TRIPLETS OF MUTUALLY UNBIASED BASES IN DIMENSION SIX

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Abstract. We prove that every triplet of mutually unbiased bases of $\mathbb{C}^6$ is equivalent, by a unitary map and a relabeling of the bases and of their vectors, up to phases, to a triplet from Zauner's construction. This is Conjecture 1 of Matolcsi, Matszangosz, Varga and Weiner, which we call the Budapest conjecture: up to equivalence, each such triplet lies in exactly one of two cases, the two-parameter family described by Szöllősi or one isolated triplet. The four authors' Conjecture 2, which they regard as the main conjecture of their paper, holds as well. Cárdenes Wuttig and Tindall have posted a different computer-assisted proof of the Budapest conjecture. We show without a computer that a triplet comes from the construction exactly when it has a unitary symmetry of order three up to a scalar that fixes every vector of one basis up to a phase and permutes the vectors of the other two up to phases; we call such a symmetry a clock, as Sun does for the Fourier family. To show that every triplet has a clock, we follow it through a finite cover of the complex Hadamard matrices of order six up to equivalence by certified balls, along an acyclic directed graph, to a ball near the Fourier family, where its clock is certified. In addition, we determine which pairs of bases lie in a triplet, and we recover that there are no four mutually unbiased bases in $\mathbb{C}^6$. The proof is computer-assisted and assumes stated hypotheses on the computations (Appendix A).

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1. Introduction

Two orthonormal bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{C}^d$ are *mutually unbiased* if $|\langle b, c\rangle|^2 = 1/d$ for all $b \in \mathcal{B}$ and $c \in \mathcal{C}$. At most $d+1$ bases of $\mathbb{C}^d$ can be pairwise mutually unbiased, and $d+1$ is reached when d is a prime power [16, 55, 1]. Six is the smallest dimension that is not a prime power. Three pairwise mutually unbiased bases of $\mathbb{C}^6$ come from $\mathbb{C}^2 \otimes \mathbb{C}^3$ [26, Lemma 3], and Zauner conjectured that there are no four [56, Vermutung 5.8], [57, 58]; the review of McNulty and Weigert gives the history and the evidence [36]. Two recent computer-assisted proofs, neither of which has appeared in a refereed journal, conclude that there are no four: a preprint of OpenAI [43], whose C++ source, numerical records and a Python program that checks those records are public [41], and our computation by its method, with independently written programs [19].

Matolcsi, Matszangosz, Varga and Weiner proposed a different approach to this bound: to determine all *triplets* of mutually unbiased bases of $\mathbb{C}^6$ [29, Section 4]. After a unitary change of coordinates a triplet consists of the standard basis and the columns of $H_1/\sqrt{6}$ and $H_2/\sqrt{6}$, where H_1 and H_2 are *complex Hadamard matrices*, matrices with entries of modulus one and $HH^* = 6I$, and $H_1^* H_2/\sqrt{6}$ is complex Hadamard as well. The construction they recall [29, Section 4] is due to Zauner [56, Proposition 5.6], [57, Section 3.3]. Szöllősi restated it [47, Proposition 4.2] and used it to construct a two-parameter family of triplets [47, Theorem 4.3]. The construction takes H_1 and H_2 in the two-parameter Fourier family. Write $\omega = \mathrm{e}^{2\pi i/3}$ and $\mathbb{T}$ for the unit circle, and let

$$F^{\mathsf{T}}(x,y) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega & x & \omega^2 x & \omega x \\ 1 & \omega & \omega^2 & y & \omega y & \omega^2 y \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & \omega^2 & \omega & -x & -\omega^2 x & -\omega x \\ 1 & \omega & \omega^2 & -y & -\omega y & -\omega^2 y \end{pmatrix} \qquad (x, y \in \mathbb{T}), \tag{1.1}$$

the matrix displayed in [29, (4.1)]. We call the family of the matrices $F^{\mathsf{T}}(x,y)$ the *Fourier family*, as [18, Section 1] does; [29, Section 4] calls it the transposed Fourier family and its transposes $F(x,y)$ the Fourier family. The construction [29, (4.4)], which we read with the matrix (1.1) (Remark 2.5), takes

$$H_1 = D_1 F^{\mathsf{T}}(x_1, y_1), \qquad H_2 = D_2 F^{\mathsf{T}}(x_2, y_2), \tag{1.2}$$

with D_1, D_2 diagonal unitary, such that $H_1^* H_2/\sqrt{6}$ is complex Hadamard. It is then 2-circulant, a 2×2 array of circulant 3×3 blocks (Theorem 2.4), and every 2-circulant complex Hadamard matrix is equivalent to one arising in this way [47, Proposition 4.2], after Zauner [56, Proposition 5.6]. Two triplets are *permutationally unitarily equivalent* if a unitary map of $\mathbb{C}^6$ carries one onto the other, up to the order of the three bases and the order and phases of the vectors in each basis. In these terms, Conjecture 1 of Matolcsi, Matszangosz, Varga and Weiner, which we call the Budapest conjecture, begins as follows.

Conjecture 1.1 ([29, Conjecture 1])**.** *Every triplet of mutually unbiased bases of* $\mathbb{C}^6$ *is permutationally unitarily equivalent to a triplet obtained by the construction* (1.2)*.*

The Budapest conjecture continues with an "in particular" sentence that describes the triplets of the construction in two cases, a two-parameter family and an isolated triplet. Matolcsi, Matszangosz, Varga and Weiner observe that the Budapest conjecture implies that there are no four mutually unbiased bases in $\mathbb{C}^6$, through the theorem of Jaming, Matolcsi,

Móra, Szöllősi and Weiner that no pair $\{I, F^{\mathsf{T}}(x,y)/\sqrt{6}\}$ lies in four mutually unbiased bases [18, Theorem 1.4]. The four authors' evidence is numerical: each of 20,000 triplets found by optimization is, up to numerical tolerance, equivalent to one of the construction [29, Empirical observation 4.1]. Earlier work settles special cases. We write $F_6 = F^{\mathsf{T}}(1,1)$, a row and column permutation of the Fourier matrix $(\mathrm{e}^{2\pi ijk/6})_{j,k}$. The pair $\{I, F_6/\sqrt{6}\}$ lies in exactly 16 triplets [12, Theorem 2], and the pair formed by the standard basis and Tao's matrix S_6 [50] lies in none [3, Section 4.1]. McNulty and Weigert showed that every triplet of product bases is equivalent to one of two [34, Theorem 3] and that no vector is unbiased to the three bases of such a triplet [34, Theorem 4], [35, Theorem 1]; the two lie in different cases of the "in particular" sentence (Section 9.2).

Theorem 1.2 (computer-assisted)**.** *Every triplet of mutually unbiased bases of* $\mathbb{C}^6$ *is permutationally unitarily equivalent to a triplet*

$$\Big(I,\ \tfrac{1}{\sqrt{6}}D_1F^{\mathsf{T}}(x_1,y_1),\ \tfrac{1}{\sqrt{6}}D_2F^{\mathsf{T}}(x_2,y_2)\Big)$$

with D_1, D_2 *diagonal unitary and* $x_1, y_1, x_2, y_2 \in \mathbb{T}$*. That is, Conjecture 1.1 holds.*

Like every statement marked computer-assisted, Theorem 1.2 assumes the hypotheses of Appendix A.1.

