquad Q(x) = x^{E_1'}(x-1)^{E_2'}(\epsilon_1(x-1) + \epsilon_2 x).$$

The four exponents are integers, since $b + c \equiv 2$ and $a - c = 2$. Let $\chi_a(x) = |\varphi(x)|^2 Q(x)$ and

$$G(x) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\chi_a(x')}{x - x'}\, d^2x'.$$

A square-integrable form is *closed* if it is closed as a current; for a form that is smooth off finitely many points this is the same as being closed off them, as a point has zero capacity in dimension two.

Lemma 7.2. *Let $a \in \Phi$ with $\mathrm{w}(a-2) = 2$, and $c = a - 2$. The integral G converges, and $\psi = (G \circ x)\,\varphi_c\, dx$ is the only square-integrable form of type $(1,0)$ and character c on X such that $\psi + (\omega_a/\omega)\bar{\omega}$ is closed.*

Proof. Write $(\omega_a/\omega)\bar{\omega} = \Theta_a\, d\bar{x}$ with $\Theta_a = |\varphi|^2\varphi_a/\varphi^2$, and $\psi = \Psi\, dx$. Off the branch points, $\psi + \Theta_a\, d\bar{x}$ is closed when $\partial_{\bar{x}}\Psi = \partial_x\Theta_a$. For $\Psi = \varphi_c H$ this reads $\partial_{\bar{x}}H = \partial_x\Theta_a/\varphi_c$, and a direct computation gives

$$\frac{\partial_x\Theta_a}{\varphi_c} = |\varphi|^2\,\frac{\varphi_a}{\varphi^2\varphi_c}\Big(\frac{\epsilon_1}{x} + \frac{\epsilon_2}{x-1}\Big) = |\varphi|^2 x^{E_1'}(x-1)^{E_2'}(\epsilon_1(x-1) + \epsilon_2 x) = \chi_a.$$

At 0, $|\chi_a| = O(|x|^{\langle ap_1\rangle - \langle cp_1\rangle - 1})$, with exponent greater than -2, and likewise at 1. At infinity $|\chi_a| = O(|x|^{\langle cp_3\rangle - \langle ap_3\rangle - 2})$, because $\mathrm{w}(a) = 1$ and $\mathrm{w}(c) = 2$ give

$$(\langle ap_1\rangle + \langle ap_2\rangle) - (\langle cp_1\rangle + \langle cp_2\rangle) = \langle cp_3\rangle - \langle ap_3\rangle - 1,$$

and this exponent lies in $(-3,-1)$. So G converges, $\partial_{\bar{x}}G = \chi_a$ as distributions, $G = O(1 + |x|^{\langle ap_1\rangle - \langle cp_1\rangle}\log(1/|x|))$ at 0 and likewise at 1, and $G = O(|x|^{\max(\langle cp_3\rangle - \langle ap_3\rangle, 0) - 1}\log|x|)$ at infinity. Hence

$$|\varphi_c G|^2 = O(|x|^{2\langle cp_1\rangle - 2} + |x|^{2\langle ap_1\rangle - 2}\log^2|x|)$$

at 0, likewise at 1, and, as $|\varphi_c|^2$ has order $|x|^{-2\langle cp_3\rangle}$ at infinity,

$$|\varphi_c G|^2 = O(|x|^{-2\min(\langle ap_3\rangle, \langle cp_3\rangle) - 2}\log^2|x|)$$

there. All are integrable, so ψ is square integrable. So is $(\omega_a/\omega)\bar{\omega}$, as ω_a is, and their sum is closed off finitely many points, hence closed.

If ψ' is another such form, then $\psi' - \psi = \varphi_c h\,dx$ with h holomorphic on $\mathbb{C}\setminus\{0,1\}$ and $\varphi_c h$ square integrable near 0, 1 and ∞. The terms x^k of the Laurent series of h at 0 are orthogonal on circles, so each term with a non-zero coefficient has $|x|^{2\langle cp_1\rangle - 2 + 2k}$ integrable at 0, that is $k > -\langle cp_1\rangle \geq -1$; likewise at 1. At infinity $|x|^{2k-2\langle cp_3\rangle}$ must be integrable, so $k < \langle cp_3\rangle - 1 \leq 0$. Hence h is entire and tends to 0 at infinity, and $h = 0$. □

For such a and $b \in \Phi$ with $a + b \equiv 4$, $(\omega_b/\omega)\bar\omega = \Theta_b\,d\bar x$ with $\Theta_b = |\varphi|^2\varphi_b/\varphi^2$, and $\varphi_c G\,\Theta_b = |\varphi|^2 G\,\varphi_c\varphi_b/\varphi^2 = |\varphi|^2 GP$. Set

$$J(a,b) = \int_{\mathbb{C}} |\varphi(x)|^2 G(x) P(x)\,d^2x.$$

The 2-form $\psi\wedge(\omega_b/\omega)\bar\omega$ has character $c + b - 2 \equiv 0$, so it is pulled back from the sphere; as x has degree N and $dx\wedge d\bar x = -2i\,d^2x$,

$$\int_X \psi\wedge\frac{\omega_b}{\omega}\,\bar\omega = -2iN\,J(a,b). \tag{7.1}$$

7.2. **The criterion.** At rank one the splitting of Lemma 2.5 can be written down completely. For a unit u put $H^1_{\pm u} = H^1_u \oplus H^1_{-u}$, and let $H_\times$ be the sum of the H^1_u over the units u.

Lemma 7.3. *Suppose* $\operatorname{rank}\mathcal{M} = 1$. *Over the Teichmüller disc there are constant subspaces of $H^1(X;\mathbb{C})$, equal at X to the planes $H^1_{\pm u}$, one for each unit u up to sign, and to*

$$H^\perp_\times = \bigoplus_{k \text{ not a unit}} H^1_k,$$

whose direct sum is $H^1(X;\mathbb{C})$, and each of which, at every X_t with $|t|$ small enough, is the direct sum of its intersections with $H^{1,0}(X_t)$ and $\overline{H^{1,0}(X_t)}$.

Proof. By Lemma 2.4, $S = \{\pm1\}$ and $p(V) = H^1_{\pm1}$. By Lemma 2.5, for each unit u the plane $H^1_{\pm u} = \sigma_u(p(V))$ is the fiber at X of a summand V_ρ of Wright's decomposition, and at every point of $\mathcal{M}\cap U$, in particular at X_t for small $|t|$, the fiber of V_ρ is the sum of its intersections with $H^{1,0}$ and $H^{0,1}$. The disc lies in the orbit of (X,ω), hence in the smooth part $\mathcal{M}_0$ (Section 2.3), and it is simply connected, so over it each V_ρ is a constant subspace of $H^1(X;\mathbb{C})$. For distinct planes these are distinct summands, hence independent, and their sum is the constant space $H_\times$. The cup product is preserved by the flat structure and, being T^*-invariant, pairs H^1_j with H^1_k to zero unless $j + k \equiv 0$; as it is non-degenerate, it pairs H^1_k perfectly with H^1_{-k}. The units are closed under $k \mapsto -k$, so the cup product is non-degenerate on $H_\times$, and the symplectic orthogonal of $H_\times$ is a constant complement, equal at X to the sum of the H^1_k with k not a unit. Finally, $H^{1,0}(X_t)$ and its conjugate are isotropic, so the orthogonal of a subspace $Y = (Y\cap H^{1,0})\oplus(Y\cap H^{0,1})$ is again such a sum: $v = v' + v''$ is orthogonal to Y if and only if $v''\perp Y\cap H^{1,0}$ and $v'\perp Y\cap H^{0,1}$, that is, if and only if v' and v'' are both orthogonal to Y. With $Y = H_\times$ this gives the last statement for $H^\perp_\times$. □

Theorem 7.4. *Let Δ be obtuse and $(a,b)\in\mathcal{P}_2$. If* $\operatorname{rank}\mathcal{M} = 1$, *in particular if Δ is a lattice triangle, then $J(a,b) = 0$.*

Proof. (i) *The expansion.* The holomorphic 1-forms on the surfaces X_t form a holomorphic vector bundle over the disc, which is trivial, so there is a holomorphic section $t\mapsto\eta_t$, η_t a holomorphic form on X_t, with $\eta_0 = \omega_a$. Off the zeros of ω, with ϑ a local primitive of ω, the functions $(t, \vartheta + t\bar\vartheta)$ are holomorphic coordinates on the total space of the family, so $\eta_t = F(t,\vartheta + t\bar\vartheta)\,(d\vartheta + t\,d\bar\vartheta)$ with F holomorphic; that is, $\eta_t = f\,(\omega + t\bar\omega)$ with $f(\cdot,\vartheta)$

holomorphic in t for each ϑ. Write $f = \sum_k t^k f_k$. Closedness of η_t is $\partial_{\bar\vartheta} f = t\,\partial_\vartheta f$, that is $\partial_{\bar\vartheta} f_k = \partial_\vartheta f_{k-1}$, with $f_0 = \omega_a/\omega$ and $f_{-1} = 0$. So $\eta_t = \sum_k t^k \eta_k$ with $\eta_k = f_k\omega + f_{k-1}\bar\omega$ closed off the zeros of ω.

For a form $\eta = f(\omega + t\bar\omega)$, pointwise $i\,\eta\wedge\bar\eta = (1-|t|^2)|f|^2\, i\,\omega\wedge\bar\omega$, while the L^2 density of η for the flat metric of X is $(1+|t|^2)|f|^2\, i\,\omega\wedge\bar\omega$. Hence

$$\|\eta_t\|^2_{L^2(X)} = \frac{1+|t|^2}{1-|t|^2}\, i\int_X \eta_t\wedge\bar\eta_t,$$

and $i\int_X \eta_t\wedge\bar\eta_t$ is the cup product of $[\eta_t]$ with its conjugate, which is continuous in t. So on $|t|\le\epsilon$ the η_t are bounded in $L^2(X)$, and $t\mapsto\eta_t$, being weakly holomorphic (against test forms supported off the zeros of ω) and locally bounded, is a holomorphic map into $L^2(X)$. Each η_t is closed as a current on X, being square integrable and closed off finitely many points, and the class of a closed square-integrable form depends continuously on it in L^2. Hence every η_k is square integrable and closed, and $[\eta_t] = \sum_k t^k[\eta_k]$ in $H^1(X;\mathbb{C})$.

(ii) *Characters.* As $T^*(\omega + t\bar\omega) = \zeta(\omega + \zeta^{-2}t\bar\omega)$, T is a holomorphic map from $X_{\zeta^{-2}t}$ to X_t, and $(T^j)^*$ maps $H^{1,0}(X_{\zeta^{2j}t})$ to $H^{1,0}(X_t)$. Replacing η_t by $\frac1N\sum_j \zeta^{-aj}(T^j)^*\eta_{\zeta^{2j}t}$, again a holomorphic section through ω_a of the same shape, replaces η_k by its projection $\frac1N\sum_j \zeta^{-(a-2k)j}(T^j)^*\eta_k$ to the ζ^{a-2k}-eigenspace of T^*. So we may assume $T^*\eta_k = \zeta^{a-2k}\eta_k$, and $[\eta_k]\in H^1_{a-2k}$.

(iii) *The first coefficient.* $\eta_1 = f_1\omega + (\omega_a/\omega)\bar\omega$ is square integrable and closed, and its two terms are orthogonal, so $f_1\omega$ is a square-integrable form of type $(1,0)$ and character c; by Lemma 7.2, $f_1\omega = \psi$.

(iv) *The splitting.* Suppose $\operatorname{rank}\mathcal{M} = 1$, and let $\mathcal{S}$ be the summand of Lemma 7.3 that contains H^1_{-b}: the one equal to $H^1_{\pm b}$ at X if b is a unit, and $H^1_\times$ if not. Each eigenspace H^1_k lies in one summand, so the projection of $[\eta_t]$ to $\mathcal{S}$ along the others is $\sum_k t^k[\eta_k]$ over the k with $H^1_{a-2k}\subset\mathcal{S}$, and it lies in $H^{1,0}(X_t)$, as every summand is split at X_t. For $k=0$, $[\eta_0] = [\omega_a]\in H^1_a$, which is not in $\mathcal{S}$: if b is a unit because $a\not\equiv\pm b$, and if not because a is a unit. For $k=1$, $[\eta_1]\in H^1_c$, which is zero or not in $\mathcal{S}$: if b is a unit, $c\equiv b$ would give $\mathrm{w}(c) = \mathrm{w}(b) = 1$, and $c\equiv -b$ would give $a+b\equiv 2$, against $a+b\equiv 4$ and $N\ge 3$; if b is not a unit, c is a unit or $H^1_c = 0$. For $k=2$, $[\eta_2]\in H^1_{a-4} = H^1_{-b}\subset\mathcal{S}$. So the projection is $t^2[\eta_2] + O(t^3)$, and t^{-2} times it lies in $H^{1,0}(X_t)$ for $t\ne 0$ and tends to $[\eta_2]$. As $H^{1,0}(X_t)$ depends continuously on t, $[\eta_2]$ is the class of a holomorphic form η' on X, and $\int_X \eta_2\wedge\omega_b = \int_X \eta'\wedge\omega_b = 0$.

(v) *The number.* By (iii) and (7.1),

$$\int_X \eta_2\wedge\omega_b = \int_X (f_2\omega + f_1\bar\omega)\wedge\omega_b = \int_X f_1\bar\omega\wedge\omega_b = -\int_X \psi\wedge\frac{\omega_b}{\omega}\,\bar\omega = 2iN\,J(a,b).$$

So $J(a,b) = 0$. A lattice triangle has rank one by Lemma 2.3. □

7.3. **The integral in closed form.**

Proposition 7.5. *Let $a,b\in\Phi$ with $a+b\equiv 4$ and $\mathrm{w}(a-2) = 2$. Then $E_1, E_2, E_1', E_2'\in\{0,1\}$, $\deg P\le 2$, $\deg Q\le 2$, and*

$$\frac{\pi J(a,b)}{I_0^2} = \frac{P'(\xi)Q(\xi) - P(\xi)Q'(\xi)}{2}.$$

Proof. Exponents. As $\langle m\rangle\in(0,1]$ and $0<2p_1/N<1$, E_1 and E_1' are integers in $(-1,2)$, and likewise E_2, E_2'. Then $\deg P = E_1+E_2 = 1 - \langle bp_3\rangle - \langle cp_3\rangle + 2p_3/N$, and $\deg Q \le E_1'+E_2'+1 = \langle cp_3\rangle - \langle ap_3\rangle + 2p_3/N$, an integer less than 3.

Absolute convergence. The function $\int_{\mathbb{C}}|\chi_a(x')|\,|x-x'|^{-1}d^2x'$ satisfies the bounds found for G in the proof of Lemma 7.2, which use only the orders of χ_a. Its product with $|\varphi|^2|P|$, of order $\langle bp_1\rangle+\langle cp_1\rangle-2$ at 0 and $\deg P-2-2p_3/N$ at infinity, is therefore $O(|x|^{\min(\langle ap_1\rangle,\langle cp_1\rangle)+\langle bp_1\rangle-2}\log(1/|x|))$ at 0, likewise at 1, and $O(|x|^{-\kappa}\log|x|)$ at infinity with $\kappa=3+2p_3/N-\deg P-\max(\langle cp_3\rangle-\langle ap_3\rangle,0)>2$ by the formula for $\deg P$, so it is integrable.

The value. By Fubini's theorem, $J=\frac{1}{\pi}\iint|\varphi(x)|^2|\varphi(x')|^2P(x)Q(x')/(x-x')\,d^2x\,d^2x'$. The measure is symmetric in (x,x'), so the kernel may be replaced by its symmetric part, the Bezoutian

$$\operatorname{Bez}(x,x')=\frac{P(x)Q(x')-P(x')Q(x)}{2(x-x')},$$

a symmetric polynomial of degree at most one in each variable, $h_{00}+h_{10}(x+x')+h_{11}xx'$, as P and Q have degree at most two. By Lemma 7.1, $\pi J=h_{00}I_0^2+2h_{10}I_0I_1+h_{11}I_1^2=I_0^2\operatorname{Bez}(\xi,\xi)$, and $\operatorname{Bez}(\xi,\xi)=(P'(\xi)Q(\xi)-P(\xi)Q'(\xi))/2$. □

7.4. **Triangles with $s\mid N$ and N/s odd.** In this subsection $s=2p_3-N$ divides N and $d=N/s$ is odd. Then $d\geq 3$, $p_3=s(d+1)/2$, $p_1+p_2=s(d-1)/2$ and $\mu=(d-1)/(2d)$.

Lemma 7.6. (a) *If $k\equiv 2 \pmod d$ and $\varrho=[kp_1]$, then $\langle kp_3\rangle=1/d$, and $k\in\Phi$ if and only if $1\leq\varrho\leq N-s-1$; then $\langle kp_1\rangle=\varrho/N$ and $\langle kp_2\rangle=1-1/d-\varrho/N$.*

(b) *If $k\equiv 0 \pmod d$ and $k\not\equiv 0 \pmod N$, then $N\mid kp_3$, $\langle kp_1\rangle+\langle kp_2\rangle=1$, $\mathrm{w}(k)=2$ and $H_k^1=0$.*

Proof. (a) $k(d+1)/2\equiv d+1\equiv 1 \pmod d$, so $kp_3=s\cdot k(d+1)/2\equiv s \pmod N$ and $\langle kp_3\rangle=s/N$. Then $kp_2\equiv-\varrho-s$. If $\varrho=0$, $\langle kp_1\rangle=1$ and $\mathrm{w}(k)\geq 2$. If $1\leq\varrho\leq N-s-1$, the representative of $-\varrho-s$ in $(0,N]$ is $N-\varrho-s$ and $\mathrm{w}(k)=1$; if $\varrho\geq N-s$, it is $2N-\varrho-s$ and $\mathrm{w}(k)=2$. (b) $kp_3=(k/d)\cdot N(d+1)/2$, so $\langle kp_3\rangle=1$, and $\langle kp_1\rangle+\langle kp_2\rangle$ is an integer in $(0,2]$; it is 2 only if N divides kp_1 and kp_2, and then $N\mid k$, as $\gcd(p_1,p_2,p_3)=1$. So $\mathrm{w}(k)=2$, and $H_k^1=0$ by Lemma 2.1. □

Proposition 7.7. *Let $a\equiv 2 \pmod d$, $a\not\equiv 2 \pmod N$, $b\equiv 4-a$, with $a,b\in\Phi$, and $c=a-2$. Then:*

(a) *$\mathrm{w}(c)=2$ and $H_c^1=0$; $E_1+E_2=1$, $E_1'+E_2'=1$ and $\epsilon_1+\epsilon_2=\mu$;*

(b) *the number $W=2\pi J(a,b)/I_0^2=(P'Q-PQ')(\xi)$ is given by*

(E_1,E_1')	P	Q	W
$(1,1)$	x	$x(\mu x-\epsilon_1)$	$-(p_1/N)^2/\mu$
$(0,0)$	$x-1$	$(x-1)(\mu x-\epsilon_1)$	$-(p_2/N)^2/\mu$
$(1,0)$	x	$(x-1)(\mu x-\epsilon_1)$	$\langle cp_1\rangle-1+p_1p_2/(N^2\mu)$
$(0,1)$	$x-1$	$x(\mu x-\epsilon_1)$	$-\langle cp_1\rangle+p_1p_2/(N^2\mu)$

(c) *$W\neq 0$;*

(d) *$(a,b)\in\mathcal{P}_2$ if and only if $a\not\equiv 2+N/2$ and at least one of a, b is a unit.*

Proof. (a) As $c\equiv 0 \pmod d$ and $c\not\equiv 0 \pmod N$, Lemma 7.6(b) applies to c. By Lemma 7.6(a), $\langle ap_1\rangle+\langle ap_2\rangle=\langle bp_1\rangle+\langle bp_2\rangle=1-1/d=2\mu$, and $\langle cp_1\rangle+\langle cp_2\rangle=1$; so $E_1+E_2=2\mu+1-2\mu=1$, $E_1'+E_2'=2\mu-1-2\mu+2=1$ and $\epsilon_1+\epsilon_2=2\mu-\mu$. By Proposition 7.5 each exponent is 0 or 1, and $\epsilon_1(x-1)+\epsilon_2x=\mu x-\epsilon_1$.

(b) Let $L=\mu x-\epsilon_1$; recall $\xi=p_1/(N\mu)$ and $\xi-1=-p_2/(N\mu)$. Row $(1,1)$: $P'Q-PQ'=xL-x(L+\mu x)=-\mu x^2$. Row $(0,0)$: likewise $-\mu(x-1)^2$. Row $(1,0)$: $(x-1)L-x(L+$

$\mu(x-1)) = -L-\mu x(x-1)$, which at ξ is $\epsilon_1 - p_1/N + p_1p_2/(N^2\mu)$; and $E_1' = 0$ says $\langle ap_1\rangle = \langle cp_1\rangle + 2p_1/N - 1$, so $\epsilon_1 = \langle cp_1\rangle + p_1/N - 1$. Row $(0,1)$: $xL-(x-1)(L+\mu x) = L-\mu x(x-1)$, which at ξ is $p_1/N - \epsilon_1 + p_1p_2/(N^2\mu)$; and $E_1' = 1$ gives $\epsilon_1 = \langle cp_1\rangle + p_1/N$.

(c) The rows $(1,1)$ and $(0,0)$ are non-zero. In the rows $(1,0)$ and $(0,1)$, $N\langle cp_1\rangle$ is an integer and $N\cdot p_1p_2/(N^2\mu) = p_1p_2/(p_1+p_2)$, so $W=0$ would need $(p_1+p_2) \mid p_1p_2$, that is $(p_1+p_2) \mid p_1^2$, and then $s \mid p_1^2$, as $s \mid p_1+p_2$. If $s \geq 2$, a prime dividing s divides p_1, p_1+p_2 and $p_3 = s(d+1)/2$, against $\gcd(p_1,p_2,p_3)=1$. If $s=1$, then $d=N$, and $a \equiv 2 \pmod N$ is excluded.

(d) Here $a \not\equiv 1$, as $a \equiv 2 \pmod d$ and $d \geq 3$; $\mathrm{w}(c)=2$; and $N \mid cp_3$. So $(a,b) \in \mathcal{P}_2$ if and only if b is a unit with $a \not\equiv \pm b$, or b is not a unit and a is. Now $a \equiv -b$ would give $N \mid 4$, impossible as $d \geq 3$ is odd, and $a \equiv b$ means $2a \equiv 4$, that is $a \equiv 2$ or $a \equiv 2+N/2$; and if $a \equiv 2+N/2$, then $b \equiv a$. □

The family $\mathcal{A}$ [17, §4] consists of the obtuse triangles with $p_1=1$, $s = 2p_3 - N$ dividing N and $d = N/s$ odd, that is the triangles $(1,(N-s)/2-1,(N+s)/2)/N$. Its members with $s=1$ are the triangles of Section 4.1, and those with $s=2$ are treated in Section 8.

Theorem 7.8. *Let $s \geq 3$, $d \geq 3$ odd, $N = sd$, and $\Delta = (1, s(d-1)/2-1, s(d+1)/2)/N$. For $1 \leq j \leq s-1$ put $a_j = 2+dj$, so that $4 - a_j \equiv a_{s-j}$, and let $j_0 = \lceil (s+3)/d \rceil$.*

(a) *The pairs $(a,b) \in \mathcal{P}_2$ with $a \equiv 2 \pmod d$ are the (a_j, a_{s-j}) with $j_0 \leq j \leq s-j_0$, $j \neq s/2$, and $\gcd(2+dj,s)=1$ or $\gcd(2-dj,s)=1$.*

(b) *For each of them $\pi J(a,b)/I_0^2 = -d/((d-1)N^2)$.*

(c) *Such a j exists for every $s \geq 3$ and odd $d \geq 3$ except the twelve pairs $(s,d) = (3,3)$, $(4,3)$, $(5,3)$, $(6,3)$, $(7,3)$, $(8,3)$, $(10,3)$, $(14,3)$, $(3,5)$, $(4,5)$, $(6,5)$, $(6,7)$, that is, except the triangles $(1,2,6)/9$, $(1,3,8)/12$, $(1,4,10)/15$, $(1,5,12)/18$, $(1,6,14)/21$, $(1,7,16)/24$, $(1,9,20)/30$, $(1,13,28)/42$, $(1,5,9)/15$, $(1,7,12)/20$, $(1,11,18)/30$, $(1,17,24)/42$.*

So, apart from these twelve, $\operatorname{rank}\mathcal{M} \geq 2$.

Proof. (a) The characters $a \equiv 2 \pmod d$ with $a \not\equiv 2 \pmod N$ are the a_j, $1 \leq j \leq s-1$, and $[a_jp_1] = a_j < N$; the pair $(2,2)$ is not in $\mathcal{P}_2$, as $c \equiv 0$ has $\mathrm{w}(0)=3$. By Lemma 7.6(a), $a_j \in \Phi$ if and only if $2+dj \leq N-s-1$, that is $d(s-j) \geq s+3$; so a_j and a_{s-j} both lie in Φ if and only if $j_0 \leq j \leq s-j_0$. A prime dividing d does not divide $a_j \equiv 2 \pmod d$, so a_j is a unit if and only if $\gcd(2+dj,s)=1$, and a_{s-j} if and only if $\gcd(2-dj,s)=1$. Finally $a_j \equiv 2+N/2$ means $j=s/2$, and Proposition 7.7(d) gives the statement.

(b) Here $\langle ap_1\rangle = (2+dj)/N$, $\langle bp_1\rangle = (2+d(s-j))/N$, $\langle cp_1\rangle = dj/N$ and $p_1/N = 1/N$, so $E_1 = E_1' = 1$. This is the row $(1,1)$ of Proposition 7.7, with $W = -(1/N)^2/\mu = -2d/((d-1)N^2)$.

(c) By the symmetry $j \leftrightarrow s-j$ it is enough to find $j \in [j_0, s-j_0]$, $j \neq s/2$, with $\gcd(2+dj,s)=1$. A prime dividing s and d never divides $2+dj$, and a prime dividing s but not d divides $2+dj$ for j in exactly one class modulo it. Let s' be the product of the primes dividing s but not d, and r their number.

Case $d \geq s+3$. Then $j_0 = 1$. The classes j modulo s with $\gcd(2+dj,s)=1$ number at least $\#(\mathbb{Z}/s)^\times \geq 2$. If s is odd, all of them except possibly the class 0 meet $[1,s-1]$, and $j \neq s/2$ holds; if s is even, the class 0 is not among them, as $\gcd(2,s)=2$, and excluding $j=s/2$ leaves at least one.

Case $d \leq s+2$ and $s \geq 404$. Then $j_0 \leq (s+2)/d + 1 \leq (s+5)/3$, so $[j_0, s-j_0] \setminus \{s/2\}$ contains a block of $L \geq (s-10)/6$ consecutive integers. If $r=0$, every j will do. Otherwise, by a sieve lemma [17, Lemma 4.10], among any L consecutive integers more

than $L\,\#(\mathbb{Z}/s')^\times/s' - 2^r + 1$ avoid one given class modulo each prime dividing s'; so the block contains a j with $\gcd(2+dj, s') = 1$ once $L \geq L_0 = (2^r-1)s'/\#(\mathbb{Z}/s')^\times$. Bounding $s'/\#(\mathbb{Z}/s')^\times$ by its value at the product of the first r primes,

$$L_0 \leq 2,\ 9,\ 26.25,\ 65.63,\ 149.2 \qquad (r = 1, \dots, 5),$$

and $L \geq L_0$ holds when $s \geq 6L_0 + 10$, that is for $s \geq 22$, 64, 168, 404, 906. For $r \geq 5$, $s \geq s' \geq 2310$, and from r to $r+1$ the bound $6L_0 + 10$ grows by a factor less than 2.3 while the least possible s' grows by a factor at least 13. So $s \geq 404$ suffices for every r.

Case $d \leq s+2$ and $3 \leq s \leq 403$. For every such pair (s, d) with d odd we find a j as required, except for the twelve pairs listed [21].

For $s \geq 3$ outside the twelve, (c) gives a pair in $\mathcal{P}_2$, (b) gives $J \neq 0$ on it, and Theorem 7.4 gives rank at least two. □

Among the twelve is $(1,3,8)/12$, of rank one (Proposition 10.2).

The family $\mathcal{B}$ [17, §4] consists of the obtuse triangles with $s = 2p_3 - N$ dividing N, $d = N/s$ odd, $2 \leq p_1 \leq s/2$, and the unit condition of [17, §4]: every unit v with $2v \not\equiv 2$ and $[vp_1] < 2p_1$ satisfies $v \equiv 2 \pmod d$.

Theorem 7.9. *Let $\Delta \in \mathcal{B}$ and $g = \gcd(p_1, N)$. There is a unit $u \equiv 2 \pmod d$ with $up_1 \equiv g \pmod N$; for it $(4-u, u) \in \mathcal{P}_2$ and*

$$\frac{\pi J(4-u,u)}{I_0^2} = -\frac{p_1^2 + (p_1-g)(p_1+p_2)}{2N(p_1+p_2)} < 0.$$

So $\operatorname{rank}\mathcal{M} \geq 2$.

Proof. By the unit condition, a unit v with $2v \not\equiv 2$ and $1 \leq [vp_1] \leq 2p_1 - 1$ is $\equiv 2 \pmod d$.

Case $g < p_1$. As p_1/g is a unit modulo N/g, there is a unit u modulo N with $u\,p_1/g \equiv 1 \pmod{N/g}$, that is $up_1 \equiv g \pmod N$. Then $[up_1] = g \leq p_1 - 1$ and $u \not\equiv 1$; and $2u \equiv 2$ would give $u \equiv 1$ or $u \equiv 1 + N/2$, and then $[up_1]$ would be p_1 or $p_1 + N/2$, not g. So $u \equiv 2 \pmod d$.

Case $g = p_1$. Then p_1 is odd: if it were even, N and $s = N/d$ would be even, and so would $p_3 = s(d+1)/2$ and $p_1 + p_2 = s(d-1)/2$, against $\gcd(p_1, p_2, p_3) = 1$. There are $\#(\mathbb{Z}/N)^\times/\#(\mathbb{Z}/(N/p_1))^\times \geq \#(\mathbb{Z}/p_1)^\times \geq 2$ units $\equiv 1 \pmod{N/p_1}$; let $u \not\equiv 1$ be one. Then $up_1 \equiv p_1$, and $2u \equiv 2$ would give $u \equiv 1 + N/2 \equiv 1 \pmod{N/p_1}$, so $(N/p_1) \mid (N/2)$ and $2 \mid p_1$. So $u \equiv 2 \pmod d$.

In both cases $k = [up_1] = g$ satisfies $1 \leq k \leq p_1$. Let $b = u$ and $a = 4 - u$. By Lemma 7.6(a), $b \in \Phi$, as $k \leq 2p_1 - 1 \leq s - 1 \leq N - s - 1$. Next $ap_1 \equiv 4p_1 - k$ with $2p_1 + 1 \leq 4p_1 - k \leq 4p_1 - 1 \leq 2s - 1 \leq N - s - 1$, so $a \in \Phi$, and $a \equiv 2 \pmod d$. And $c = 2 - u$ has $cp_1 \equiv 2p_1 - k \in [1, 2p_1 - 1]$, so $c \not\equiv 0$ and $\langle cp_1 \rangle = (2p_1 - k)/N$. If $a \equiv 2 + N/2$, then $a \equiv b$ and $2u \equiv 4$, so N divides $4p_1 - 2k$; but $2 \leq 4p_1 - 2k < 4p_1 - k < N$. As b is a unit, $(a, b) \in \mathcal{P}_2$ by Proposition 7.7(d). Finally $E_1 = (k + (2p_1 - k) - 2p_1)/N = 0$ and $E_1' = ((4p_1 - k) - (2p_1 - k) - 2p_1)/N + 1 = 1$: this is the row $(0, 1)$ of Proposition 7.7, with

$$W = -\frac{2p_1 - k}{N} + \frac{p_1 p_2}{N(p_1+p_2)} = -\frac{p_1^2 + (p_1-k)(p_1+p_2)}{N(p_1+p_2)}.$$

Theorem 7.4 gives the conclusion. □

8. Obtuse triangles

This section proves that an obtuse triangle has rank at least two unless it is a lattice triangle or one of $(1,3,6)/10$, $(1,3,8)/12$. In it $\Delta = (p_1, p_2, p_3)/N$ is obtuse, written with

$p_1 \le p_2 < p_3$, so that $p_3 > N/2$. Isosceles triangles are allowed; for them the result also follows from Apisa's determination of their orbit closures [2, Theorems 1.4 and 7.7].