The proof rests on a reformulation that does not involve the parameters. A *clock* of a triplet is a unitary map U of $\mathbb{C}^6$, not a scalar multiple of the identity, with U^3 scalar, that fixes every vector of one of the three bases up to a phase and maps every vector of each of the other two bases to a multiple of a vector of the same basis. The name follows Sun, whose clock of the Fourier pairs [46, (6.1)] is, in other coordinates, the matrix Z below, and who studies the triplets whose third basis it preserves [46, Section 8].

Theorem 1.3 (the clock criterion)**.** *A triplet of mutually unbiased bases of* $\mathbb{C}^6$ *is permutationally unitarily equivalent to a triplet of the construction* (1.2) *if and only if it has a clock.*

Theorem 1.3 answers, for triplets, the question of Matszangosz and Szöllősi how to decide whether a triplet comes from Zauner's construction [32, Section 1]; at the level of transition matrices, they characterize the Fourier family and the family of 2-circulant matrices through entries equal to -1 [32, Theorem 1.1]. The proof of Theorem 1.3 in Section 2 uses no computer. Every triplet of the construction has the clock

$$Z = \operatorname{diag}(1, \omega, \omega^2, 1, \omega, \omega^2),$$

a symmetry of every pair $\{I, F^{\mathsf{T}}(x,y)/\sqrt{6}\}$ [18, Section 4.3], and conversely a clock fixing the standard basis forces both transition matrices from it into the form (1.1) up to equivalence. The clock also explains why the Fourier pairs extend to triplets at all: together with an antiunitary map J with $J^2 = -I$ that preserves the two bases, it completes every vector v unbiased to the standard basis and to the columns of $F^{\mathsf{T}}(x,y)/\sqrt{6}$ to a third basis, the six vectors $Z^a v, Z^a J v$ $(a = 0, 1, 2)$ (Proposition 2.6; this completion is due to Sun [46, Theorem 6.1], and we include a short proof). Such a vector v exists for every pair [46, Corollary 6.2], by a theorem of Idel and Wolf [14, Theorem 2], so every Fourier pair lies in a triplet, as the searches of [18, Numerical Evidence 1.3] had found. A computer-assisted argument (Theorem 2.8) shows that every third basis of a Fourier pair is one of these completions $\{Z^a v, Z^a J v\}$, except at the twelve points at which the pair is equivalent to $\{I, F_6/\sqrt{6}\}$. Theorem 1.2 is therefore the statement that every triplet has a clock, and the rest of the paper proves it.

By Theorem 1.2, the "in particular" sentence holds as well, up to equivalence (Remark 9.11): its two cases are exhaustive and exclusive, the class of the exceptional triplet of [29, Section 4] is an isolated point of the space of classes, and the triplets of the first case are parametrized up to equivalence, continuously, by Szöllősi's parameter and include a disc of dimension two of pairwise inequivalent triplets (Theorem 9.8). For every triplet with a clock, which includes the triplets of the construction, Proposition 9.12 proves the four authors' Conjecture 2, on the Hadamard cubes of the triplets, which they regard as the main conjecture of their paper [29, Section 4], and the identity [29, (4.9)]; by Theorem 1.2, both hold for every triplet of mutually unbiased bases of $\mathbb{C}^6$.

No four mutually unbiased bases.

Corollary 1.4. *There are no four mutually unbiased bases in* $\mathbb{C}^6$.

Proof. Four such bases would contain a triplet, and by Theorem 1.2 and Lemma 3.1 (d) one of its transition matrices, $H = \sqrt{6}\,U^*V$ for two of its bases U and V, is $\cong$ some $F^{\mathsf{T}}(x,y)$. With $x = \mathrm{e}^{2\pi ia}$ and $y = \mathrm{e}^{2\pi ib}$, $\sqrt{6}$ times the matrix $F(a,b)$ of [18, (3)] is a row and column permutation of $F^{\mathsf{T}}(x,y)$ (the proof of the counts after Proposition 2.10), so $H = P_1D_1(\sqrt{6}\,F(a,b))D_2P_2$ with permutation matrices P_i and diagonal unitary matrices D_i. The unitary map $(P_1D_1)^{-1}U^*$ carries the lines of U onto those of the standard basis and the lines of V onto those of the columns of $F(a,b)$. As unbiasedness depends only on the lines, the standard basis, the columns of $F(a,b)$ and the images of the other two bases are four mutually unbiased bases, contradicting [18, Theorem 1.4], which is computer-assisted. □

As in the work of Cárdenes Wuttig and Tindall [5], described below, Corollary 1.4 reaches Zauner's bound through the triplets; it also gives [29, Conjecture 3], that no pair $\{I, S/\sqrt{6}\}$ with S in Szöllősi's family $X(\alpha)$ lies in four mutually unbiased bases, and so do the conclusions of [43, Theorem 1.1], [19, Theorem 1.1] and a second argument of [5], which exclude a fourth basis for every pair of bases at once. By an improvement of Delsarte's linear programming bound, Matolcsi and Weiner show that no pair $\{I, F^{\mathsf{T}}(x,y)/\sqrt{6}\}$ lies in a complete set of seven mutually unbiased bases [31, Theorem 4.1].

Pairs that lie in a triplet. Let $\mathrm{circ}(a,b,c)$ denote the circulant matrix with the rows (a,b,c), (c,a,b) and (b,c,a).

Corollary 1.5. *Let H be a complex Hadamard matrix of order six. The pair $\{I, H/\sqrt{6}\}$ lies in a triplet of mutually unbiased bases if and only if permutations of the rows and of the columns of H and multiplication of its rows and columns by numbers of modulus one carry H to a matrix $F^{\mathsf{T}}(x,y)$ or to its transpose $F(x,y)$ for some $x, y \in \mathbb{T}$, or to*

$$\begin{pmatrix} \mathrm{circ}(1, \bar{x}, \overline{xy}) & \mathrm{circ}(1, \overline{x'}, \overline{x'y'}) \\ \mathrm{circ}(1, \overline{x'}, \overline{x'y'})^* & -\mathrm{circ}(1, \bar{x}, \overline{xy})^* \end{pmatrix} \tag{1.3}$$

for some $x, y, x', y' \in \mathbb{T}$ with

$$x + y + \overline{xy} + x' + y' + \overline{x'y'} = 0. \tag{1.4}$$

The "if" direction holds without Theorem 1.2.

The matrices (1.3) are those of Szöllősi's form [47, (1) and (5)] with circulant blocks whose first entries are 1, and (1.4) is his condition [47, (7)] for them to be complex Hadamard. In the notation of [36] (Section 9.1), Corollary 1.5 is the conjecture that McNulty and Weigert base on the numerical evidence of [29] [36, Conjecture 8.1]; with Theorem 9.8 it answers [36,

Problem 10.5] up to permutational unitary equivalence: for complex Hadamard matrices H_1, H_2 with $H_1^* H_2/\sqrt{6}$ complex Hadamard, the triplet $(I, H_1/\sqrt{6}, H_2/\sqrt{6})$ is permutationally unitarily equivalent to the exceptional triplet of [29, Section 4] or to one of Szöllősi's triplets [47, Proposition 4.2 and Theorem 4.3], and the matrices H_1, H_2, $H_1^* H_2/\sqrt{6}$ that occur are those of Corollary 1.5. With a computed lower bound on the Haagerup set of one of its members, Corollary 1.5 implies that the two-dimensional regions of Karlsson's family $K_6^{(2)}$ in which Goyeneche found triplets numerically [11, Fig. 1] consist, up to a set of measure zero, of approximate triplets (Remark 9.3).