Theorem 8.1. *Every obtuse rational triangle other than* $(1,1,N-2)/N$, $(1,2,2k-3)/2k$ *with* $k \ge 4$, $(1,4,7)/12$, $(1,3,6)/10$ *and* $(1,3,8)/12$ *has rank at least two.*

The first three are lattice triangles, and every other obtuse triangle is not [17, Theorem 1.1]. We sort the triangles by witnesses (Section 8.1); the families without a witness are treated by residues and the involution (Section 8.2), by the second-order criterion (Section 7) and by cylinder deformations (Section 8.3), and finitely many triangles by computation (Section 8.4). Section 8.5 puts these together.

8.1. **Triangles without a witness.** By Theorem 2.6 a triangle with a witness has rank at least two, and for obtuse triangles Lemma 2.7(b) is the form of the criterion used by Larsen, Norton and Zykoski [27, Proposition 2.1]. When the obtuse angle exceeds $2\pi/3$ they found the triangles without a witness [27, Theorem 4.1]: with $p_1 \le p_2 < p_3$ and $p_3 > 2N/3$, Δ has no witness if and only if

(1) $p_1 = p_2 = 1$; or
(2) $p_1 = 1$, $p_2 = 2$ and N is even; or
(3) $p_1 = 1$, $p_2 = 4$ and $p_3 \equiv 7 \pmod 8$; or
(4) $p_1 = 1$ and $p_3 = 3p_2 + 1$; or
(5) $p_1 = 2$ and $p_3 = 3p_2 + 2$; or
(6) $p_1 = 4$ and $p_3 = 3p_2 + 4$; or
(7) (p_1,p_2,p_3) is one of $(1,4,11)$, $(1,3,16)$, $(2,3,17)$, $(1,4,21)$, $(1,8,19)$, $(3,8,29)$, $(2,11,29)$.

For $N/2 < p_3 \le 2N/3$ let $s = 2p_3 - N$. The families $\mathcal{A}$ and $\mathcal{B}$ of [17, §4] are defined in Section 7.4, and $\mathcal{C}$ consists of the triangles with $s = 2$, $4 \mid N$, and either $p_1 \in \{1,2\}$ or $p_1 = 4$ and $N \equiv 4 \pmod 8$, that is the $(p_1, N/2 - p_1 - 1, N/2 + 1)/N$; let $\mathcal{E}$ be the list of thirty triangles with $N \le 48$ of [17, Table 1]. The family $\mathcal{A}$ with $s = 2$ consists of the $(1, d-2, d+1)/2d$ with d odd. By [17, Theorem 4.1(a)], a triangle with $N/2 < p_3 \le 2N/3$ has no witness if and only if it lies in $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$ or is one of the triangles of $\mathcal{E}$.

8.2. **Residues and the involution.** The triangles $(1,(N-3)/2,(N+1)/2)/N$ of $\mathcal{A}$ with $s = 1$ are treated in Proposition 4.4. For the family (4) let $q \ge 3$, $N = 4q+2$ and $\Delta = (1, q, 3q+1)/N$. Then $g_1 = 1$, $g_2 = \gcd(q,2)$, and $g_3 = \gcd(3q+1, 4q+2)$, which divides $4(3q+1) - 3(4q+2) = -2$, is 2 for odd q and 1 for even q. The point above vertex 1 is regular and p_2, p_3 do not divide N, so $I = \{2,3\}$, and by Lemma 2.2 $Z = Z_0 \oplus Z_{N/2}$, with vertex 3 for odd q and vertex 2 for even q as the vertex of $N/2$. In the computations below we use that $(2q+1)k$ is a multiple of N for every even k.

Lemma 8.2. (a) *If* $q \ge 3$ *is odd, then* $(2q-1,3) \in \mathcal{P}(N/2)$.
(b) *If* $q \ge 6$ *is even, then* $(2q-3,5) \in \mathcal{P}(N/2)$.

Proof. The table gives, for the pairs (a,b) of (a) and (b) and that of Lemma 8.5, the character c, its vertex, and the residues $[ap_i]$ and $[bp_i]$. In the last row $N = 2m$ and $(p_1,p_2,p_3) = (1, m-2, m+1)$.

	c	vertex	(a,b)	$[ap_1]$, $[ap_2]$, $[ap_3]$	$[bp_1]$, $[bp_2]$, $[bp_3]$
(a)	$2q+1$	3	$(2q-1,3)$	$2q-1$, 1, $2q+2$	3, $3q$, $q-1$
(b)	$2q+1$	2	$(2q-3,5)$	$2q-3$, 2, $2q+3$	5, $q-2$, $3q-1$
Lemma 8.5	$m/2$	2	$(m/2-2, m+3)$	$m/2-2$, 4, $3m/2-2$	$m+3$, $m-6$, 3

In each row, a is a unit, $b \not\equiv a$ and $a + b \equiv 1 - c$. Both triples have sum N, so $a, b \in \Phi$, and at the vertex of c the two residues add up to its numerator. So $(a, b) \in \mathcal{P}(c)$. We check the first row, and the others in the same way.

In (a), $\gcd(2q-1, 4q+2) = \gcd(2q-1, 4) = 1$. As $(2q-1)q - 1 = (2q+1)(q-1)$ and $(2q-1)(3q+1) - (2q+2) = (2q+1)(3q-3)$, the residues of $(2q-1) \cdot (1, q, 3q+1)$ are $2q-1$, 1, $2q+2$. Those of $3 \cdot (1, q, 3q+1)$ are 3, $3q$, $q-1$. Both sums are N, and $3 \neq 2q-1$. Then $(2q-1) + 3 = 2q+2 \equiv 1 - N/2$, and at vertex 3, $(2q+2) + (q-1) = 3q+1 = p_3$. □

Proposition 8.3. *For every $q \geq 3$ with $q \neq 4$, $(1, q, 3q+1)/(4q+2)$ has rank at least two.*

Proof. The obtuse angle exceeds $2\pi/3$, so the triangle is not a lattice triangle [17, Theorem 1.3]. It is scalene and N is even, and the only odd c with $Z_c \neq 0$ is $N/2 = 2q+1$. Lemma 8.2 gives $\mathcal{P}(N/2) \neq \emptyset$, and Corollary 6.3(iii) applies. □

For six further families the involution alone suffices: their rel is zero.

Lemma 8.4. *In each of the following triangles, with the vertices numbered as written, N is even and N/g_i is even for every $i \in I$, so* $\operatorname{rel} \mathcal{M} = 0$ *by Corollary 6.3*(ii).

(a) $(1, 4, N-5)/N$, $N \equiv 4 \pmod 8$, $N \geq 12$: $g_1 = 1$, $g_2 = 4$, $g_3 = \gcd(5, N)$.
(b) $(2, q, 3q+2)/(4q+4)$, q *odd*, $q \geq 3$: $g_1 = 2$, $g_2 = g_3 = 1$.
(c) $(4, q, 3q+4)/(4q+8)$, q *odd*, $q \geq 3$: $g_1 = 4$, $g_2 = g_3 = 1$.
(d) $(1, m-2, m+1)/2m$, $4 \mid m$, $m \geq 4$: $g_1 = 1$, $g_2 = 2$, $g_3 = 1$.
(e) $(2, m-3, m+1)/2m$, m *even*, $m \geq 4$: $g_1 = 2$, $g_2 = \gcd(m-3, 6)$, $g_3 = 1$.
(f) $(4, m-5, m+1)/2m$, $m \equiv 2 \pmod 4$, $m \geq 6$: $g_1 = 4$, $g_2 = \gcd(m-5, 10)$, $g_3 = 1$.

Proof. Each triangle is scalene, is not $(1, 2, 3)/6$, and has N even. The g_i of the statement follow from Euclid's algorithm, as in (b): $N = 4(q+1)$, $\gcd(q, 4q+4) = \gcd(q, 4) = 1$, and $\gcd(3q+2, 4q+4) = \gcd(3q+2, q+2) = \gcd(4, q+2) = 1$, as $q+2$ is odd. In every case each g_i is odd, so that N/g_i is even, or $g_i = p_i$, so that $i \notin I$, except $g_2 = 2$ in (d), where $N/g_2 = m$ is even. □

For $(1, m-2, m+1)/2m$ with $m \equiv 2 \pmod 4$ and $m \geq 10$, $g_2 = 4$ and $N/g_2 = m/2$ is odd, so $Z^- \neq 0$ and Corollary 6.3(ii) does not give rel zero; there residues are needed.

Lemma 8.5. *Let $m \equiv 2 \pmod 4$, $m \geq 10$, and $\Delta = (1, m-2, m+1)/2m$. The odd characters c with $Z_c \neq 0$ are $m/2$ and $3m/2 \equiv -m/2$, and $(m/2 - 2,\ m+3) \in \mathcal{P}(m/2)$.*

Proof. Let $k = m/2$, which is odd and at least 5, so that $N = 4k$ and $(p_1, p_2, p_3) = (1, 2k-2, 2k+1)$. Then $g_1 = 1$, $g_2 = \gcd(2k-2, 4) = 4$, $g_3 = \gcd(2k+1, 2) = 1$, and $p_2 \nmid N$. So $I = \{2, 3\}$, the non-zero characters c with $Z_c \neq 0$ are k, $2k$, $3k$, all with vertex 2, and the odd ones are k and $3k$. The pair is the last row of the table in the proof of Lemma 8.2. □

Theorem 8.6. *Each of the following triangles has rank at least two:*

(a) $(1, 4, N-5)/N$, $N \equiv 4 \pmod 8$, $N \geq 20$;
(b) $(2, q, 3q+2)/(4q+4)$, q *odd*, $q \geq 3$;
(c) $(4, q, 3q+4)/(4q+8)$, q *odd*, $q \geq 3$;
(d) $(1, m-2, m+1)/2m$, m *even*, $m \geq 8$;
(e) $(2, m-3, m+1)/2m$, m *even*, $m \geq 6$;
(f) $(4, m-5, m+1)/2m$, $m \equiv 2 \pmod 4$, $m \geq 10$.

Proof. None of them is a lattice triangle: (a) by [17, Theorem 1.3], as its largest angle $(N-5)\pi/N$ exceeds $2\pi/3$ for $N > 15$; (b), (c), (d), (e), (f) by [17, Theorems 5.9, 5.11, 5.15, 5.8 and 5.10] respectively. In (a), (b), (c), (e), (f), and in (d) with $4 \mid m$, $\operatorname{rel}\mathcal{M} = 0$ by Lemma 8.4; so if the rank were one, then $\dim_{\mathbb{C}}\mathcal{M} = 2$ and Δ would be a lattice triangle by Lemma 2.3. In (d) with $m \equiv 2 \pmod 4$, so $m \geq 10$, Lemma 8.5 and Corollary 6.3(iii) give the conclusion. □

8.3. **The family $\mathcal{A}$ with $s = 2$.** For these triangles we write down one completely periodic direction and its cylinders.

Theorem 8.7. *For every odd $d \geq 7$, $(1, d-2, d+1)/2d$ has rank at least three.*

We use the star model of [17, §5.3], placed and labeled as in the proof of [17, Theorem 5.15], where the same triangles are treated for even d. The gluing rule and the heights are derived there, under "Gluing" and "Heights", without using the parity of d, and we take both from there. Our d, v_j and e_j are the m, V_j and Ξ_j of that proof, and the cylinder Y_1 below is its cylinder Z'. Let $d \geq 5$ be odd, $\psi = \pi/2d$ and $u = 2\cos 2\psi = 2\cos(\pi/d)$, and let the triangle have vertices A, B, C with angles ψ, $(d-2)\psi$, $(d+1)\psi$ and $|BC| = 1$. For $j \in \mathbb{Z}/4d$ let

$$v_j = \rho_j\,(\sin j\psi, \cos j\psi), \qquad \rho_j = \frac{\cos 2\psi}{\sin\psi}\ (j \text{ even}), \quad \rho_j = \cot\psi\ (j \text{ odd}),$$

so that v_j has argument $\pi/2 - j\psi$. The star $P_\star$ is the polygon with vertices $v_0, v_{-1}, v_{-2}, \dots$ in counterclockwise order, the union of $4d$ copies of the triangle with the vertex A at the origin O; the even vertices are copies of C and the odd ones copies of B. Let e_j be the side from v_j to v_{j-1}. By [17, Lemma 5.12], in the form of the proof of [17, Theorem 5.15], X is $P_\star$ with its sides glued by translation as follows, and Figure 1 draws it for $d = 9$:

$$\text{for even } i,\ e_i \text{ is glued to } e_{i+2d+3},\ \text{with } v_i \leftrightarrow v_{i+2d+2} \text{ and } v_{i-1} \leftrightarrow v_{i+2d+3}. \tag{8.1}$$

The rule is invariant under $j \mapsto j + 2d$, so the half-turn $z \mapsto -z$ of $P_\star$ is an automorphism of X that multiplies ω by -1; by Theorem 6.2 it is the involution $\iota = T^{N/2}$. The copies of B form one zero z_B of cone angle $2\pi(d-2)$, the copies of C two zeros of cone angle $\pi(d+1)$, and O is a regular point. The star is (X, ω) rotated and scaled by a positive factor, which multiplies all heights and circumferences by that factor and preserves the moduli, so Lemma 2.8 applies to the horizontal direction of $P_\star$ with the heights and circumferences measured in $P_\star$.

Let y_j be the height of v_j, $x(v_j) = \rho_j \sin j\psi$, and $w_j = y_{j-1} - y_j$ for $1 \leq j \leq d$. Then $w_j = \sin((j+1)\psi)$ for even j and $w_j = \sin((j-2)\psi)$ for odd j, and

$$0 = y_d < y_{d-1} < \cdots < y_3 < y_2 < y_0 < y_1, \qquad y_1 - y_0 = \sin\psi, \qquad y_0 - y_2 = u\sin\psi.$$

A horizontal line at a height $h > 0$ that is not the height of a vertex meets the upper half of $P_\star$ as follows. For $h \in (y_j, y_{j-1})$, $3 \leq j \leq d$, in one chord from e_{1-j} to e_j; these chords form a strip Π_j of width w_j. For $h \in (y_2, y_0)$, in one chord from e_{-1} to e_2, passing below v_0; these form a strip Π_2 of width $u\sin\psi$. For $h \in (y_0, y_1)$, in two chords, from e_1 to e_2 and from e_{-1} to e_0, forming strips Π_1^{r} and Π_1^{l} of width $\sin\psi$. That these chords lie in $P_\star$ follows from [17, Lemma 5.13] and the paragraph after it: for each strip, the vertices on the counterclockwise arc of $\partial P_\star$ between the upper ends of its two sides lie at or above its top; for Π_j they are the v_i with $|i| \leq j-1$, and for Π_2 they are v_1, v_0, v_{-1}. The lower half consists of the images of these strips under ι: the $\Pi_j' = \iota(\Pi_j)$, $2 \leq j \leq d$, with left side e_{2d+j} and right side e_{2d+1-j}, and the images of Π_1^{r} and Π_1^{l}. So $P_\star$, minus finitely many horizontal segments, is the disjoint union of these strips.