The idea of the proof. Fix a triplet and make one of its bases standard, so that it reads $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ with H and B complex Hadamard. Its *transition matrices*, $\sqrt{6}$ times U^*V for two of its bases U and V written as unitary matrices, are H, $H^*B/\sqrt{6}$ and B^* in cyclic order, and moving the standard basis to $B/\sqrt{6}$ replaces H by B^*. Every complex Hadamard matrix of order six may a priori be a transition matrix, and we use no explicit description of these matrices, such as the classification of [4], which leaves the joint conditions on transition matrices sharing a basis [4, Section 6]. Our computation excluding four mutually unbiased bases [19], which carries out the method of [43], built on the discretization of [18, Section 3] and [17, Sections 2–4], provides finitely many *records*, certified balls in phase coordinates whose members cover every complex Hadamard matrix of order six up to equivalence [19, Sections 4.2 and 4.3]. We extend its cover to 8,630 records of the common radius 0.016 cycles (Proposition 4.1) and run on every record its exhaustive search for third bases of the pairs $\{I, H/\sqrt{6}\}$, H a member. On 7,065 records it finds none, so no member of them is a transition matrix of a triplet; the other 1,565 form the list $\mathcal{L}^+$ (Proposition 4.3) and are of three kinds.

(a) The 978 records whose centers are close to 2-circulant matrices: the transition B^* of a triplet through such a record is expected near the Fourier family, and certified distances give a directed graph without cycles (Theorem 5.1) on these and the 48 other records with centers within 0.05 of the point D of (b).

(b) Thirty records near the point D of the Fourier family where it meets Szöllősi's family $X(\alpha)$ [47], sorted into three *tiers*: every triplet through a member has a clock, or its transition B^* is close to F_6 and so lies in no record of (a) and in none of the same or a later tier (Theorem 7.3).

(c) The remaining 557 records, near the Fourier family (Remark 4.4), are *closed*: every triplet through a member has a clock (Sections 6 and 8).

A descent lemma (Lemma 3.2), a minimal-counterexample argument on a directed graph without cycles whose vertices are the 1,008 records of the first two kinds, turns these three facts into the theorem: as one follows the transition matrices around a triplet, each step either finds a clock or moves along an edge of this graph, so the descent ends at a clock.

The closures of the third kind rest on a rigidity of the clock (Proposition 6.2): near a triplet of the Fourier family that has the clock Z and at which the derivative of the triplet equations is injective on the deformations that break Z (rephasings aside), every triplet in an explicit neighborhood, its tube (Section 6.2), has the clock Z. On a record of the third kind, Stage C of the search (Section 4.3) produces candidate sets of six lines for every third basis of every member, and each candidate is excluded or matched to a branch of lines, certified by a Krawczyk test [27], that lies in such a tube (Theorem 6.12). At F_6, where the Fourier family meets the family of its transposes, a certified chart in which both families are coordinate planes replaces the tube (Section 8.2).

While this paper was being completed, Cárdenes Wuttig and Tindall [5] posted a different computer-assisted classification of the triplets, by which every triplet is equivalent to a member of a two-parameter family or to an isolated triplet; as they state, it proves Conjecture 1.1. The triplets of their family [5, (31)] have a clock, their symmetry [5, (32)], and their isolated triplet [5, (30)] is equivalent to the exceptional triplet of [29, Section 4], as they state [5, p. 1] (Section 9.2); so their classification also determines the triplets of [36, Problem 10.5] up to permutational unitary equivalence. They also state, from their classification, that every triplet is strongly unextendible in the sense of [36, Definition 6.4] and that this resolves [36, Problem 10.3] [5, Section IV.A]. Their proof rests on the connectedness, which they formalize in Lean, of the space of equivalence classes of complex Hadamard matrices of order six other than Tao's; it covers this space by balls in four phase coordinates and a chart at F_6, and certifies by computer that every triplet has an order-three symmetry, which up to a permutation of the coordinates is the clock Z. Our proof covers the complex Hadamard matrices of order six by the records of [19], extended in Proposition 4.1, in the phase coordinates of [43], without their connectedness theorem; both proofs use the local tests of [43].

The computation. The computer-assisted steps use Arb's ball arithmetic [22], exact arithmetic and, in some steps, binary64 arithmetic. The computations for the exclusions of Proposition 4.3 and for Propositions 6.13 and 8.10 and Theorem 7.3 are checks: an independent program decides a finite list that a search proposed, so a wrong proposal can only make a check fail; Appendix A.2 names the computations that have no independent check. The proof assumes that the computers and their software worked as specified and that the deposit [21] holds the programs of the runs and the outputs that the proofs read, together with three hypotheses on the computed values (Appendix A.1). Appendix C restates the statements of [19] that we use, and Lemma 6.8 proves what we use of its Stage C; where [19] uses a statement of [43], it proves it or a replacement, and we use neither the programs [41] nor the outputs of [43].

Plan of the paper. Section 2 proves the clock criterion and the completion of Fourier pairs. Section 3 proves the descent lemma, without a computer, for any finite cover by sets of matrices. Section 4 states the cover and the list, Sections 5, 6, 7 and 8 treat the three kinds of records (Section 7 uses the tools of Section 6), Section 9 assembles the proof of Theorem 1.2 and proves Corollary 1.5, and Appendix A describes the computations. The argument can be followed in Sections 2 and 3, Sections 6.1 and 6.2, the openings of Sections 5, 7 and 8 and the proof of Theorem 1.2; the rest of Sections 4, 5, 6, 7 and 8 states the finite checks and proves the lemmas that make them sufficient.

2. Triplets, clocks and the construction

This section proves the clock criterion (Theorem 2.4) and the completion of Fourier pairs (Proposition 2.6), both without a computer. Section 2.4 states what is known about all third bases of a Fourier pair; of this, the proof of Theorem 1.2 uses only part (b) of Theorem 2.8, proved by an exact computation, for the records near F_6 (Section 8).

2.1. **Conventions.** A basis of $\mathbb{C}^6$ is an orthonormal basis, written as the unitary matrix whose columns are its vectors. Its *lines* are its vectors up to a factor of modulus one. A *triplet* is an unordered set of three pairwise mutually unbiased bases. If one basis of a triplet is the standard basis, the other two are the columns of $H/\sqrt{6}$ and $B/\sqrt{6}$ for complex Hadamard matrices H and B, and we write the triplet $\{I, H/\sqrt{6}, B/\sqrt{6}\}$. For two complex

Hadamard matrices H and B, the columns of $B/\sqrt{6}$ form a *third basis* of the pair $\{I, H/\sqrt{6}\}$ if they are unbiased to the columns of $H/\sqrt{6}$, that is, if $H^*B/\sqrt{6}$ is complex Hadamard as well. Let $\mathcal{H}$ be the set of complex Hadamard matrices of order six.

Throughout, $\omega = \mathrm{e}^{2\pi i/3}$,

$$Z = \operatorname{diag}(1, \omega, \omega^2, 1, \omega, \omega^2), \tag{2.1}$$

$F_3 = (\omega^{jk})_{0\le j,k\le 2}$, $F_2 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, and $F^{\mathsf{T}}(x, y)$ is the matrix (1.1). In blocks,

$$F^{\mathsf{T}}(x, y) = \begin{pmatrix} \overline{F_3} & E\,\overline{F_3} \\ \overline{F_3} & -E\,\overline{F_3} \end{pmatrix}, \qquad E = \operatorname{diag}(1, x, y). \tag{2.2}$$

Multiplying row j of $\overline{F_3}$ by ω^j replaces its column k by its column $k-1$, so

$$Z\,F^{\mathsf{T}}(x, y) = F^{\mathsf{T}}(x, y)\,(I_2 \otimes S), \tag{2.3}$$

where S is the permutation matrix with $Se_k = e_{k-1}$, the cyclic shift $k \mapsto k-1$ of $\{0, 1, 2\}$. This identity is [18, Section 4.3], where Z is the matrix U^2 and (2.3) reads $U^2 f_k = f_{k+2}$ for the columns f_k of their matrix $F(a, b)$.

2.2. **Clocks.** A unitary map U is a *symmetry* of a set of bases if it maps every vector of each basis to a multiple of a vector of the same basis. It then permutes the lines of each basis. A *clock* of a triplet is a symmetry U of the triplet that is not a scalar multiple of the identity, has U^3 scalar and fixes every line of one of the three bases. Multiplying a clock by a scalar of modulus one gives a clock, so a clock may be normalized to $U^3 = I$.

Lemma 2.1. *Let $\mathcal{B}$ and $\mathcal{C}$ be mutually unbiased bases and U a unitary map that fixes every line of $\mathcal{B}$ and one line of $\mathcal{C}$. Then U is a scalar multiple of the identity.*

If U fixes every line of $\mathcal{C}$ as well, the lemma is the injectivity statement of [18, Theorem 4.4].

Proof. Write $U = \sum_{b\in\mathcal{B}} \mu_b\, bb^*$ and $Uc = \nu c$ for the fixed line c. Then $\mu_b\langle b, c\rangle = \langle U^*b, c\rangle = \langle b, Uc\rangle = \nu\langle b, c\rangle$, and $\langle b, c\rangle \neq 0$, so $\mu_b = \nu$ for every b. □

So a clock with $U^3 = I$ that fixes the lines of one basis permutes the lines of each of the other two bases without fixed points, in two cycles of length three.

Lemma 2.2. *Let U be unitary with $U^3 = I$, and let $\mathcal{C}$ be a basis of $\mathbb{C}^6$ whose lines U permutes in two cycles of length three. Then the vectors of $\mathcal{C}$ can be multiplied by scalars of modulus one so that $\mathcal{C} = \{U^a c_1, U^a c_2 : a = 0, 1, 2\}$, and each of $1, \omega, \omega^2$ is an eigenvalue of U of multiplicity two.*

Proof. If c is a vector of $\mathcal{C}$ in one cycle, the vectors c, Uc, U^2c represent the three lines of the cycle, and $U(U^2c) = c$. On their span U is the cyclic permutation of this orthonormal triple, with eigenvalues $1, \omega, \omega^2$. The spans of the two cycles are orthogonal and together make up $\mathbb{C}^6$, so each eigenvalue has multiplicity two. □

Proposition 2.3. *Let $H \in \mathcal{H}$, and let U be a unitary map with $U^3 = I$ that fixes every line of the standard basis and permutes the lines of the columns of H in two cycles of length three. Then there are a permutation matrix P, diagonal unitary matrices Δ and Δ', a permutation matrix Q and $x, y \in \mathbb{T}$ such that*

$$PH = \Delta\, F^{\mathsf{T}}(x, y)\, \Delta' Q \qquad \text{and} \qquad PUP^{-1} = Z^2.$$

Moreover P may be chosen as any permutation matrix with $PUP^{-1} = Z^2$.

Proof. Since U fixes the lines of the standard basis, $U = \operatorname{diag}(\lambda_0, \dots, \lambda_5)$ with $\lambda_i^3 = 1$. By Lemma 2.2 applied to the columns of H, each cube root of unity occurs twice among the λ_i, and after the columns are permuted and multiplied by scalars of modulus one they read p, Up, U^2p, q, Uq, U^2q with $p, q \in \mathbb{T}^6$. Row i of H is then $(p_i, \lambda_i p_i, \lambda_i^2 p_i, q_i, \lambda_i q_i, \lambda_i^2 q_i)$. The columns p and $U^k q$ are orthogonal for $k = 0, 1, 2$, that is $\sum_m \omega^{mk} G_m = 0$ with $G_m = \sum_{\lambda_i = \omega^m} \overline{p_i} q_i$, and the Vandermonde matrix of $1, \omega, \omega^2$ is invertible, so $G_m = 0$ for each m. If i_m and i'_m are the two indices with $\lambda_i = \omega^m$, this says that $t_m := q_{i_m}/p_{i_m} = -q_{i'_m}/p_{i'_m}$. Divide row i by p_i, put the rows in the order $i_0, i_2, i_1, i'_0, i'_2, i'_1$, and multiply the last three columns by $\overline{t_0}$. The result is $F^{\mathsf{T}}(x, y)$ with $x = t_2/t_0$ and $y = t_1/t_0$, and the eigenvalues of U in the new order of the rows are $1, \omega^2, \omega, 1, \omega^2, \omega$, the diagonal of Z^2. The orthogonality relations not used here hold automatically. Any permutation P with $PUP^{-1} = Z^2$ orders the rows as $i_0, i_2, i_1, i'_0, i'_2, i'_1$ for some choice of the labels i_m, i'_m within each eigenvalue, and the argument holds for every such choice, since exchanging i_m and i'_m replaces t_m by $-t_m$. □

Theorem 2.4 (the clock criterion)**.**

(a) *Let D_1, D_2 be diagonal unitary, $x_1, y_1, x_2, y_2 \in \mathbb{T}$ and $H_r = D_r F^{\mathsf{T}}(x_r, y_r)$. If $H_1^* H_2/\sqrt{6}$ is complex Hadamard, then Z is a clock of the triplet $(I, H_1/\sqrt{6}, H_2/\sqrt{6})$, and $H_1^* H_2/\sqrt{6}$ is 2-circulant.*

(b) *A triplet with a clock is permutationally unitarily equivalent to such a triplet, with the clock's fixed basis as the standard basis.*

Hence a triplet is permutationally unitarily equivalent to a triplet of the construction (1.2) if and only if it has a clock, and Conjecture 1.1 holds if and only if every triplet of mutually unbiased bases of $\mathbb{C}^6$ has a clock.

Proof. (a) Z is diagonal, so it fixes the lines of the standard basis; $Z^3 = I$ and Z is not scalar. Since Z commutes with D_r, (2.3) gives $ZH_r = H_r(I_2 \otimes S)$, so Z permutes the columns of each H_r. The matrix $H_1^* H_2/\sqrt{6} = \sqrt{6}\, H_1^{-1} H_2$ therefore commutes with $I_2 \otimes S$, which says that each of its four 3×3 blocks commutes with S, that is, is circulant, as in [47, Proposition 4.2] and [56, Proposition 5.6].

(b) Let U be a clock fixing the lines of the basis B_0, normalized to $U^3 = I$. The unitary B_0^* maps the triplet to $\{I, H_1/\sqrt{6}, H_2/\sqrt{6}\}$ with $H_1, H_2 \in \mathcal{H}$, and maps U to $U' = B_0^* U B_0$, a symmetry of the new triplet that fixes the lines of the standard basis. By Lemma 2.1, U' permutes the lines of the columns of H_1 and of H_2 in two cycles of length three. Fix a permutation matrix P with $PU'P^{-1} = Z^2$, possible by Lemma 2.2. By Proposition 2.3, applied to H_1 and to H_2 with this P, $PH_r = \Delta_r F^{\mathsf{T}}(x_r, y_r) \Delta'_r Q_r$ for $r = 1, 2$. The unitary P fixes the set of lines of the standard basis, and right multiplication by $\Delta'_r Q_r$ only reorders and rephases the columns. So the triplet is permutationally unitarily equivalent to $(I, \Delta_1 F^{\mathsf{T}}(x_1, y_1)/\sqrt{6}, \Delta_2 F^{\mathsf{T}}(x_2, y_2)/\sqrt{6})$, whose transition matrix $(\Delta_1 F^{\mathsf{T}}(x_1, y_1))^* \Delta_2 F^{\mathsf{T}}(x_2, y_2)/\sqrt{6}$ is complex Hadamard since the two bases are unbiased.