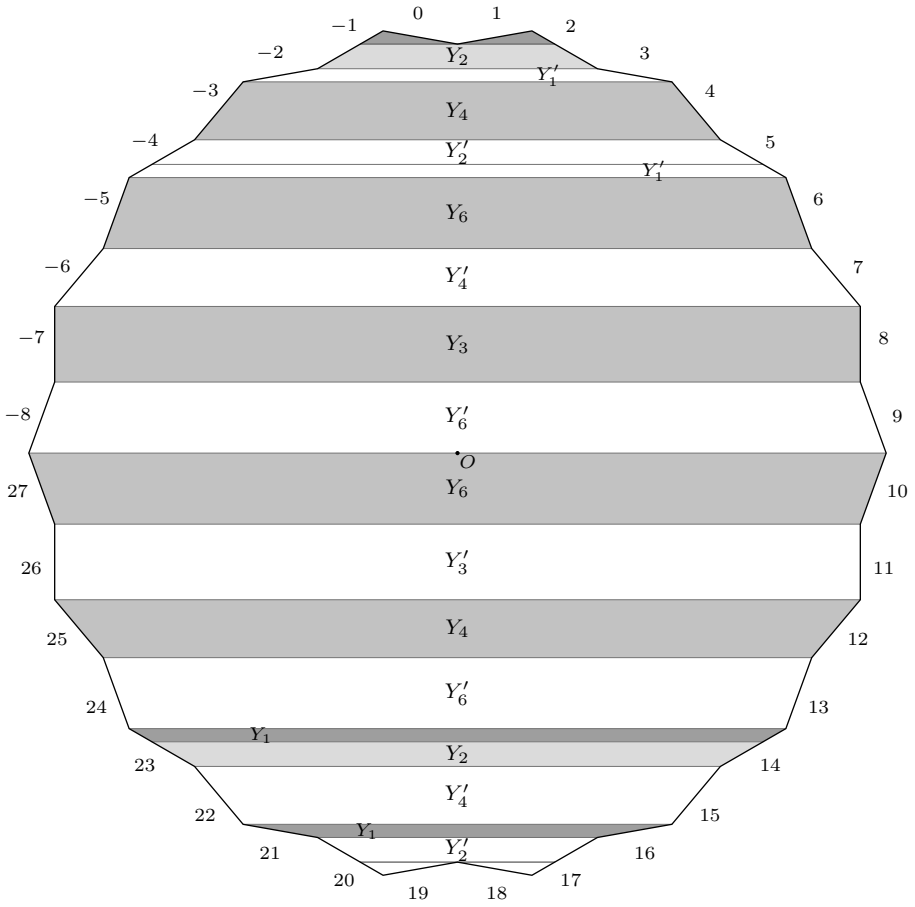

FIGURE 1. The star $P_\star$ for $d = 9$, the triangle $(1, 7, 10)/18$, with the origin O. The side e_j is marked j; for even i it is glued to e_{i+21}, indices modulo 36. The horizontal lines cut $P_\star$ into the strips Π_1^{r}, Π_1^{l}, $\Pi_2, \dots, \Pi_9$ and their images under ι, with Π_5 and Π_5' cut into two parts. A piece marked Y lies in the cylinder Y of Proposition 8.8; Π_1^{r} and Π_1^{l} lie in Y_1, and their images in Y_1'. The cylinders Y_1, Y_2, Y_3, Y_4 and Y_6 are shaded.

Proposition 8.8. *Let $d \geq 5$ be odd. The horizontal direction of $P_\star$, the direction of AB, is completely periodic, with exactly $d + 1$ maximal cylinders: the cylinders Y_1, Y_2, Y_3 and Y_j for even j with $4 \leq j \leq d - 3$ of the table, and their images Y_1', Y_2', Y_3', Y_j' under ι.*

cylinder	*strips*	*height*	*circumference*
Y_1	Π_1^{r}, *top part of* Π_5', Π_1^{l}, Π_3'	$\sin\psi$	$2u(u+2)\cos\psi$
Y_2	Π_2, *bottom part of* Π_5'	$u\sin\psi$	$2u(u+1)\cos\psi$
Y_3	Π_{d-1}	1	$u\cot\psi$
Y_j, j *even*, $4 \leq j \leq d-3$	Π_j, Π_{j+3}'	$\sin((j+1)\psi)$	$2u\cot\psi\,\sin((j+1)\psi)$

There is no Y_j with $j \geq 4$ for $d = 5$. The moduli are $\tan\psi/(2u(u+2))$, $\tan\psi/(2(u+1))$, $\tan\psi/u$ and $\tan\psi/(2u)$. Let z_0 and z_2 be the zeros formed by the v_i with $i \equiv 0$, respectively $i \equiv 2 \pmod 4$, which ι exchanges. The zeros on the top and the bottom boundary are z_B

and z_0 for Y_1; z_0 and z_2 for Y_2; z_B and the zero of v_{d-1} for Y_3; and z_B and the zero of v_j for Y_j. For the images, top and bottom are exchanged and so are z_0 and z_2.

Proof. Each step uses (8.1) and [17, Lemma 5.14]: strips glued in a cycle, side to side at equal heights, form an annulus of closed leaves, and it is a maximal cylinder if each boundary component contains a zero. The strip Π'_k has top at height $-y_k$ and bottom at $-y_{k-1}$. Figure 1 shows the strips and the cylinders for $d = 9$.

Y_1: Π^{r}_1 leaves through $e_2 \to e_{2d+5}$, the left side of Π'_5, with $v_1 \mapsto v_{2d+5}$, the top of Π'_5; as $\sin\psi < w_5 = \sin 3\psi$, it fills the top part of Π'_5, of width $\sin\psi$. This part leaves through $e_{2d-4} \to e_{-1}$, with $v_{2d-5} \mapsto v_{-1}$, and fills Π^{l}_1, which leaves through $e_0 \to e_{2d+3}$, the left side of Π'_3, with $v_{-1} \mapsto v_{2d+3}$. As $w_3 = \sin\psi$, it fills Π'_3, which leaves through $e_{2d-2} \to e_1$, with $v_{2d-3} \mapsto v_1$. On the upper boundary the strips Π^{r}_1, Π^{l}_1 have width zero, so the circumference is $2x(v_5) + 2x(v_3) = 2\cot\psi(\sin 5\psi + \sin 3\psi) = 16\cos^3\psi\cos 2\psi = 2u(u+2)\cos\psi$.

Y_2: Π_2 leaves through $e_2 \to e_{2d+5}$ with the translation of Π^{r}_1; it has heights $(y_1 - \sin 3\psi, y_1 - \sin\psi)$ and fills the rest of Π'_5, which leaves through $e_{2d-4} \to e_{-1}$ and arrives at heights (y_2, y_0) on e_{-1}, the left side of Π_2. On the lower boundary the circumference is $2x(v_2) + 2x(v_4) = 2\rho_0(\sin 2\psi + \sin 4\psi) = 2u(u+1)\cos\psi$.

Y_3: $e_{d-1} \to e_{3d+2} = e_{2-d}$, the left side of Π_{d-1} itself, with $v_{d-1} \mapsto v_{1-d}$; the height is $w_{d-1} = \sin d\psi = 1$ and the circumference $2x(v_{d-1}) = u\cot\psi$.

Y_j: $e_j \to e_{2d+j+3}$, the left side of Π'_{j+3}, with v_{j-1}, the top of Π_j, going to v_{2d+j+3}, the top of Π'_{j+3}; the widths agree, $w_j = \sin((j+1)\psi) = w_{j+3}$; and Π'_{j+3} leaves through $e_{2d-j-2} \to e_{1-j}$, the left side of Π_j. On the upper boundary the circumference is $2x(v_{j-1}) + 2x(v_{j+3}) = 2\cot\psi[\sin((j-1)\psi) + \sin((j+3)\psi)] = 4\cot\psi\cos 2\psi\sin((j+1)\psi)$.

Boundaries. An even vertex v_i is an end of e_i and e_{i+1}; the gluings identify it with v_{i+2d+2} and v_{i+2d-2}. As d is odd, $2d \pm 2 \equiv 0 \pmod 4$, and these shifts generate the multiples of 4 modulo $4d$; so the copies of C form the two zeros z_0, z_2. An odd vertex v_j is identified with v_{j+2d-4} and v_{j+2d+4}, shifts that generate all even shifts, as $\gcd(2d-4, 2d+4, 4d) = 2$; so the copies of B form the single zero z_B. The boundary components of Y_1, Y_2, Y_3, Y_j contain v_1 and v_0; v_0 and $v_{\pm 2}$; v_{d-2} and v_{d-1}; v_{j-1} and v_j. All are zeros, so the annuli are maximal cylinders, with the zeros stated. The half-turn shifts indices by $2d \equiv 2 \pmod 4$ and exchanges top and bottom.

Exhaustion. The cylinders use the upper strips Π^{r}_1, Π^{l}_1 and Π'_3 (Y_1); their images and Π_3 (Y'_1); Π_2 and Π'_2 (Y_2, Y'_2); the two parts of Π'_5 (Y_1, Y_2) and of Π_5 (Y'_1, Y'_2); Π_{d-1} and Π'_{d-1} (Y_3, Y'_3); Π_j and Π'_j for even $j \in [4, d-3]$ (Y_j, Y'_j); and Π_j, Π'_j for odd $j \in [7, d]$ (Y'_{j-3}, Y_{j-3}). These are all the strips, each once, so the cylinders cover X up to finitely many horizontal segments. There are $2(3 + (d-5)/2) = d+1$ of them. The moduli follow from $\tan\psi = \sin\psi/\cos\psi$ and $4\cos^2\psi = u + 2$. □

Proof of Theorem 8.7. Rational relations. $u = \zeta + \zeta^{-1}$ with $\zeta = e^{i\pi/d}$ a primitive $2d$-th root of unity, so $[\mathbb{Q}(u) : \mathbb{Q}] = \#(\mathbb{Z}/2d)^\times/2 = \#(\mathbb{Z}/d)^\times/2$. For odd $d \geq 7$, $\#(\mathbb{Z}/d)^\times \geq 6$: if every prime factor ℓ of d had $\ell - 1 \leq 4$, then $d = 3^a5^b$, and $\#(\mathbb{Z}/d)^\times \leq 4$ forces $d \in \{1, 3, 5\}$. So 1, u, u^2 are linearly independent over $\mathbb{Q}$. Multiplying the moduli of Y_1, Y_2, Y_3 by $2u(u+1)(u+2)/\tan\psi$ gives $u+1$, $u(u+2)$ and $2(u+1)(u+2)$, which are independent over $\mathbb{Q}$: a relation $r_1(u+1) + r_2(u^2+2u) + 2r_3(u^2+3u+2) = 0$ gives $r_2 + 2r_3 = 0$, $r_1 + 2r_2 + 6r_3 = 0$ and $r_1 + 4r_3 = 0$, so $r_1 = r_2 = r_3 = 0$. The modulus of each Y_j with $j \geq 4$ is half that of Y_3, and ι preserves moduli, so the rational relations among the moduli are spanned by $m(Y_1) = m(Y'_1)$, $m(Y_2) = m(Y'_2)$ and $m(Y_3) = m(Y'_3) = 2m(Y_j) = 2m(Y'_j)$. So $\mathcal{T}$ is the space of t with $t_{Y_1} = t_{Y'_1} = \tau_1$, $t_{Y_2} = t_{Y'_2} = \tau_2$, $t_{Y_j} = t_{Y'_j} = \tau_3$ and $t_{Y_3} = t_{Y'_3} = 2\tau_3$,

for $\tau \in \mathbb{R}^3$. The classes Ξ_i are linearly independent, each being 1 on a path crossing its own cylinder and 0 on paths crossing the others, so $\dim L = 3$.

No relative part. Let $t \in \mathcal{T}$ with $\sum_i t_i c_i \Xi_i \in Z$, and λ as in Lemma 2.8. From Y_1 and Y_1', $\tau_1 c(Y_1) = \lambda(z_B) - \lambda(z_0) = \lambda(z_2) - \lambda(z_B) =: \kappa$. From Y_j, $\tau_3 c(Y_j)$ is $\lambda(z_B)$ minus the value of λ at the zero of v_j, which is κ if $j \equiv 0 \pmod 4$ and $-\kappa$ if $j \equiv 2 \pmod 4$; from Y_3, likewise $2\tau_3 c(Y_3) = \pm\kappa$, according to $d - 1 \bmod 4$. If $d \geq 9$, Y_4 and Y_6 exist and give $\tau_3(c(Y_4)+c(Y_6)) = 0$; if $d = 7$, Y_4 and Y_3, with $d-1 \equiv 2 \pmod 4$, give $\tau_3(c(Y_4)+2c(Y_3)) = 0$. Circumferences are positive, so $\tau_3 = 0$; then $\kappa = 0$ and $\tau_1 = 0$, and Y_2 gives $\tau_2 c(Y_2) = \lambda(z_0) - \lambda(z_2) = -2\kappa = 0$. So $L \cap Z = 0$, $\dim_{\mathbb{R}} p(L) = 3$, and Lemma 2.8, with $\Sigma' = \Sigma$, gives $\operatorname{rank} \mathcal{M} \geq 3$. □

At $d = 5$ the moduli of this direction span a $\mathbb{Q}$-space of dimension two, and the same equations have a line of solutions in Z. The triangle is $(1, 3, 6)/10$, of rank one.

8.4. **Finitely many triangles.**

Proposition 8.9. *Each of the following fifty-one obtuse triangles has rank at least two:* $(1, 3, 5)/9$ *and* $(1, 6, 8)/15$*;* $(1, 4, 13)/18$*;* $(1, 4, 15)/20$*, also covered by Theorem 8.6*(a)*; the seven triangles of* (7)*; the twenty-nine triangles of* $\mathcal{E}$ *other than* $(2, 3, 6)/11$*; and the eleven triangles of Theorem 7.8*(c)* other than* $(1, 3, 8)/12$*.*

Proof. We follow the procedure by which Delecroix and Rüth bound orbit closures from below [8, §3]. For each of the fifty-one triangles we find finitely many completely periodic directions, among the directions of the saddle connections found by the program of the deposit, whose spaces L, together with $\operatorname{Re}[\omega]$ and $\operatorname{Im}[\omega]$, span a subspace whose image in $H^1(X; \mathbb{R})$ has dimension at least four, certified by a non-vanishing minor [21]. So Lemma 2.9 gives rank at least two. □

Proposition 8.10. *Let* Δ *be* $(1, 4, 11)/16$ *or* $(1, 4, 15)/20$*, and* $a = (1, 2, 2, 11)$*,* $a = (1, 2, 2, 15)$*, respectively. Then* $\mathcal{M}$ *is the locus* ΩG_a *of Eskin, McMullen, Mukamel and Wright* [9, Theorem 1.1]*; it has rank two, rel zero and complex dimension four.*

Proof. The linear parts of the reflections in the sides of $\Delta = (1, 4, 11)/16$ generate a dihedral group G_Δ of order 32. Reflecting Δ in its side joining the vertices of angles $\pi/16$ and $11\pi/16$ gives the non-convex quadrilateral Q with angles $\pi/8$, $\pi/4$, $11\pi/8$, $\pi/4$, whose group G_Q is dihedral of order 16 and does not contain the reflection σ in that side. So $G_\Delta = G_Q \sqcup G_Q\sigma$, the 16 copies of Q are the 32 copies of Δ with the same gluings, and the two unfoldings are the same translation surface. Eskin, McMullen, Mukamel and Wright cut Hooper's triangle out of a symmetric quadrilateral in the same way [9, §8, Remark 5]. The unfolding has one point of cone angle 22π and five of cone angle 2π, which are not marked. Q is a quadrilateral of type $a = (1, 2, 2, 11)$ [9, §1], so its unfolding is a cyclic form of type a [9, §1], [31, proof of Lemma 6.3], and lies in the locus $\Omega G_a \subset \Omega\mathcal{M}_6(10)$, which is irreducible of dimension four [9, Theorem 1.1] and of rank two, so of rel zero [9, §5, General properties 2]. The same holds for $(1, 4, 15)/20$, with Q of angles $\pi/10$, $\pi/5$, $3\pi/2$, $\pi/5$, $a = (1, 2, 2, 15)$, groups of orders 40 and 20, five zeros of order two and five points of cone angle 2π, and $\Omega G_a \subset \Omega\mathcal{M}_6(2^5)$. Since ΩG_a is closed and $\mathrm{GL}^+(2, \mathbb{R})$-invariant [9, §1 and proof of Theorem 5.3], and the $\mathrm{GL}(2, \mathbb{R})$-orbit of the unfolding is its $\mathrm{GL}^+(2, \mathbb{R})$-orbit (Section 2.3), $\mathcal{M} \subset \Omega G_a$, so $\operatorname{rank} \mathcal{M} \leq 2$ and $\operatorname{rel} \mathcal{M} = 0$. By Theorem 8.6(a) for $(1, 4, 15)/20$ and by Proposition 8.9 for $(1, 4, 11)/16$, $\operatorname{rank} \mathcal{M} = 2$, so $\dim_{\mathbb{C}} \mathcal{M} = 4$, and as ΩG_a is irreducible of dimension four, $\mathcal{M} = \Omega G_a$. □