For the last sentence: a unitary map W carries a clock U of a triplet to the clock WUW^{-1} of the image, and relabeling bases or their vectors does not change which maps are clocks. □

The matrix $U^2 = Z$ of [18, Section 4.3] is a clock of the triplets of [18, Theorem 2.4] with $t \neq \pi/4$: it maps column k of their $F(0, b)$ to column $k + 2$ modulo 6, and column k of $C(t)$ to column $k + 2$ modulo 3 within each half of the columns. For $m = 3$ the matrices E_1 and E_2 of Zauner's triplets [18, Appendix B] are $F^{\mathsf{T}}(x_1, y_1)\Delta_1 Q/\sqrt{6}$ and $D_2 F^{\mathsf{T}}(x_2, y_2)\Delta_2 Q/\sqrt{6}$, with D_2, Δ_r diagonal unitary and Q a permutation matrix. So, by Theorem 2.4, the two

families compared in [18, Remark 2.5] are, up to permutational unitary equivalence, triplets of the same construction (1.2).

Remark 2.5. The second arXiv version and the journal text of [29] write $F_i = D_i F(x_i, y_i)$ with "members $F(x_1, y_1)$ and $F(x_2, y_2)$ of the Fourier family (4.1)", but their (4.1) displays $F^{\mathsf{T}}(x, y)$ and the next sentence calls its transpose $F(x, y)$. The first arXiv version calls the displayed matrix $F(x, y)$ in its construction, numbered (4.3) there, and the factorization of [47, (24) and Proposition 4.2] uses the same matrix. We therefore read (1.2) with F^{T}.

2.3. **The completion of Fourier pairs.** For $x, y \in \mathbb{T}$ and $E = \operatorname{diag}(1, x, y)$ define the antiunitary map

$$J \begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} E\,\overline{r} \\ -E\,\overline{p} \end{pmatrix} \qquad (p, r \in \mathbb{C}^3). \tag{2.4}$$

Then $J^2 = -I$ and $JZ = Z^{-1}J$.

Proposition 2.6 (after [46, Theorem 6.1])**.** *Let $x, y \in \mathbb{T}$ and let v be a unit vector unbiased to the standard basis and to the columns of $F^{\mathsf{T}}(x, y)/\sqrt{6}$. Then the six vectors $Z^a v$ and $Z^a J v$ $(a = 0, 1, 2)$ form a third basis of the pair $\{I, F^{\mathsf{T}}(x, y)/\sqrt{6}\}$, and Z is a clock of the triplet it makes with the pair.*

The first assertion is Sun's prescribed-ray completion [46, Theorem 6.1] for $m = 3$ and $\eta = (1, \overline{x}, \overline{y})$, in the coordinates of (1.1); the clock assertion is not part of Sun's statement, and follows from [46, (6.2) and (6.3)]. That v, Zv and $Z^2 v$ are orthonormal is [18, Corollary 4.7]. We include the short proof in these coordinates, since the clock Z and the map J are used again in Section 6.

Proof. Both Z and J map the lines of the standard basis to lines of the standard basis. By (2.3) Z permutes the columns of $F^{\mathsf{T}}(x, y)$. Write the columns of (2.2) as $c_k = (g_k; g_k)$ and $c'_k = (Eg_k; -Eg_k)$, where g_k is column k of $\overline{F_3}$, so that $\overline{g_k} = g_{-k}$. Then $Jc_k = c'_{-k}$ and $Jc'_k = -c_{-k}$, so J permutes the column lines as well. Unitary and antiunitary maps preserve $|\langle u, w\rangle|$, so the six vectors are unbiased to both bases. Every coordinate of v and of Jv has modulus $1/\sqrt{6}$, so $\langle Z^a v, Z^b v\rangle$ and $\langle Z^a Jv, Z^b Jv\rangle$ equal $\frac{1}{3}\sum_{j=0}^{2} \omega^{(b-a)j} = \delta_{ab}$. For $v = (p; r)$,

$$\langle Z^a v, Z^b Jv\rangle = \sum_{j=0}^{2} \omega^{(b-a)j} \left(\overline{p_j}\, E_{jj} \overline{r_j} - \overline{r_j}\, E_{jj} \overline{p_j}\right) = 0.$$

So the six vectors are orthonormal. The basis is invariant under Z, which is therefore a clock of the triplet. □

Remark 2.7. The map J is a quaternionic structure ($J^2 = -I$) that maps each of the two bases to itself up to phases, and the third bases of Proposition 2.6, the bases $E_\eta(g)$ of [46, (6.2)], which we call *Kramers bases*, are exactly the orbits of the group generated by Z and J on the common unbiased lines: the orbit of the line of v consists of the six lines of $Z^a v$ and $Z^a Jv$. The name recalls Kramers' theorem, whose key step is that an antiunitary map with square $-I$, such as J, maps every vector to an orthogonal one. A pair $\{I, F^{\mathsf{T}}(x, y)/\sqrt{6}\}$ has at most 48 common unbiased lines (Theorem 2.8), hence at most eight Kramers bases.

2.4. **The Fourier fiber.** The *Fourier fiber* is the set of the pairs $(F^{\mathsf{T}}(x, y), B)$, $x, y \in \mathbb{T}$, such that $B/\sqrt{6}$ is a third basis of $\{I, F^{\mathsf{T}}(x, y)/\sqrt{6}\}$: the fiber over the Fourier family of the map $(H, B) \mapsto H$ on the set of $(H, B) \in \mathcal{H}^2$ for which $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ is a triplet.

Theorem 2.8 (computer-assisted)**.**

(a) *For $x, y \in \mathbb{T}$ with $(x^2, y^2) \notin \{(1,1), (\omega, \omega^2), (\omega^2, \omega)\}$, the pair $\{I, F^{\mathsf{T}}(x,y)/\sqrt{6}\}$ has at most 48 common unbiased lines, and every third basis of the pair is a Kramers basis $\{Z^a v, Z^a Jv\}$; in particular Z is a clock of the triplet it forms with the pair.*

(b) *At the twelve remaining points, those with $x, y \in \mu_6$ and $xy = \pm 1$, the pair is equivalent to $\{I, F_6/\sqrt{6}\}$ with $F_6 = F^{\mathsf{T}}(1,1)$; it has exactly 48 common unbiased lines and exactly 16 third bases. At $(1,1)$, ten of them are invariant under Z, eight of these being Kramers bases, and the other six are invariant under $T_3 = I_2 \otimes S$, which fixes every column line of $F^{\mathsf{T}}(1,1)$; at the other eleven points the same holds for the images of Z and T_3 under the equivalence.*

(c) *Every triplet of mutually unbiased bases that contains a pair mapped by a unitary or antiunitary map onto $\{I, F^{\mathsf{T}}(x,y)/\sqrt{6}\}$ or $\{I, F(x,y)/\sqrt{6}\}$, for some $x, y \in \mathbb{T}$, has a clock.*

Appendix B proves Theorem 2.8: part (a) by hand and by a computation in ball arithmetic (Proposition B.7), part (b) by an exact computation, and part (c) from (a), (b) and Lemma 3.1. The counts of (b) are known: the 48 lines were counted by Björck and Fröberg and by Haagerup, as unimodular cyclic 6-roots (see [2, Section 6] and [36, Section 7.3 and Theorem 7.6]), and Grassl found the 48 lines and the 16 bases by computer algebra [12, Theorem 2]. That there are 48 lines near $(1,1)$ is [18, Theorem 4.6]; numerically there are 48 at every sampled parameter, with eight third bases at generic ones [18, Numerical Evidence 1.3], [3, Section 4.2 and Table 2].

Of the triplets containing $\{I, (\mathrm{e}^{2\pi ijk/6})_{j,k}/\sqrt{6}\}$ that Bengtsson et al. list [2, Section 6], the two for which they had no explanation, those whose third basis is equivalent to their $\mathbf{F}^{\mathsf{T}}(1/6, 0)$, have Kramers bases as third bases (after the permutation $n \mapsto 3(n \bmod 2) + (n \bmod 3)$ of $\mathbb{Z}/6$, which carries this pair onto $\{I, F^{\mathsf{T}}(1,1)/\sqrt{6}\}$); as these are invariant under Z, Lemma 2.1 and Proposition 2.3 show that these bases are equivalent to members of the Fourier family. For each entry of their list, the deposit [21] records which kind of third basis of Theorem 2.8 (b) it is.

Two complex Hadamard matrices X and Y are *equivalent in the usual sense*, written $X \cong Y$, if $Y = P_1 D_1 X D_2 P_2$ with permutation matrices P_1, P_2 and diagonal unitary matrices D_1, D_2 [29, (2.2)], [48, Definition 2.2]; the equivalence $\sim$ of Section 3.1 also allows transposition and complex conjugation. If the entries of a matrix M all have modulus $\rho > 0$, the numbers $m_{ij} m_{kl} \overline{m_{il} m_{kj}}/\rho^4$ form the *Haagerup set* of M [13]; it does not change under $\cong$ or under transposition [48, (11), Lemma 2.5 and the remark after it]. For a triplet whose bases, in the order (X_1, X_2, X_3), are written as unitary matrices, the transition matrices in cyclic order are $\sqrt{6} X_1^* X_2$, $\sqrt{6} X_2^* X_3$ and $\sqrt{6} X_3^* X_1$. The triplet has *property P* if, of these three, one is equivalent in the usual sense to a complex Hadamard matrix of the form [29, (4.2)], $\left(\begin{smallmatrix} A & B \\ B^* & -A^* \end{smallmatrix}\right)$ with A and B circulant 3×3 matrices (equivalently, by Lemma 9.2 (c), to a matrix (1.3) with (1.4)), another to some $F(x,y)$, and the third to some $F^{\mathsf{T}}(x', y')$. Let $\mathcal{T}_b = (I, F_6/\sqrt{6}, B_b/\sqrt{6})$, $b = 0, \ldots, 15$, be the 16 triplets that contain $\{I, F_6/\sqrt{6}\}$ (Theorem 2.8 (b)), with the bases in this order and numbered as in the deposit [21]. Their transition matrices in cyclic order are F_6, $F_6^* B_b/\sqrt{6}$ and B_b^*.

Proposition 2.9 (computer-assisted)**.**

(a) *Exactly two of the triplets $\mathcal{T}_b$, namely $\mathcal{T}_0$ and $\mathcal{T}_3$, have all three transition matrices $\cong F_6$. $\mathcal{T}_3$ is the complex conjugate of $\mathcal{T}_0$, and $\mathcal{T}_0$ and $\mathcal{T}_3$ are permutationally unitarily equivalent. For both, every entry of the Hadamard cube of* [29, (3.3)] *(that is, of* [29, (4.6)] *evaluated with the bases I, $F_6/\sqrt{6}$, $B_b/\sqrt{6}$ and multiplied by $6^{3/2}$; its entries have modulus one) is a 24th root of unity: $w_{jkl} = (F_6)_{jk}(F_6^* B_b)_{kl}\overline{(B_b)_{jl}}$ satisfies $w_{jkl}^{24} = 6^{12}$.*

(b) *Each of the other 14 has property P. They form two classes under permutational unitary equivalence:*

(i) *$\{\mathcal{T}_1, \mathcal{T}_2\}$: one transition $\cong F_6$; $F_6^* B_b/\sqrt{6}$ is $\cong F^{\mathsf{T}}(x,y)$ exactly at the 24 points $x, y \in \mu_6$, $xy \neq \pm 1$, among them $(e^{i\pi/3}, 1)$ and (ω, ω), and $\cong$ no $F(x,y)$; B_b^* is $\cong F(x,y)$ exactly at the same 24 points and $\cong$ no $F^{\mathsf{T}}(x,y)$; both are $\cong$ matrices (1.3), neither is $\cong F_6$.*

(ii) *$\{\mathcal{T}_4, \dots, \mathcal{T}_{15}\}$: two transitions $\cong F_6$; the third is $\cong$ a matrix (1.3), its Haagerup set has 16 elements, and it is $\cong$ no $F^{\mathsf{T}}(x,y)$ and no $F(x,y)$.*

Allowing antiunitary maps merges none of the three classes. Every $\mathcal{T}_b$ is permutationally unitarily equivalent to its complex conjugate.

(c) *F_6 is $\cong$ no matrix (1.3) with $x, y, x', y' \in \mathbb{T}$ satisfying (1.4); hence it is $\cong$ no complex Hadamard matrix of the form* [29, (4.2)]*. So $\mathcal{T}_0$ and $\mathcal{T}_3$ do not have property P.*

(d) *Property P and "all three transition matrices $\cong F_6$" are invariant under permutational unitary equivalence and under complex conjugation. Hence every triplet of mutually unbiased bases of $\mathbb{C}^6$ one of whose transition matrices is $\cong F_6$ either is permutationally unitarily equivalent to $\mathcal{T}_0$ or has property P, and not both. The triplets permutationally unitarily equivalent to $\mathcal{T}_0$ form one class, closed under complex conjugation.*

Proof. A program in the deposit [21] checks each statement below about explicit matrices in exact arithmetic, over the field $\mathbb{Q}(\zeta_{12}, 12^{1/4})$ of degree 8, which contains the entries of F_6 and of the matrices B_b built from the 16 bases of the proof of Theorem 2.8 (b), and over $\mathbb{Q}(\zeta_{18})$.

Two tests. The Haagerup set of F_6 lies in μ_6, so a matrix whose Haagerup set does not is not $\cong F_6$. The matrix $F^{\mathsf{T}}(x,y)$ has first row and first column equal to 1. If $M = P_1 D_1 F^{\mathsf{T}}(x,y) D_2 P_2$ and (r,c) is the position in M of the entry $(0,0)$ of $F^{\mathsf{T}}(x,y)$, the matrix $N_{ij} = M_{ij}M_{rc}/(M_{ic}M_{rj})$ does not depend on D_1 and D_2, and it is $F^{\mathsf{T}}(x,y)$ with its rows and columns permuted, row r and column c going to 0. The program tries all 36 positions (r,c) and all 5! orders of the other columns and reads x and y off the rows; so it finds every (x,y) with $M \cong F^{\mathsf{T}}(x,y)$ and, applied to M^{T}, every (x,y) with $M \cong F(x,y)$. It checks each match as an identity $D_1 M' D_2 = F^{\mathsf{T}}(x,y)$, with M' a row and column permutation of M and D_1, D_2 diagonal. A match at $(1,1)$ shows $M \cong F_6$; the Haagerup set, or the absence of such a match, shows $M \ncong F_6$.