8.5. **Proof of Theorem 8.1.** Let Δ be obtuse and not one of the five excluded triangles. If it has a witness, it has rank at least two by Theorem 2.6. Otherwise it lies in one of the families (1)–(7) of Section 8.1 if its obtuse angle exceeds $2\pi/3$, and in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ or $\mathcal{E}$ if not. The table says where each member is treated. Here $s = 2p_3 - N$ and $d = N/s$; for (2)–(6) the third column starts with the range in which the obtuse angle exceeds $2\pi/3$; in $\mathcal{C}$, $N = 2m$ with $4 \mid N$.

family	triangles	where
(1)	$(1, 1, N-2)/N$	excluded
(2)	$(1, 2, N-3)/N$, N even	$N \geq 10$; excluded
(3)	$(1, 4, N-5)/N$, $N \equiv 4 \pmod 8$	$N \geq 20$; Theorem 8.6(a)
(4)	$(1, q, 3q+1)/(4q+2)$	$q \geq 2$; $q = 2$: $(1,2,7)/10$, excluded; $q = 4$: Proposition 8.9; $q \geq 3$, $q \neq 4$: Proposition 8.3
(5)	$(2, q, 3q+2)/(4q+4)$, q odd	$q \geq 3$; Theorem 8.6(b)
(6)	$(4, q, 3q+4)/(4q+8)$, q odd	$q \geq 5$; Theorem 8.6(c)
(7)	seven triangles	Proposition 8.9
$\mathcal{A}$, $s = 1$	$(1, (N-3)/2, (N+1)/2)/N$, N odd	$N = 5$: $(1,1,3)/5$, excluded; $N = 7, 11$: Section 4.2; $N = 9, 15$: Proposition 8.9; $N \geq 13$, $N \neq 15$: Proposition 4.4
$\mathcal{A}$, $s = 2$	$(1, d-2, d+1)/2d$, d odd	$d = 3$: $(1,1,4)/6$ and $d = 5$: $(1,3,6)/10$, excluded; $d \geq 7$: Theorem 8.7
$\mathcal{A}$, $s \geq 3$		Theorem 7.8, except its twelve triangles: $(1,3,8)/12$ is excluded, the other eleven are in Proposition 8.9
$\mathcal{B}$		Theorem 7.9
$\mathcal{C}$, $p_1 = 1$	$(1, m-2, m+1)/2m$, m even	$m \geq 8$: Theorem 8.6(d); $m = 4$: $(1,2,5)/8$ and $m = 6$: $(1,4,7)/12$, excluded
$\mathcal{C}$, $p_1 = 2$	$(2, m-3, m+1)/2m$, m even	$m \geq 6$: Theorem 8.6(e); $m = 4$: $(1,2,5)/8$, excluded
$\mathcal{C}$, $p_1 = 4$	$(4, m-5, m+1)/2m$, $m \equiv 2 \pmod 4$	$m \geq 10$: Theorem 8.6(f); $m = 6$: $(1,4,7)/12$, excluded
$\mathcal{E}$	thirty triangles	$(2,3,6)/11$: Section 4.2; the others: Proposition 8.9

In every case Δ has rank at least two. □

9. Acute triangles

This section proves Theorem 1.7, which finds the acute triangles without a witness for every denominator, and then shows that the acute scalene triangles of rank one are $(2,3,4)/9$, $(3,4,5)/12$ and $(3,5,7)/15$ (Theorem 9.19). Here $\Delta = (p_1, p_2, p_3)/N$ is acute, written with $p_1 \leq p_2 \leq p_3 < N/2$; isosceles triangles are allowed until Section 9.4. We write $\mathcal{F}_1, \ldots, \mathcal{F}_4$ for the four families of Theorem 1.7, in the order listed there:

$$(1,k,k)/(2k+1), \quad (2, m-1, m-1)/2m, \quad (4, m-2, m-2)/2m, \quad (4, m-3, m-1)/2m,$$

with $k \geq 1$; m even, $m \geq 4$; m odd, $m \geq 5$; and $m \equiv 2 \pmod 4$, $m \geq 6$. In $\mathcal{F}_3$ and $\mathcal{F}_4$ the numerators are not in increasing order for the least m: $m = 5$ in $\mathcal{F}_3$ is $(3,3,4)/10$, and $m = 6$ in $\mathcal{F}_4$ is $(3,4,5)/12$.

We bound the denominator of a triangle without a witness by hand (Proposition 9.13) and then enumerate below the bound. The bound uses Jacobsthal's function, as in [32]: for an integer $n \geq 1$, $\mathrm{j}(n)$ is the least $J \geq 1$ such that any J consecutive integers contain one prime to n. It depends only on the primes dividing n, and $\mathrm{j}(n') \leq \mathrm{j}(n)$ when $n' \mid n$. Kanold proved that $\mathrm{j}(n) \leq 2^r$ for squarefree n with r prime factors [23, Satz 4, p. 324]; so

$$\mathrm{j}(n) \leq 2^r, \qquad r \text{ the number of distinct prime factors of } n. \tag{9.1}$$

Lemma 9.1 (Puchta [32, Lemma 4 (2)])**.** *Let $n \geq 1$, and let c_0, c_1 be integers with $\gcd(c_0, c_1, n) = 1$. Any $\mathrm{j}(n)$ consecutive terms of the progression $c_0 + ic_1$, $i \in \mathbb{Z}$, contain one prime to n.*

9.1. **Witnesses of acute triangles.** For $i = 1, 2, 3$ let A_i be the set of $a \in \mathbb{Z}/N$ with $[ap_i] < 2p_i$, and let $\mathcal{U}$ be the set of units a with $2a \not\equiv 2$, that is $a \neq 1$ and, for even N, $a \neq 1 + N/2$. By Lemma 2.7(a), $a \in \mathcal{U}$ is a witness if and only if $a \in A_1 \cap A_2 \cap A_3$; this does not use the order of p_1, p_2, p_3. Throughout, $\mathrm{j} = \mathrm{j}(N)$, and

$$\gamma_2 = N - 2p_2, \qquad \gamma_3 = N - 2p_3, \qquad \text{so that} \qquad \gamma_2 + \gamma_3 = 2p_1, \quad 1 \leq \gamma_3 \leq p_1 \leq \gamma_2 \leq 2p_1 - 1;$$

here $\gamma_3 \geq 1$ as $p_3 < N/2$, and $\gamma_3 = p_1 + p_2 - p_3 \leq p_1$ as $p_2 \leq p_3$. Also $N/4 < p_2 < N/2$, since $p_1 + p_2 = N - p_3 > N/2$ and $p_2 \geq p_1$; and $N \geq 3p_1$. Let $g_1 = \gcd(p_1, N)$, $p_1 = g_1 p_1'$ and $N = g_1 N'$, so that $\gcd(p_1', N') = 1$. A prime dividing g_1 and p_2 divides p_1, p_2 and N, hence p_3; so $\gcd(p_2, g_1) = \gcd(p_3, g_1) = 1$. Put

$$\upsilon = g_1 \prod_{\ell \mid g_1,\ \ell \nmid N'} \Big(1 - \frac{1}{\ell}\Big) \qquad (\ell \text{ prime}).$$

Lemma 9.2. *Let a be a unit with $r_1 = [ap_1] < 2p_1$, and $r_2 = [ap_2]$. Then $a \in A_2 \cap A_3$ if and only if $r_2 \in \mathcal{I}(r_1)$, where*

$$\mathcal{I}(t) = \big[\max(1, \gamma_3 - t + 1),\ N - 1 - \max(\gamma_2, t)\big].$$

For $1 \leq t \leq 2p_1 - 1$ the complement $\mathcal{I}(t)^{\mathrm{c}} = [0, \max(0, \gamma_3 - t)] \cup [N - \max(\gamma_2, t), N - 1]$ of $\mathcal{I}(t)$ in $[0, N-1]$ is one cyclic interval of at most $2p_1$ residues, containing 0.

Proof. Let $r_3 = [ap_3]$, so that $r_1 + r_2 + r_3 \equiv 0$ and $r_3 \geq 1$. If $r_2 < 2p_2$ and $r_3 < 2p_3$, then $r_1 + r_2 + r_3 < 2N$, so $r_3 = N - r_1 - r_2$; then $r_3 < 2p_3$ means $r_1 + r_2 > \gamma_3$, $r_3 \geq 1$ means $r_1 + r_2 \leq N - 1$, and $r_2 < 2p_2$ means $r_2 \leq N - 1 - \gamma_2$; so $r_2 \in \mathcal{I}(r_1)$. Conversely, if $r_2 \in \mathcal{I}(r_1)$, then $r_2 < 2p_2$ and $r_1 + r_2 \leq N - 1$, so $r_3 = N - r_1 - r_2 < N - \gamma_3 = 2p_3$. The complement has $\gamma_3 - t + 1 + \gamma_2 \leq 2p_1$ elements if $t \leq \gamma_3$; $1 + \gamma_2 \leq 2p_1$ if $\gamma_3 < t \leq \gamma_2$; and $1 + t \leq 2p_1$ if $t > \gamma_2$. □

Lemma 9.3. *For an integer t let $D_t = \{a \in \mathbb{Z}/N :\ ap_1 \equiv g_1 t \pmod N\}$.*

(a) *D_t is the class $a \equiv t p_1'^{-1} \pmod{N'}$, with g_1 elements. The units in A_1 are the units in the D_t with $1 \leq t \leq 2p_1' - 1$, and these D_t are pairwise distinct.*

(b) *$a \mapsto [ap_2]$ maps D_t bijectively onto the g_1 residues $\equiv t p_2 p_1'^{-1} \pmod{N'}$.*

(c) *$a \in D_t$ is a unit if and only if $\gcd(t, N') = 1$ and $\gcd([ap_2], g_1) = 1$. If $\gcd(t, N') = 1$, D_t contains exactly υ units; $\upsilon = 1$ if and only if $g_1 = 1$, or $g_1 = 2$ and N' is odd.*

(d) *The unit 1 lies in $D_{p_1'}$. For even N, $1 + N/2$ lies in $D_{p_1'}$ if p_1 is even, and in no D_t with $1 \leq t \leq 2p_1' - 1$ if p_1 is odd.*

Proof. (a) $ap_1 \equiv g_1 t \pmod{g_1 N'}$ if and only if $ap_1' \equiv t \pmod{N'}$. For a unit a, $[ap_1]$ is a non-zero multiple of g_1, so $a \in A_1$ means $[ap_1] = g_1 t$ with $1 \leq t \leq 2p_1' - 1$; and $2p_1' - 1 < N'$

as $2p_1 < N$. (b) On $a_0 + N'i$, $ap_2 = a_0p_2 + N'p_2i$, and $i \mapsto p_2i$ is a bijection of $\mathbb{Z}/g_1$. (c) $a \equiv tp_1'^{-1} \pmod{N'}$ is prime to N' if and only if t is. For a prime $\ell \mid g_1$, $\ell \nmid p_2$, so $\ell \mid a$ if and only if $\ell \mid [ap_2]$, as $\ell \mid N$. If also $\ell \mid N'$, then $\ell \nmid a$ already; if $\ell \nmid N'$, ℓ divides $a_0 + N'i$ for i in one class modulo ℓ; count by the Chinese remainder theorem. Finally $\upsilon \geq \#(\mathbb{Z}/g_1)^\times \geq 2$ for $g_1 \geq 3$. (d) $1 \cdot p_1 = g_1p_1'$; and $[(1 + N/2)p_1]$ is p_1 for even p_1 and $p_1 + N/2 \geq 2p_1$ for odd p_1, while $g_1t \leq 2p_1 - g_1$ for $t \leq 2p_1' - 1$. □

Lemma 9.4. *If $4p_1p_2 - \gamma_3^2 > 4N\mathrm{j}^2$, the triangle has a witness. So a triangle without a witness has $p_1 < 16\mathrm{j}^2/3$.*

Proof. The map $a \mapsto (ap_1, ap_2) \bmod N$ is injective on $\mathbb{Z}/N$: if $ap_1 \equiv ap_2 \equiv 0$, then $ap_3 \equiv 0$, and $\gcd(p_1, p_2, p_3) = 1$ gives $a \equiv 0$. Hence $\Gamma = \{(x, y) \in \mathbb{Z}^2 : (x, y) \equiv (ap_1, ap_2) \bmod N \text{ for some } a\}$ is a lattice of index N in $\mathbb{Z}^2$. The set B of $(x, y) \in \mathbb{R}^2$ with $|x| < p_1/\mathrm{j}$, $|y| < p_2/\mathrm{j}$ and $|x + y| < p_3/\mathrm{j}$ is open, convex and symmetric: the box $|x| < p_1/\mathrm{j}$, $|y| < p_2/\mathrm{j}$ minus two corner triangles with legs $(p_1 + p_2 - p_3)/\mathrm{j} = \gamma_3/\mathrm{j}$. Its area is $(4p_1p_2 - \gamma_3^2)/\mathrm{j}^2 > 4N$, so by Minkowski's theorem B contains a point $(u, v) \neq 0$ of Γ, say $(u, v) \equiv (a'p_1, a'p_2)$. For $1 \leq i \leq \mathrm{j}$ let $a_i = 1 + ia'$. Then $(a_ip_1, a_ip_2) \equiv (p_1 + iu, p_2 + iv)$ with $0 < p_1 + iu < 2p_1$ and $0 < p_2 + iv < 2p_2$, and $a_ip_3 \equiv p_3 - i(u + v)$ with $0 < p_3 - i(u + v) < 2p_3 < N$. So $a_i \in A_1 \cap A_2 \cap A_3$. By Lemma 9.1 with $c_0 = 1$, one a_i is a unit. It lies in $\mathcal{U}$: $a_i \equiv 1$ would give $(iu, iv) \equiv (0, 0)$ with $|iu| < p_1$ and $|iv| < p_2$, so $(u, v) = 0$; and $a_i \equiv 1 + N/2$ would give $(iu, iv) \equiv (p_1N/2, p_2N/2)$, each coordinate $\equiv 0$ or $N/2$, and $|iu| < p_1 < N/2$, $|iv| < p_2 < N/2$ again force $(u, v) = 0$. So it is a witness. For the second sentence, $4p_1p_2 - \gamma_3^2 \geq 4p_1p_2 - p_1^2 \geq 3p_1p_2 > 3p_1N/4$. □

Lemma 9.4 is the acute counterpart of Larsen, Norton and Zykoski's Proposition 7.5 [27]; Minkowski's theorem takes the place of their pigeonhole argument.

9.2. **Four cases.** For $p_1 \geq 2$, by Lemma 9.3(c), exactly one of the following holds: *Case 1*, $p_1 \mid N$; *Case 2*, $p_1 \nmid N$ and $\upsilon \geq 2$; *Case 3*, $g_1 = 1$; *Case 4*, $g_1 = 2$, $N/2$ odd and $p_1' \geq 2$.

Case 1.