(a) and (b). The two tests give the statements of (a) and (b) on $\cong F_6$, $F^{\mathsf{T}}(x,y)$ and $F(x,y)$ for the three transition matrices of every $\mathcal{T}_b$, and the program computes the Haagerup sets with 16 elements. The 24 points in (b) are those of Proposition 2.10. Each $\mathcal{T}_b$ is invariant under Z or under T_3 (Theorem 2.8 (b)): the four with $b \leq 3$ under both, $\mathcal{T}_4$, $\mathcal{T}_8$ and $\mathcal{T}_{12}, \dots, \mathcal{T}_{15}$ under Z only, the other six under T_3 only. Z fixes the standard lines and T_3 the column lines of F_6, and each permutes the lines of the other two bases in two cycles of length three; after the monomial changes of basis built from these cycles, the transition matrix between the other two bases, $F_6^* B_b/\sqrt{6}$ for Z and B_b for T_3, is 2-circulant. The program

follows the proof of Lemma 9.2 for this matrix and for its adjoint: where the numbers ε_k there are $\varepsilon_0\omega^{nk}$, it reads off x, y, x', y', checks that monomial matrices carry the matrix to (1.3), and checks (1.4). This happens for every clock of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_4, \dots, \mathcal{T}_{15}$, and for no clock of $\mathcal{T}_0$ and $\mathcal{T}_3$. So each $\mathcal{T}_b$ with $b \neq 0, 3$ has property P: in $\mathcal{T}_1$ and $\mathcal{T}_2$, $F_6^* B_b/\sqrt{6}$ is $\cong$ a matrix (1.3), $B_b^* \cong F(\mathrm{e}^{i\pi/3}, 1)$ and $F_6 = F^{\mathsf{T}}(1,1)$; in the other twelve, the transition matrix in cyclic order that the clock makes 2-circulant is $\cong$ a matrix (1.3) and the other two are $\cong F_6$, which is $F^{\mathsf{T}}(1,1)$ and $F(1,1)$. For $\mathcal{T}_0$ and $\mathcal{T}_3$ the program computes the 216 numbers w_{jkl} and checks $w_{jkl}^{24} = 6^{12}$.

The classes. Let G be the group of unitary maps, modulo scalars, that map the set of the lines of I and of $F_6/\sqrt{6}$ onto itself. Lines of different bases are not orthogonal, so an element of G maps the standard lines onto the standard lines or onto the column lines of F_6; one that maps them onto themselves is monomial and maps $(1, \dots, 1)$, a column of F_6, to a multiple of a column of F_6; the program tests all $6 \cdot 720$ such monomial matrices and finds a group G_1 of 72. As $F_6/\sqrt{6}$ exchanges the two sets of lines, $G = G_1 \sqcup (F_6/\sqrt{6})G_1$ has 144 elements. A permutational unitary equivalence Φ from $\mathcal{T}_b$ to $\mathcal{T}_{b'}$ maps $\{I, F_6/\sqrt{6}\}$ onto two bases X, Y of $\mathcal{T}_{b'}$ with $\sqrt{6}X^*Y \cong F_6$, say $\sqrt{6}X^*Y = P_1 D_1 F_6 D_2 P_2$. The unitary map $W = (P_1 D_1)^{-1} X^*$ carries X to the standard lines and Y to the column lines of F_6, hence $\mathcal{T}_{b'}$ onto some $\mathcal{T}_{b''}$, and $W\Phi \in G$, so b'' lies in the orbit of b under G. So the classes are the connected components of the graph on $\{0, \dots, 15\}$ with the edges $\{b, gb\}$, $g \in G$, and $\{b', b''\}$ for every pair of bases of $\mathcal{T}_{b'}$ whose transition matrix is $\cong F_6$. The program computes each edge as an explicit map and finds the components $\{0, 3\}$, $\{1, 2\}$ and $\{4, \dots, 15\}$. Complex conjugation maps both sets of lines onto themselves and acts on the 16 bases by $0 \leftrightarrow 3$, $4 \leftrightarrow 8$, $6 \leftrightarrow 7$, $9 \leftrightarrow 11$, $12 \leftrightarrow 13$, $14 \leftrightarrow 15$, fixing 1, 2, 5 and 10; so it joins no two components, $\mathcal{T}_3 = \overline{\mathcal{T}_0}$, and every $\mathcal{T}_b$ is permutationally unitarily equivalent to its complex conjugate.

(c). Let S be a matrix (1.3) with $x, y, x', y' \in \mathbb{T}$ and $S \cong F_6$; then the Haagerup set of S lies in μ_6. Numbering rows and columns from 0, its elements with $(i, j, k, l) = (0, 0, 1, 1)$ and $(0, 0, 1, 2)$ are x^2y and xy^2, and those with $(0, 3, 1, 4)$ and $(0, 3, 1, 5)$ are x'^2y' and $x'y'^2$. So $x^3 = (x^2y)^2(xy^2)^{-1} \in \mu_6$, likewise $y^3, x'^3, y'^3 \in \mu_6$, and $x, y, x', y' \in \mu_{18}$. The program runs over all 18^4 quadruples, in integer exponents modulo 18, keeps the 36 with these four elements in μ_6 that satisfy (1.4) (decided in $\mathbb{Q}(\zeta_{18})$), and finds by the search above, run in exponents against F_6, that none of them is $\cong F_6$. By Lemma 9.2 (c) every complex Hadamard matrix of the form [29, (4.2)] is $\cong$ a matrix (1.3) with (1.4).

(d). By Lemma 3.1 (d), the transition matrices in cyclic order of two permutationally unitarily equivalent triplets are $\cong$ each other or $\cong$ each other's adjoints. The adjoint of $\left(\begin{smallmatrix} A & B \\ B^* & -A^* \end{smallmatrix}\right)$ has the same form, with A^* for A; and $F^{\mathsf{T}}(x,y)^* \cong F(\bar{x}, \bar{y})$, $F(x,y)^* \cong F^{\mathsf{T}}(\bar{x}, \bar{y})$ and $F_6^* \cong F_6$, since $\overline{F^{\mathsf{T}}(x,y)}$ is $F^{\mathsf{T}}(\bar{x}, \bar{y})$ with its columns permuted (proof of Lemma B.5). Complex conjugation of a triplet conjugates its transition matrices, and the conjugate of a matrix of each of these kinds is of the same kind. So both properties are invariant.

Now let a triplet have a transition matrix $\sqrt{6}X^*Y = P_1 D_1 F_6 D_2 P_2 \cong F_6$. As above, the unitary map $W = (P_1 D_1)^{-1} X^*$ carries the triplet onto some $\mathcal{T}_b$. By (a)–(c) and the invariance, $\mathcal{T}_b$, and hence the triplet, is permutationally unitarily equivalent to $\mathcal{T}_0$ or has property P, and not both. The triplets permutationally unitarily equivalent to $\mathcal{T}_0$ form one class, which is closed under complex conjugation since $\overline{\mathcal{T}_0} = \mathcal{T}_3$ lies in it. □

Proposition 2.10 (computer-assisted)**.** *At* $(x,y)=(\mathrm{e}^{i\pi/3},1)$ *the pair* $\{I,F^{\mathsf T}(x,y)/\sqrt{6}\}$ *has exactly* 42 *common unbiased lines and lies in exactly* 7 *triplets. Its third bases are the Kramers bases* $\{Z^a v, Z^a Jv\}$*, which partition the* 42 *lines, and* Z *is a clock of each triplet. The same holds, up to a unitary or antiunitary map, at each of the* 24 *points* $x,y\in\mu_6$ *with* $xy\neq\pm1$*, among them* $(x,y)=(1,\mathrm{e}^{i\pi/3})$*, where* $F^{\mathsf T}(x,y)$ *has, in another order, the columns of the matrix* D *of Section 7, and for the pairs formed by the standard basis and the matrix* $F(1/6,0)$ *of* [18, (3)] *and of* [3, Appendix A.2, (35)] *or its transpose.*