Lemma 9.5. *Let $p_1 \mid N$, $p_1 \geq 2$, and $N' = N/p_1$. The triangle has a witness if and only if some integer k with $0 < |k| < p_1p_2/N$ has $\gcd(p_2 + kN', p_1) = 1$.*

Proof. Here $p_1' = 1$, so by Lemma 9.3(a) the units in A_1 lie in D_1, where $r_1 = p_1$ and $\mathcal{I}(p_1) = [1, 2p_2 - 1]$, as $\gamma_3 \leq p_1 \leq \gamma_2$ and $N - \gamma_2 = 2p_2$. By Lemma 9.3(b) and (c), the units of D_1 correspond to the $y \equiv p_2 \pmod{N'}$ with $\gcd(y, p_1) = 1$, and $y \in \mathcal{I}(p_1)$ if and only if $y = p_2 + kN'$ with $|k|N' < p_2$. The value $k = 0$ is the unit 1. For even p_1 the element $1 + N/2$ of D_1 has $y = [p_2 + p_2N/2] = p_2 + N/2 > 2p_2 - 1$, as p_2 is odd. □

Lemma 9.6. *If $n \geq 15$ has two distinct odd prime factors, then $\mathrm{j}(n) \leq \lfloor n/4 \rfloor$.*

Proof. Let r be the number of prime factors of n; by (9.1), $\mathrm{j}(n) \leq 2^r$. If $r = 2$: $4 \leq \lfloor n/4 \rfloor$ for $n \geq 16$, and $\mathrm{j}(15) = 3$ [21]. If $r = 3$: $8 \leq \lfloor n/4 \rfloor$ for $n \geq 32$, and the only smaller such n is 30, with $\mathrm{j}(30) = 6 \leq 7$ [32, Lemma 4 (5)]. If $r \geq 4$: $n \geq 210$, and $2^r \leq \lfloor n/4 \rfloor$, as the product of the first r primes is at least $4 \cdot 2^r$ for $r \geq 4$. □

Proposition 9.7. *Let $p_1 \mid N$ and $p_1 \geq 2$. The triangle has no witness if and only if $p_1 = 2$; or $p_1 = 4$ and $N \equiv 4 \pmod 8$; or it is $(3,4,5)/12$ or $(3,5,7)/15$.*

Proof. Let n_0 be the number of integers k with $0 < k < p_1p_2/N$. As $p_1/4 < p_1p_2/N < p_1/2$, $\lfloor p_1/4 \rfloor \leq n_0 \leq \lceil p_1/2 \rceil - 1$. As $\gcd(p_1, p_2) = 1$, p_2 is odd when p_1 is even.

(0) If $n_0 = 0$, that is $p_1p_2 \leq N$, there is no witness by Lemma 9.5, and $p_1 < 4$. For $p_1 = 2$ this always holds. For $p_1 = 3$, $3p_2 \leq N$ and $p_2 = N - 3 - p_3 > N/2 - 3$ give $N < 18$, so $N \in \{9, 12, 15\}$; $N = 9$ gives $(3,3,3)/9$, not reduced, $N = 12$ gives $p_2 = 4$, and $N = 15$ gives $p_2 = 5$.

(1) Let $n_0 \geq 1$ and suppose there is no witness. By Lemma 9.5 there are primes $\ell_+ \mid \gcd(p_2 + N', p_1)$ and $\ell_- \mid \gcd(p_2 - N', p_1)$. An odd prime dividing both would divide $2p_2$ and p_1. So either (1a) p_1 is even and N' is odd, and then $p_2 \pm N'$ are even; or (1b) p_1 is odd, or N' is even and $p_2 \pm N'$ are odd, and then ℓ_+, ℓ_- are odd and distinct, so p_1 has two distinct odd prime factors and $p_1 \geq 15$.

(2) Let $n_0 \geq 2$ and suppose there is no witness. There are also primes $\ell'_\pm \mid \gcd(p_2 \pm 2N', p_1)$. They are odd, as p_2 is odd when p_1 is even, and distinct, as a common one divides $4N'$ and $2p_2$, hence p_2. So p_1 has two distinct odd prime factors, $p_1 \geq 15$, and $\mathrm{j}(p_1) \leq \lfloor p_1/4 \rfloor \leq n_0$ by Lemma 9.6. By Lemma 9.1, as $\gcd(p_2, N', p_1) = 1$, some $k \in \{1, \dots, n_0\}$ has $\gcd(p_2 + kN', p_1) = 1$, and there is a witness. So a triangle without a witness has $n_0 \leq 1$.

(3) Let $n_0 = 1$ and suppose there is no witness. Then $p_1p_2 \leq 2N$ and $p_2 > N/4$ give $p_1 < 8$, so (1a) holds: $p_1 \in \{4, 6\}$ and N' is odd. For $p_1 = 4$, $n_0 = 1$ always, and $p_2 \pm N'$ is even if and only if N' is odd: there is no witness if and only if $N/4$ is odd. For $p_1 = 6$, $n_0 = 1$ means $p_2 \leq N/3$, so $p_3 \geq 2N/3 - 6$, and $p_3 < N/2$ gives $N < 36$, so $N \in \{18, 30\}$; then $p_2 \leq 6 = p_1$ forces $(6,6,6)/18$, and $N = 30$ forces $(6,10,14)/30$; neither is reduced.

Conversely, the triangles listed have no witness by (0) and (3). □

Case 2.

Lemma 9.8. *Let $p_1 \nmid N$. If $\upsilon > \lceil 2p_1g_1/N \rceil$, then D_1 contains a witness. So a triangle without a witness with $p_1 \nmid N$ and $\upsilon \geq 2$ has $N < 2p_1g_1/(\upsilon - 1) \leq 4p_1\mathrm{j}$.*

Proof. Here $p'_1 \geq 2$, so $D_1 \neq D_{p'_1}$, and by Lemma 9.3(d) all υ units of D_1 lie in $\mathcal{U}$. They have $r_1 = g_1$, and by Lemma 9.3(b) their values r_2 are distinct residues in one class modulo N'. By Lemma 9.2 the cyclic interval $\mathcal{I}(g_1)^{\mathrm{c}}$ has at most $2p_1$ residues, hence at most $\lceil 2p_1/N' \rceil = \lceil 2p_1g_1/N \rceil$ residues of one class modulo N'. So some unit of D_1 has $r_2 \in \mathcal{I}(g_1)$ and is a witness. Without a witness, $\upsilon - 1 < 2p_1g_1/N$. Finally $\upsilon \geq \#(\mathbb{Z}/g_1)^\times \geq g_1/\mathrm{j}(g_1)$, since each of the g_1 blocks $\{(k-1)\mathrm{j}(g_1) + 1, \dots, k\mathrm{j}(g_1)\}$, $1 \leq k \leq g_1$, contains an integer prime to g_1, and these $g_1\mathrm{j}(g_1)$ integers contain exactly $\mathrm{j}(g_1)\#(\mathbb{Z}/g_1)^\times$ integers prime to g_1; moreover $\mathrm{j}(g_1) \leq \mathrm{j}$ and $\upsilon - 1 \geq \upsilon/2$. □

Case 3. Here $\gcd(p_1, N) = 1$. The units in A_1 are the $a_t = tp_1^{-1}$ with $1 \leq t \leq 2p_1 - 1$ and t prime to N, and $[a_tp_1] = t$. By Lemma 9.3(d), $a_t \in \mathcal{U}$ if and only if $t \neq p_1$; for even N, p_1 is odd. Let $\nu = [p_2p_1^{-1}]$, so that $[a_tp_2] = [t\nu]$ and $p_1\nu \equiv p_2 \pmod N$. By Lemma 9.2, a_t with $t \neq p_1$ is a witness if and only if $[t\nu] \notin \mathcal{I}(t)^{\mathrm{c}}$.

Lemma 9.9. *Let $\gcd(p_1, N) = 1$ and $p_1 \geq 2$. Exactly one of the following holds.*

(a) *$\gamma_3 \leq \nu \leq N - \gamma_2 - 1$; then $a_1 = p_1^{-1}$ is a witness.*
(b) *$1 \leq \nu \leq \gamma_3 - 1$; then $N < 2p_1^2$, and every a_t with t prime to N, $t \neq p_1$, $1 \leq t \leq 2p_1 - 1$ and $\gamma_3 < t(\nu + 1) < N - \gamma_2$ is a witness. One exists if $p_1 \geq \mathrm{j} + 1$ and $N > (2\mathrm{j} + 2)p_1$.*

(c) $\nu = N - \nu'$ *with* $1 \le \nu' \le \gamma_2$*; then* $\nu' \ge 2$*,* $N < 4p_1^2$*, and every* a_t *with* t *prime to* N*,* $t \ne p_1$*,* $1 \le t \le 2p_1 - 1$ *and* $\gamma_2 < t\nu' < N - \gamma_3$ *is a witness. One exists if* $p_1 \ge \mathrm{j} + 1$ *and* $N > 2p_1 + 2\mathrm{j}(2p_1 - 1)$*.*

So a triangle of Case 3 without a witness has $N < 4p_1^2$*, and* $p_1 \le \mathrm{j}$ *or* $N \le 2p_1 + 2\mathrm{j}(2p_1 - 1)$*.*

Proof. The three ranges partition $[1, N-1]$, and $\nu \ne 0$. (a) says $\nu \notin \mathcal{I}(1)^{\mathrm{c}} = [0, \gamma_3 - 1] \cup [N - \gamma_2, N - 1]$, and $a_1 \in \mathcal{U}$ as $p_1 \ge 2$.

(b) Write $p_1\nu = p_2 + hN$ with $h \ge 0$. If $h \ge 1$, then $\nu > N/p_1$ and $\nu < \gamma_3 \le p_1$, so $N < p_1^2$. If $h = 0$, then $p_2 = p_1\nu \le p_1(p_1 - 1)$ and $N < 2(p_1 + p_2) \le 2p_1^2$. If $t(\nu + 1) < N - \gamma_2$, then $[t\nu] = t\nu < N - \gamma_2 - t$, which is at most $N - 1 - \max(\gamma_2, t)$, and $t\nu > \gamma_3 - t$; so $[t\nu] \in \mathcal{I}(t)$. Existence: the allowed t are the integers of $E = \big(\gamma_3/(\nu+1),\ \min(2p_1, (N - \gamma_2)/(\nu + 1))\big)$ prime to N and different from p_1, and $\gamma_3/(\nu + 1) \le p_1/2$. If $(N - \gamma_2)/(\nu + 1) \ge 2p_1$, E contains $p_1 + 1, \dots, 2p_1 - 1$, which are $p_1 - 1 \ge \mathrm{j}$ consecutive integers, one of them prime to N. Otherwise E is an open interval of length $(N - 2p_1)/(\nu + 1) \ge (N - 2p_1)/p_1 > 2\mathrm{j}$, so it contains $2\mathrm{j}$ consecutive integers, hence j consecutive integers avoiding p_1, one of them prime to N.

(c) $p_1\nu' + p_2 \equiv 0$ and is positive, so $p_1\nu' \ge N - p_2 > N/2$; so $\nu' > N/(2p_1) > 1$ and $N < 2p_1\nu' \le 2p_1(2p_1 - 1)$. If $t\nu' < N$, then $[t\nu] = N - t\nu'$, which lies in $\mathcal{I}(t)^{\mathrm{c}}$ if and only if $t(\nu' - 1) \ge N - \gamma_3$ or $t\nu' \le \max(\gamma_2, t)$, the latter meaning $t\nu' \le \gamma_2$ as $\nu' \ge 2$. For $\gamma_2 < t\nu' < N - \gamma_3$ neither holds. Existence follows as in (b), with $E = \big(\gamma_2/\nu',\ \min(2p_1, (N - \gamma_3)/\nu')\big)$, $\gamma_2/\nu' < p_1$, and length $(N - 2p_1)/\nu' \ge (N - 2p_1)/(2p_1 - 1) > 2\mathrm{j}$ in the second alternative.

The last sentence: without a witness (a) fails, and in (b) and (c) the existence statements fail; note $(2\mathrm{j} + 2)p_1 \le 2p_1 + 2\mathrm{j}(2p_1 - 1)$. □

Case 4. Here $N = 2m$ with m odd, $p_1 = 2p_1'$ with $\gcd(p_1', m) = 1$ and $p_1' \ge 2$, and p_2, p_3 are odd, so $4 \mid \gamma_2$ and $4 \mid \gamma_3$. By Lemma 9.3(c), each D_t with t prime to m has one unit, its odd element, and this unit lies in $\mathcal{U}$ if and only if $t \not\equiv p_1' \pmod m$. Let a_1 be the unit of D_1 and $\nu = [a_1p_2]$, which is odd; $p_1a_1 \equiv 2$, so $p_1\nu \equiv 2p_2 \pmod N$. For odd t the unit of D_t is $a_t = ta_1$, with $[a_tp_2] = [t\nu]$; for even t it is $a_t = ta_1 + m$, with $[a_tp_2] = [t\nu + m]$.

Lemma 9.10. *In Case 4, if* $\gamma_2 = p_1$*, so that* $p_2 = p_3 = m - p_1'$ *and* p_1' *is even, then* $\nu = N - 1$*. For* $p_1' = 2$ *the triangle is* $(4, m-2, m-2)/2m$*. For* $p_1' \ge 3$ *the unit* $a_2 = 2a_1 + m$ *is a witness.*

Proof. $2p_2 = N - p_1$ gives $2p_2a_1 \equiv -2$, so $p_2a_1 \equiv -1 \pmod m$; as ν is odd, $\nu = N - 1$. For $p_1' \ge 3$, $t = 2 \le 2p_1' - 1$ is prime to m and different from p_1'; $[a_2p_1] = 4$, $[a_2p_2] = [2\nu + m] = m - 2$, and $\mathcal{I}(4) = [\max(1, p_1 - 3), N - 1 - p_1]$, as $\gamma_3 = \gamma_2 = p_1 \ge 6$, contains $m - 2$, since $p_1 < m$. □

Lemma 9.11. *Let* $N = 2m$ *with* m *odd,* $g_1 = 2$*,* $p_1' \ge 2$ *and* $\gamma_2 \ne p_1$*. Exactly one of the following holds.*

(a) $\gamma_3 - 1 \le \nu \le N - \gamma_2 - 1$*; then* a_1 *is a witness.*

(b) $\nu \le \gamma_3 - 2$*; then* $N < p_1^2$*, and every* a_t *with* t *odd, prime to* m*,* $t \ne p_1'$*,* $t \le 2p_1' - 1$ *and* $\gamma_3 < t(\nu + 2) < N - \gamma_2$ *is a witness. One exists if* $p_1' \ge \mathrm{j} + 1$ *and* $N > 2p_1 + 2\mathrm{j}p_1$*.*

(c) $\nu = N - \nu'$ *with* $1 \le \nu' \le \gamma_2$*; then* ν' *is odd,* $\nu' \ge 5$*,* $N < 2p_1^2$*, and every* a_t *with* t *odd, prime to* m*,* $t \ne p_1'$*,* $t \le 2p_1' - 1$ *and* $\gamma_2 < t\nu' < N - \gamma_3$ *is a witness. One exists if* $p_1' \ge \mathrm{j} + 1$ *and* $N > 2p_1 + 2\mathrm{j}(2p_1 - 1)$*.*

So a triangle of Case 4 with $\gamma_2 \ne p_1$ *and without a witness has* $N < 2p_1^2$*, and* $p_1' \le \mathrm{j}$ *or* $N \le 2p_1 + 2\mathrm{j}(2p_1 - 1)$*.*

Proof. For odd t, $[a_t p_1] = 2t$, and the condition of Lemma 9.2 is $[t\nu] \in \mathcal{I}(2t)$. (a) is $\nu \notin \mathcal{I}(2)^{\mathrm{c}} = [0, \gamma_3 - 2] \cup [N - \gamma_2, N - 1]$, as $\gamma_3, \gamma_2 \geq 4$.

(b) Write $p_1\nu = 2p_2 + hN$ with $h \geq 0$. If $h \geq 1$, then $\nu > N/p_1$ and $\nu < \gamma_3 \leq p_1$, so $N < p_1^2$. If $h = 0$, then $2p_2 = p_1\nu \leq p_1(\gamma_3 - 2)$ and $N < 2(p_1 + p_2) \leq p_1\gamma_3 \leq p_1^2$. If $t(\nu + 2) < N - \gamma_2$, then $[t\nu] = t\nu < N - \gamma_2 - 2t$, below $N - \gamma_2$ and $N - 2t$, and $t\nu > \gamma_3 - 2t$; so $[t\nu] \in \mathcal{I}(2t)$. Existence follows as in Lemma 9.9, with $E = (\gamma_3/(\nu + 2),\ \min(2p_1', (N - \gamma_2)/(\nu + 2)))$, $\gamma_3/(\nu + 2) \leq p_1/3 < p_1'$, and length $(N - 2p_1)/(\nu + 2) \geq (N - 2p_1)/p_1$ in the second alternative; the odd integers prime to m are the integers prime to N.

(c) ν' is odd as ν is. $p_1\nu' + 2p_2 = hN$ with $h \geq 1$, so $p_1\nu' = (h - 1)N + \gamma_2$. If $h = 1$, $p_1\nu' = \gamma_2 < 2p_1$ gives $\nu' = 1$ and $\gamma_2 = p_1$, which is excluded. So $p_1\nu' > N$, $\nu' > N/p_1 \geq 3$, $\nu' \geq 5$, and $N < p_1\gamma_2 < 2p_1^2$. If $t\nu' < N$, then $[t\nu] = N - t\nu'$, which lies in $\mathcal{I}(2t)^{\mathrm{c}}$ if and only if $t(\nu' - 2) \geq N - \gamma_3$ or $t\nu' \leq \max(\gamma_2, 2t)$, that is $t\nu' \leq \gamma_2$. For $\gamma_2 < t\nu' < N - \gamma_3$ neither holds. Existence with $E = (\gamma_2/\nu',\ \min(2p_1', (N - \gamma_3)/\nu'))$ and $\gamma_2/\nu' \leq \gamma_2/5 < p_1'$. □

9.3. **The reduction, and the proof of Theorem 1.7.**

Lemma 9.12. *The triangles of $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$ have no witness.*

Proof. We use Lemma 2.7(a), which does not use the order of the numerators. $\mathcal{F}_1$: $[a] < 2$ forces $a = 1$. $\mathcal{F}_2$: $[2a] < 4$ forces $2a \equiv 2$. $\mathcal{F}_4$: $N = 2m \equiv 4 \pmod 8$, so $[4a]$ is a multiple of 4, and $[4a] < 8$ forces $4a \equiv 4 \pmod{2m}$, that is $a \equiv 1 \pmod{m/2}$; as $m/2$ is odd, $1 + m/2$ and $1 + 3m/2$ are even, and $a \in \{1, 1 + m\}$ is not in $\mathcal{U}$. $\mathcal{F}_3$: a is odd, and $[4a] < 8$ means $2a \equiv c \pmod m$ with $c \in \{1, 2, 3\}$. For $c = 2$, $a \equiv 1 \pmod m$ and $a = 1$. For $c = 1, 3$, $2a \equiv c + m \pmod{2m}$, as $2a$ and $c + m$ are even, and $a(m - 2) \equiv m - 2a \equiv -c$, so $[a(m - 2)] = 2m - c \geq 2(m - 2)$ and $a \notin A_2$. □

Proposition 9.13. *An acute triangle without a witness lies in $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$, or is $(3, 5, 7)/15$, or has $3N < 32\,\mathrm{j}^2(2\mathrm{j} + 1)$.*

Proof. If $p_1 = 1$, $p_2 + p_3 = N - 1$ with $p_2, p_3 < N/2$ forces $p_2 = p_3$, and the triangle is in $\mathcal{F}_1$. Let $p_1 \geq 2$. In Case 1, by Proposition 9.7, either $p_1 = 2$, and then $p_2 = p_3 = N/2 - 1$, as both are odd with $p_3 \leq N/2 - 1$, which is $\mathcal{F}_2$; or $p_1 = 4$ and $N = 2m \equiv 4 \pmod 8$, and then $(p_2, p_3) = (m - 3, m - 1)$, as p_2, p_3 are odd with $p_2 + p_3 = 2m - 4$ and $p_3 \leq m - 1$, which is $\mathcal{F}_4$; or the triangle is $(3, 4, 5)/12$, in $\mathcal{F}_4$, or $(3, 5, 7)/15$. In Case 2, $N < 4p_1\mathrm{j}$ by Lemma 9.8. In Case 3, by Lemma 9.9, $N < 4p_1^2 \leq 4\mathrm{j}^2$ or $N \leq 2p_1 + 2\mathrm{j}(2p_1 - 1) < (4\mathrm{j} + 2)p_1$. In Case 4, Lemma 9.10 gives $\mathcal{F}_3$ if $\gamma_2 = p_1$, and otherwise Lemma 9.11 gives $N < 2p_1^2 \leq 8\mathrm{j}^2$ or $N < (4\mathrm{j} + 2)p_1$. By Lemma 9.4, $p_1 < 16\mathrm{j}^2/3$, so $4p_1\mathrm{j} < \frac{64}{3}\mathrm{j}^3$ and $(4\mathrm{j} + 2)p_1 < \frac{32}{3}\mathrm{j}^2(2\mathrm{j} + 1)$; and $4\mathrm{j}^2$, $8\mathrm{j}^2$ and $\frac{64}{3}\mathrm{j}^3$ are all below $\frac{32}{3}\mathrm{j}^2(2\mathrm{j} + 1)$, as $\mathrm{j} \geq 2$. □

Let $\mathcal{L}$ be the set of $N \geq 2$ with $3N < 32\,\mathrm{j}(N)^2(2\mathrm{j}(N) + 1)$.

Lemma 9.14. *Every $N \in \mathcal{L}$ has at most 10 distinct prime factors, and $3N < 32 \cdot 4^r(2^{r+1} + 1)$, where r is their number. So $\mathcal{L}$ is finite.*

Proof. By (9.1), $3N < B_r = 32 \cdot 4^r(2^{r+1} + 1)$, and $N \geq P_r$, the product of the first r primes. For $r = 11$, $3P_{11} = 601{,}681{,}470{,}390 > B_{11} = 549{,}890{,}031{,}616$, and from r to $r + 1$ the left side grows by a factor at least 37 and the right side by a factor less than 8. □

Proof of Theorem 1.7. We compute $\mathrm{j}(N)$ exactly, as one more than the longest interval of integers that can be covered by choosing one residue class modulo each prime dividing N, for every N allowed by Lemma 9.14, and test every acute triangle with $N \in \mathcal{L}$ and $3p_1 < 16\,\mathrm{j}(N)^2$ for a witness. We find that $\mathcal{L}$ has 4,627 elements, the largest being $232{,}050 = 2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$,

and that for every $N \in \mathcal{L}$ the acute triangles of denominator N with $3p_1 < 16\,\mathrm{j}(N)^2$ and no witness are exactly the members of $\mathcal{F}_1, \dots, \mathcal{F}_4$ of denominator N and those of the nine triangles of Theorem 1.7 of denominator N [21].

The families have no witness by Lemma 9.12. The nine have $N \le 42$, so $N \in \mathcal{L}$, as every N with $2 \le N \le 213$ lies in $\mathcal{L}$ because $\mathrm{j}(N) \ge 2$; and they have $3p_1 < 16\,\mathrm{j}(N)^2$, as $p_1 \le 8$, so they have no witness. Conversely, by Proposition 9.13 an acute triangle without a witness is in a family, or is $(3,5,7)/15$, or has $N \in \mathcal{L}$; in the last case $3p_1 < 16\,\mathrm{j}(N)^2$ by Lemma 9.4, and it is in a family or one of the nine. □

9.4. **Acute scalene triangles of rank one.** The members of $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are isosceles, while those of $\mathcal{F}_4$ and the nine triangles of Theorem 1.7 are scalene. So it remains to treat $\mathcal{F}_4$ and seven of the nine.

Lemma 9.15. *For $m \equiv 2 \pmod 4$ with $m \ge 14$, $(4, m-3, m-1)/2m$ is not a lattice triangle.*

Proof. Kenyon and Smillie prove that if an acute rational triangle with angles $\alpha \ge \beta \ge \gamma$ is a lattice triangle and n is the least common denominator of α/π, β/π, γ/π, then $\cos(k\beta + k\gamma)\sin(k\beta)\sin(k\gamma) < 0$ for every k prime to n [24, Theorem 24]; their unfolding is ours (Section 2.1). Here $n = 2m$, $\beta = (m-3)\pi/2m$ and $\gamma = 4\pi/2m$. Write $m = 4t+2$ and $k = m/2 - 2 = 2t - 1$, which is prime to $2m$, as $\gcd(2t-1, 8t+4) = \gcd(2t-1, 8) = 1$. Then $k\gamma = \pi - 4\pi/m$, so $\sin k\gamma = \sin(4\pi/m) > 0$; $k\beta = \pi(t-2) + \pi(3/4 + 3/m)$, so $\sin k\beta = (-1)^t \sin(3\pi/4 + 3\pi/m)$, with $\sin(3\pi/4 + 3\pi/m) > 0$ as $3/m < 1/4$; and $k\beta + k\gamma = \pi(t-1) + \pi(3/4 - 1/m)$, so $\cos(k\beta + k\gamma) = (-1)^{t-1}\cos(3\pi/4 - \pi/m)$, with $\cos(3\pi/4 - \pi/m) < 0$. The product is $(-1)^{2t-1}$ times a negative number, hence positive, and the condition fails at k. □

At $m = 10$, the triangle $(4,7,9)/20$, the condition of Kenyon and Smillie's Theorem 24 holds for all eight units k, and it does not decide.

Lemma 9.16. *For $m \equiv 2 \pmod 4$ with $m \ge 6$, with the vertices numbered in the order written, $(4, m-3, m-1)/2m$ has $g_1 = 4$, $g_2 = \gcd(m-3, 6)$, $g_3 = 1$, and N/g_i is even for every $i \in I$; so $\operatorname{rel}\mathcal{M} = 0$ by Corollary 6.3(ii).*

Proof. The triangle is scalene, is not $(1,2,3)/6$, and has N even. Its g_2 is that of Lemma 8.4(e), whose p_2 is also $m-3$, and $g_1 = \gcd(4, 2m) = 4$ and $g_3 = \gcd(m-1, 2) = 1$, as $m \equiv 2 \pmod 4$. So g_2 and g_3 are odd, and N/g_2, N/g_3 are even. Also $g_1 = p_1$, so $1 \notin I$. □

Proposition 9.17. *For $m \equiv 2 \pmod 4$ with $m \ge 14$, $(4, m-3, m-1)/2m$ has rank at least two.*

Proof. If the rank were one, then $\dim_{\mathbb{C}} \mathcal{M} = 2$, as $\operatorname{rel}\mathcal{M} = 0$ by Lemma 9.16, and the triangle would be a lattice triangle by Lemma 2.3, against Lemma 9.15. □

Proposition 9.18. *Each of the eight triangles $(4,7,9)/20$, $(3,5,6)/14$, $(5,6,7)/18$, $(3,8,9)/20$, $(5,8,11)/24$, $(7,9,14)/30$, $(8,9,13)/30$, $(5,17,20)/42$ has rank at least two.*

Proof. For each of them we find, as in the proof of Proposition 8.9, completely periodic directions for which Lemma 2.9 gives rank at least two [21]. □

Theorem 9.19. *An acute scalene rational triangle of rank one is $(2,3,4)/9$, $(3,4,5)/12$ or $(3,5,7)/15$.*

Proof. Let Δ be acute and scalene, and not one of the three. If it has a witness, it has rank at least two by Theorem 2.6. Otherwise, by Theorem 1.7, it is $(4, m-3, m-1)/2m$ with $m \equiv 2 \pmod 4$, $m \geq 6$, or one of the nine. In the family, $m = 6$ gives $(3,4,5)/12$, excluded; $m = 10$ gives $(4,7,9)/20$, in Proposition 9.18; and $m \geq 14$ is Proposition 9.17. Of the nine, $(2,3,4)/9$ and $(3,5,7)/15$ are excluded and the other seven are in Proposition 9.18. □

10. Right triangles and two Prym eigenforms

This section treats the right triangles by witnesses, proves that $(1,3,6)/10$ and $(1,3,8)/12$ have rank one, describes their orbit closures, and assembles the proof of Theorem 1.1.

10.1. **Right triangles.** Let Δ be a right rational triangle, with the right angle at the third vertex. Then $N = 2n$, $p_3 = n$, and by Apisa's Lemma 7.1 [2], after renumbering the two acute vertices, $\Delta = (a, n-a, n)/2n$ with $\gcd(a, 2n) = 1$, so a is odd. For odd n, a is determined by Δ; for even n, a and $n-a$ both qualify, and the conditions below do not depend on the choice.

Proposition 10.1. *Let $\Delta = (a, n-a, n)/2n$ with a odd, $\gcd(a, 2n) = 1$ and $a \notin \{1, n-1, n-2\}$. Put $\varepsilon = 1$ if n is even and $\varepsilon = 2$ if n is odd, and let*

$$\alpha \equiv a^{-1} \pmod{2n} \text{ if } 2a \leq n, \qquad \alpha \equiv (n-\varepsilon)\, a^{-1} \pmod{2n} \text{ if } 2a > n.$$

Then α is a witness, and Δ has rank at least two.

Proof. This is Apisa's choice in the proof of his Lemma 7.4 [2]: $[\alpha a]$ and $[(2-\alpha)a]$ are 1 and $2a-1$ if $2a \leq n$, and $n-\varepsilon$ and $2a-n+\varepsilon$ if $2a > n$. They lie in $[1, n-1]$, as $3 \leq a \leq n/2$, respectively $n/2 < a \leq n-3$. Here α is a unit, as $n-\varepsilon$ is odd and prime to n, so α and $2-\alpha$ are odd. For odd ℓ with $[\ell a] < n$ the residues of $\ell \cdot (a, n-a, n)$ are $[\ell a]$, $n-[\ell a]$ and n, of sum $2n$, so $\alpha, 2-\alpha \in \Phi$. Finally $2\alpha \not\equiv 2$, as $\alpha \equiv 1 \pmod n$ would give $a = 1$, respectively $a = n-\varepsilon$. So α is a witness, and Theorem 2.6 gives rank at least two. □

The excluded values give two families. For $a \in \{1, n-1\}$ the triangle is $(1, n-1, n)/2n = (1, k, k+1)/(2k+2)$ with $k = n-1$, scalene if and only if $k \geq 2$; for $a = n-2$ with n odd it is $(2, n-2, n)/2n = (2, 2k+1, 2k+3)/(4k+6)$ with $n = 2k+3$, and $k = 0$ gives $(1,2,3)/6$ again. Their smallest angles are $\pi/(2k+2)$ and, for $k \geq 1$ in the second family, $\pi/(2k+3)$.

10.2. **Two Prym eigenforms.** We recall McMullen's definitions [29, §3]. Let ρ be a holomorphic involution of a compact Riemann surface Y. It splits the holomorphic 1-forms as $\Omega(Y)^+ \oplus \Omega(Y)^-$ and $H_1(Y;\mathbb{Z})$ into even and odd cycles, and $\mathrm{Prym}(Y,\rho) = (\Omega(Y)^-)^*/H_1(Y;\mathbb{Z})^-$. For a discriminant $D > 0$ let $\mathcal{O}_D$ be the real quadratic order of discriminant D. If $\dim \Omega(Y)^- = 2$, the Prym variety has real multiplication by $\mathcal{O}_D$ if its endomorphisms contain a proper subring isomorphic to $\mathcal{O}_D$, generated by one endomorphism that is self-adjoint for the intersection pairing; a form $\eta \in \Omega(Y)^-$ with $0 \neq \mathcal{O}_D \cdot \eta \subset \mathbb{C}\eta$ is a Prym eigenform, and ΩE_D is the locus of Prym eigenforms. McMullen proves that ΩE_D is a closed $\mathrm{SL}(2,\mathbb{R})$-invariant subset of $\Omega\mathcal{M}_g$ [29, Theorem 3.2]; positive scalars preserve it as well. For a stratum $\mathcal{H}(\kappa)$ let $\Omega E_D(\kappa) = \Omega E_D \cap \mathcal{H}(\kappa)$.

Proposition 10.2. *Let Δ be $(1,3,6)/10$ or $(1,3,8)/12$, and let $\iota = T^{N/2}$ and $\Theta = T^* + (T^*)^{-1}$.*

(a) *(X,ω) is a Prym eigenform for ι and real multiplication by $\mathbb{Z}[\Theta]$: it lies in $\Omega E_5(2,2,2)$ for $(1,3,6)/10$ and in $\Omega E_{12}(1,1,1,1)$ for $(1,3,8)/12$.*

(b) $\operatorname{rank} \mathcal{M} = 1$.

(c) *For* $(1,3,6)/10$, $\mathcal{M}$ *is a connected component of* $\Omega E_5(2,2,2)$, *of complex dimension three, so* $\operatorname{rel}\mathcal{M}=1$.

Neither triangle is a lattice triangle.

The last sentence is part of the classification of obtuse lattice triangles [17, Theorem 1.1]; for $(1,3,8)/12$ it is also Vorobets' Theorem 4.5 [37].

Proof. (a) We use Lemma 2.1. For $(1,3,6)/10$ the characters 0 and 5 are not in $\Phi\cup-\Phi$ and the other eight are, so $H^1(X;\mathbb{C})$ is the sum of the eight lines H^1_a, $a\neq 0,5$, and the genus is 4, in accordance with (2.1). The involution ι acts on H^1_a by $(-1)^a$, and $\iota^*\omega=-\omega$. The odd lines are $a=1,3,7,9$, of which 1 and 7 lie in Φ, as $1+3+6=10$ and $7+1+2=10$, and 3, 9 do not. So $\Omega(X)^-=H^{1,0}_1\oplus H^{1,0}_7$ has dimension two. Θ acts on H^1_a by $\zeta^a+\zeta^{-a}=2\cos(2\pi a/10)$, which is $2\cos(\pi/5)$ on the lines 1, 9 and $2\cos(3\pi/5)$ on the lines 3, 7; both are roots of x^2-x-1, which is therefore the minimal polynomial of Θ on $H^1(X;\mathbb{Q})^-$. For $(1,3,8)/12$ the characters $0,3,4,6,8,9$ are not in $\Phi\cup-\Phi$ and the other six are, and the genus is 3; the odd lines are $1,5,7,11$, of which 1 and 5 lie in Φ, as $1+3+8=12$ and $5+3+4=12$; Θ acts by $\sqrt{3}$ on the lines 1, 11 and by $-\sqrt{3}$ on the lines 5, 7, with minimal polynomial x^2-3.

In both cases T^* is integral, preserves the intersection pairing and commutes with ι^*; so Θ preserves $H^1(X;\mathbb{Z})^-$, is self-adjoint, as $\langle T^*x,y\rangle=\langle x,(T^*)^{-1}y\rangle$, and preserves $\Omega(X)^-$. Dually it acts on $H_1(X;\mathbb{Z})^-$ as a complex-linear endomorphism of $\operatorname{Prym}(X,\iota)$. As the minimal polynomial is monic and irreducible, $\mathbb{Z}[\Theta]\cong\mathbb{Z}[x]/(x^2-x-1)=\mathcal{O}_5$, respectively $\mathbb{Z}[x]/(x^2-3)=\mathcal{O}_{12}$, the ring of integers of $\mathbb{Q}(\sqrt{5})$, respectively $\mathbb{Q}(\sqrt{3})$. If an endomorphism F has $mF\in\mathbb{Z}[\Theta]$ for an integer $m\neq 0$, then $F\in\mathbb{Q}[\Theta]$, and F is integral over $\mathbb{Z}$, as its characteristic polynomial on the lattice is monic with integer coefficients; so $F\in\mathbb{Z}[\Theta]$, and the subring is proper. Finally $\Theta\omega=(\zeta+\zeta^{-1})\omega$, so ω is a Prym eigenform. The zeros, by the cone angles $2\pi p_i/g_i$ of Section 2.1: for $(1,3,6)/10$, one regular point above the first vertex, one zero of order two above the second and two above the third, so the stratum is $\mathcal{H}(2,2,2)$; for $(1,3,8)/12$, regular points above the first two vertices and four simple zeros above the third, so $\mathcal{H}(1,1,1,1)$.

(b) By McMullen's theorem the orbit of (X,ω) lies in the closed subset $\Omega E_D(\kappa)$ of the stratum, hence so does $\mathcal{M}$. Take U and V as in Section 2.3, and identify the surfaces of $V\cap U$ with X by the marking. Each $w\in V\cap U$ is a Prym eigenform: there are an involution ρ_w, whose action ρ_w^* on $H^1(X,\Sigma;\mathbb{Z})$ is an integer matrix, as it permutes the zeros; an endomorphism Θ_w of $H^1(X;\mathbb{Q})$ that is rational, preserves the ρ_w-odd part and vanishes on the even part; and a root μ_w of its minimal polynomial, such that $\rho_w^*w=-w$ and $p(w)$ is an eigenvector of Θ_w for μ_w. There are countably many triples (ρ^*,Θ',μ), and each condition is linear in w; so by Baire's theorem one triple serves for a non-empty open subset of $V\cap U$, hence for all of V. The minimal polynomial of Θ' on the four-dimensional odd part is that of a generator of $\mathcal{O}_D$, irreducible as D is not a square. So each of its eigenspaces there has dimension two, and $p(V)$ lies in one of them. As $p(V)$ contains the independent classes $[\omega]$ and $[\bar\omega]$, $\dim p(V)=2$ and $\operatorname{rank}\mathcal{M}=1$.

(c) The involution $\iota=T^5$ maps y to $-y$, so its fixed points lie above 0, 1 and ∞. As T permutes the g_i points above vertex i cyclically, ι fixes the single points above the first two vertices and exchanges the two above the third. By the Riemann–Hurwitz formula X/ι has genus two, and ω^2 descends to a quadratic differential on it with a simple pole at the image of the regular fixed point, a simple zero at the image of the fixed double zero, and a zero of order four at the image of the exchanged pair. So (X,ω) lies in the Prym locus attached to the stratum $\mathcal{Q}(-1,1,4)$, which Lanneau and Nguyen list in their Table 1 [26], as

$\mathrm{Prym}(2,2,2)^{\mathrm{even}}$. The same holds for every point of $\Omega E_5(2,2,2)$. A Prym involution ρ of a surface of genus four in $\mathcal{H}(2,2,2)$ has a quotient of genus two, hence two fixed points by the Riemann–Hurwitz formula; a fixed point is a regular point or a zero of even order, since if $\rho(z) = -z$ and $\eta = z^k u(z)\,dz$ with $u(0) \neq 0$, then $\rho^*\eta = -\eta$ forces k even; and the zeros that are not fixed are exchanged in pairs. So ρ fixes one zero and one regular point and exchanges the other two zeros, the quotient lies in $\mathcal{Q}(-1,1,4)$, and $\Omega E_5(2,2,2)$ is contained in a Prym locus of their Table 1. Lanneau and Nguyen prove that if $\Omega E_D(\kappa)$ has complex dimension three, that is, is contained in one of the Prym loci of their Table 1, then the $\mathrm{GL}^+(2,\mathbb{R})$-orbit of a Prym eigenform in $\Omega E_D(\kappa)$ is closed, or its closure is a connected component of $\Omega E_D(\kappa)$ [26, Theorem 1.1]. As $(1,3,6)/10$ is not a lattice triangle, the second alternative holds, and $\mathcal{M}$ has dimension three; with rank one, $\operatorname{rel}\mathcal{M} = 1$. □

Proposition 10.3. *For* $(1,3,8)/12$, $\operatorname{rank}\mathcal{M} = 1$, $\operatorname{rel}\mathcal{M} = 2$ *and* $\dim_{\mathbb{C}}\mathcal{M} = 4$*; near* (X,ω), $\mathcal{M}$ *coincides with* $\Omega E_{12}(1,1,1,1)$*, and it is an irreducible component of* $\Omega E_{12}(1,1,1,1)$.

Proof. The rank is one by Proposition 10.2, so $p(V) = H^1_1 \oplus H^1_{11}$ by Lemma 2.4. Here $I = \{3\}$ and Σ consists of the $g_3 = 4$ simple zeros above the third vertex, so $Z = Z_3 \oplus Z_6 \oplus Z_9$, three lines (Lemma 2.2). By Lemma 2.4, $V \cap Z$ is the sum of its intersections with them; it lies in $Z^- = Z_3 \oplus Z_9$ by Corollary 6.3, and it is not zero by Lemma 2.3, as the triangle is not a lattice triangle. As V and Z are defined over $\mathbb{R}$ and complex conjugation exchanges Z_3 and Z_9, $V \cap Z = Z^-$. So $\operatorname{rel}\mathcal{M} = 2$, $\dim_{\mathbb{C}}\mathcal{M} = 4$, and V is the space L of the $w \in E^-$ with $p(w) \in H^1_1 \oplus H^1_{11}$, since L contains V and has dimension four, as p maps E^- onto $H^1(X;\mathbb{C})^-$ with kernel Z^-.

Every $w \in L \cap U$ is a Prym eigenform for $\mathcal{O}_{12}$: the involution ι deforms with the surface, as $w \in E^-$, and Θ has $p(w)$ as an eigenvector, so it preserves $\Omega(X_w)^-$, which is spanned by w and a form in the other eigenspace of Θ, the symplectic orthogonal of $p(w)$ and $p(\bar{w})$ in $H^1(X;\mathbb{C})^-$ (as $\Omega(X_w)^-$ is isotropic, it lies in the symplectic orthogonal of $p(w)$ in the odd part, which is $\mathbb{C}p(w)$ plus that eigenspace). Conversely, let $w_n \to [\omega]$ be Prym eigenforms for $\mathcal{O}_{12}$, with involutions ρ_n and generators Θ_n of $\mathcal{O}_{12}$ with $\Theta_n^2 = 3$. The ρ_n^* are bounded, as ρ_n is an isometry of the flat metric, and so are the Θ_n, which are self-adjoint for the Hodge norm with eigenvalues $\pm\sqrt{3}$; so along a subsequence both are constant, ρ^* and Θ', and, as in the proof of [29, Theorem 3.2], ρ_n converges to an automorphism ρ of X with $\rho^*\omega = -\omega$, which is ι (Theorem 6.2). In the limit $p[\omega]$ is an eigenvector of Θ'; the eigenspace of Θ' containing it is a plane containing $p[\bar{\omega}]$, hence $H^1_1 \oplus H^1_{11}$, and it contains the $p(w_n)$. So near $[\omega]$ every such eigenform lies in L, and $\Omega E_{12}(1,1,1,1)$ and $\mathcal{M}$ coincide near (X,ω). The same argument at any point (Y,η), in period coordinates of marked surfaces, gives one linear subspace of dimension four for each of the finitely many Prym involutions for which the surface is an eigenform, as, whichever generator of $\mathcal{O}_{12}$ is used, the eigenspace containing $p[\eta]$ is the span of $p[\eta]$ and $p[\bar{\eta}]$; so $\Omega E_{12}(1,1,1,1)$ is an analytic set of pure dimension four. Finally $\mathcal{M}$ is a closed subset of the same dimension, and it is irreducible, as its regular locus contains the smooth part $\mathcal{M}_0$ of Section 2.3, which contains the dense, connected orbit; so it is one of the irreducible components of $\Omega E_{12}(1,1,1,1)$. □

Question 10.4. Is $\Omega E_{12}(1,1,1,1)$ irreducible, so that it is the orbit closure of the unfolding of $(1,3,8)/12$? If not, is that orbit closure a connected component of it?

10.3. **Proof of Theorem 1.1.** *The listed triangles have rank one.* The triangles $(1,k,k+1)/(2k+2)$ and $(2,2k+1,2k+3)/(4k+6)$ are right triangles with smallest angle $\pi/(2k+2)$, respectively $\pi/(2k+3)$; they are lattice triangles by Veech [35]. The triangles $(1,2,2k-3)/2k$, $k \geq 4$, were announced as lattice triangles by Vorobets [36, Theorem 5] and

proved to be in [37, Proposition 4.2 and Theorem 4.4]. The triangle $(1,4,7)/12$ is a lattice triangle by Hooper [16, Theorem 1]. Among acute triangles, $(3,4,5)/12$ was announced by Veech [35] and proved to be a lattice triangle by Vorobets, who also treats $(3,5,7)/15$ [37, Theorem 4.4]; $(2,3,4)/9$ is a lattice triangle by Kenyon and Smillie [24, Theorem 25]. By Lemma 2.3 a lattice triangle has rank one. For $k=2$ the first family gives $(1,2,3)/6$, whose unfolding is a flat torus, outside the setting of Lemma 2.3 as it has no zeros; its orbit is its whole stratum, of dimension two, so its rank is one. The triangles $(1,3,6)/10$ and $(1,3,8)/12$ have rank one and are not lattice triangles by Proposition 10.2.

The other scalene triangles have rank at least two. A scalene triangle is right, obtuse or acute. A right one is $(a,n-a,n)/2n$ as in Section 10.1; if $a \notin \{1,n-1,n-2\}$ it has rank at least two by Proposition 10.1, and otherwise it is in one of the two right families of the list. An obtuse one other than $(1,1,N-2)/N$, $(1,2,2k-3)/2k$ with $k \geq 4$, $(1,4,7)/12$, $(1,3,6)/10$ and $(1,3,8)/12$ has rank at least two by Theorem 8.1; the first of these is isosceles, and $(1,2,2k-3)/2k$ is obtuse if and only if $k \geq 4$. An acute scalene triangle of rank one is $(2,3,4)/9$, $(3,4,5)/12$ or $(3,5,7)/15$ by Theorem 9.19.

So the scalene triangles of rank one are exactly those of Theorem 1.1, and all of them but $(1,3,6)/10$ and $(1,3,8)/12$ are lattice triangles. □

Proof of Corollary 1.3. A right triangle is a lattice triangle if and only if its smallest angle is π/m with $m \geq 4$, and an acute isosceles triangle if and only if its apex angle is π/m with $m \geq 3$ [24, Theorem 2]. The obtuse lattice triangles are $(1,1,N-2)/N$ with $N \geq 5$, $(1,2,2k-3)/2k$ with $k \geq 4$, and $(1,4,7)/12$ [17, Theorem 1.1]. A lattice triangle has rank one, so by Theorem 1.1 the acute scalene lattice triangles are $(2,3,4)/9$, $(3,4,5)/12$ and $(3,5,7)/15$. In McMullen's notation a type lists integers proportional to the angles. In his type A, $(1,1,1)$ and $(2,m,m)$ with $m \geq 2$ are the acute isosceles triangles with apex $\pi/3$ and $\pi/(m+1)$, and $(1,1,m)$ with $m \geq 3$ are the obtuse isosceles ones. Its other members, $(2,1,1)$, $(1,1,2)$, $(2,1,3)$ and $(2,m,m+2)$ with $m \geq 2$, are the right triangles with smallest angle $\pi/4$, $\pi/4$, $\pi/6$ and $\pi/(m+2)$. His types B and D, $(1,2,m)$ with $m \geq 5$ odd and $(1,4,7)$, are the other obtuse ones, and his type C consists of the three acute scalene ones. □

11. Billiards in the triangles of rank one

Corollary 11.1. *Let Δ be one of the triangles of Theorem 1.1 and θ a direction. If one billiard trajectory in Δ that starts in direction θ is periodic, then so is every billiard trajectory in Δ that starts in direction θ and does not hit a vertex.*

For the lattice triangles of Theorem 1.1, all but $(1,3,6)/10$ and $(1,3,8)/12$ (Proposition 10.2), this follows from Veech's Proposition 2.10 [35]. For these two it follows from Proposition 10.2 and Wright's Corollary 1.6 [40], that every surface in a Prym eigenform locus is completely periodic.

Proof. A billiard trajectory in Δ that starts in direction θ and misses the vertices is periodic if and only if the straight line on X that starts at the same point of the copy Δ in direction θ (Section 2.1) is closed, so it is enough to show that if one straight line in direction θ that avoids the points above the vertices is closed, then so is every such line.

If $\Delta = (1,2,3)/6$, then X is a flat torus (Section 2.1), and every straight line parallel to a closed one is closed. Otherwise $\Sigma \neq \emptyset$, and a closed line in direction θ that avoids Σ lies, with the parallel lines near it, in a cylinder in direction θ. By Theorem 1.1, $\operatorname{rank}\mathcal{M} = 1$, that is, $\dim_{\mathbb{C}} p(V) = 2$ for the tangent space V of $\mathcal{M}$ at (X,ω) (Section 2.3). So by Wright's theorem [40, Theorem 1.5], X is the union of the closures of its maximal cylinders in direction

θ, which are bounded by saddle connections between points of Σ (Section 2.6). A line in direction θ that avoids the points above the vertices avoids Σ, so it does not lie on one of these saddle connections; it lies in the interior of one of the cylinders and is closed. □

Declarations

Use of AI. This paper and its results were produced by an AI research system that the author designed, built and directs, running on Anthropic's Claude Fable 5.2 in Claude Code and OpenAI's GPT-6 Astra in Codex. The author chooses the problems and prioritizes the work. Separate AI agents checked each result, and a second program confirmed the computations the proofs use, with any exceptions listed in the public deposit. The system also searched the literature and checked the citations against reliable sources. The author read the paper, reran the computations from the deposit and takes full responsibility for all of the paper's contents.

Data availability. The programs and data used in this paper, with instructions for rerunning them, are deposited at Zenodo [21].

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