At the matrix $F(1/6,0)$ of [18, (3)], Proposition 2.10 gives 42 common unbiased lines and 7 triplets. There the computer-algebra search of Brierley and Weigert reports 48 unbiased vectors and at most 70 triplets [3, Section 4.2 and Table 2]; Durt, Englert, Bengtsson and Życzkowski list both numbers from that search [10, Section 5.8 and Table 3], and 70 is given as the number of third bases at this point in [18, Numerical Evidence 1.3], which credits Brierley and Weigert in its footnote 2, and in [36, Section 7.3]. Crooks's numerical recount at the matrix of [3, Appendix A.2, (35)] finds 42 distinct unbiased vectors, 48 when six pairs of merging roots are counted twice, and 7 bases [6, Results 3 and 4]. This answers the conjecture of [18, Section 3] that the number is 48 at every point of the family: it is at most 48 (Theorem 2.8) and equals 42 at the 24 points of Proposition 2.10.

Proof of the counts. At $(x,y)=(\mathrm{e}^{i\pi/3},1)$ we have $(x^2,y^2)=(\omega,1)$, not an F_6-type point (these are the points excluded in Theorem 2.8 (a); see Appendix B), and $\eta=(\frac12,\frac12,-\frac12)$ in the notation of Appendix B.1. By Lemma B.1 (d) the number of common unbiased lines is three times the number of real solutions of $\Sigma(\frac12,\frac12,-\frac12)$. This system is solved exactly: its ideal in $\mathbb{Q}[C_0,C_1,C_2,S_0,S_1,S_2]$ is zero-dimensional of degree 92, with 38 distinct complex solutions, of which exactly 14 are real: the characteristic polynomial of multiplication by the linear form of the proof of Theorem 2.8 (b) on the quotient by the radical is squarefree of degree 38 and has 14 real roots. They are $C=0$, $S=\pm(1,-1,1)$, with $N_0=N_1=N_2=\frac32$, and, with $\varepsilon,\varepsilon'\in\{\pm1\}$,

$$C=\varepsilon\tfrac{\sqrt{21}}{14}(1,1,-2),\ S=\varepsilon'\tfrac{\sqrt7}{14}(5,-5,-4);\qquad C=\varepsilon\tfrac{\sqrt{21}}{14}(-1,2,-1),\ S=\varepsilon'\tfrac{\sqrt7}{14}(-5,-4,-5);$$
$$C=\varepsilon\tfrac{\sqrt{21}}{14}(2,-1,-1),\ S=\varepsilon'\tfrac{\sqrt7}{14}(4,5,-5),$$

with $N_0=N_1=N_2=\frac{75}{98}$. So there are 42 common unbiased lines. An exact computation in the field $\mathbb{Q}(\zeta_{12},\sqrt7)$, which contains the ratio vectors of the lines (Appendix B) and in which the orthogonality of two of the lines is decided, finds exactly 159 orthogonal pairs among the 861 pairs of lines and exactly seven six-cliques; these are the third bases. By Proposition 2.6 every Kramers basis is one of them, and by Lemma B.2 the Kramers bases partition the 42 lines, so there are seven of them, and they are the seven third bases. Each is invariant under Z, and Z is a clock of each triplet by Proposition 2.6.

The 24 points $x,y\in\mu_6$ with $xy\neq\pm1$ are the points of $\frac\pi3\mathbb{Z}^2$ modulo $2\pi\mathbb{Z}^2$ outside the lattice Λ of Appendix B. As Λ has index 3 in $\frac\pi3\mathbb{Z}^2$ and $(\frac\pi3,0)\notin\Lambda$, they form the two classes $(\frac\pi3,0)+\Lambda$ and $-(\frac\pi3,0)+\Lambda$. The translations by Λ and the map $(\alpha,\beta)\mapsto(-\alpha,-\beta)$ lie in $\mathcal{W}$, so by Lemma B.5 each of the 24 pairs is carried onto the pair at $(\mathrm{e}^{i\pi/3},1)$ by a unitary or antiunitary map that preserves lines, orthogonality, third bases and orbits of Z, and clocks go along by Lemma 3.1 (a). The matrix $F(a,b)$ of [18, (3)] is $\frac{1}{\sqrt6}$ times the matrix with the entries $\mathrm{e}^{2\pi ijk/6}d^{\,k \bmod 2}_{j \bmod 3}$, $j,k=0,\dots,5$, with $d=(1,\mathrm{e}^{2\pi ia},\mathrm{e}^{2\pi ib})$; since $\mathrm{e}^{2\pi ijk/6}=(-1)^{jk}\omega^{-jk}$, the latter is the matrix $F^{\mathsf T}(\mathrm{e}^{2\pi ia},\mathrm{e}^{2\pi ib})$ with the row and the column $3(j \bmod 2)+(j \bmod 3)$ moved to position j. So $\sqrt6\,F(\frac16,0)$ is a row and column permutation of $F^{\mathsf T}(\mathrm{e}^{i\pi/3},1)$. The

matrix $F(\frac{1}{6},0)$ of [3, (35)], built from the Fourier matrix of [3, (15)] without the factor $\frac{1}{\sqrt{6}}$, is the transpose of $\sqrt{6}\,F(\frac{1}{6},0)$. By Lemma 3.1 (b) a unitary or antiunitary map carries the pair formed by the standard basis and any of these matrices onto $\{I, F^{\mathsf{T}}(e^{i\pi/3},1)/\sqrt{6}\}$. □

3. The reduction to finitely many sets of matrices

This section reduces Theorem 1.2 to statements about finitely many sets of complex Hadamard matrices, without a computer. The sets will be the records of Section 4, but here they are arbitrary: the reduction uses only that they cover $\mathcal{H}$ up to equivalence. Its conclusion is Lemma 3.2, which Section 9 applies.

3.1. **Equivalence and transport.** Let G be the group of maps of 6×6 matrices generated by the permutations of rows, the permutations of columns, transposition and complex conjugation. For $X, Y \in \mathcal{H}$ write $X \sim Y$, and call X and Y *equivalent*, if $Y = D_1 g(X) D_2$ for some $g \in G$ and diagonal unitary D_1, D_2. This is coarser than the usual equivalence of complex Hadamard matrices, which allows only permutations and rephasings, and it is the equivalence of [19, Section 2.2]. A map of $\mathbb{C}^6$ is *antiunitary* if it is the composition of a unitary map with coordinatewise complex conjugation.

Lemma 3.1.

(a) *A unitary or antiunitary map W carries every triplet $\mathcal{T}$ to a triplet $W\mathcal{T}$, and $\mathcal{T}$ has a clock if and only if $W\mathcal{T}$ has one.*

(b) *If $H, H' \in \mathcal{H}$ and $H' \sim H$, a unitary or antiunitary map carries the lines of the standard basis and of the columns of $H/\sqrt{6}$ onto the lines of the standard basis and of the columns of $H'/\sqrt{6}$, possibly exchanging the two bases. In particular $\{I, H/\sqrt{6}\}$ has a third basis if and only if $\{I, H'/\sqrt{6}\}$ has one, and every triplet containing the first pair is carried to a triplet containing the second.*

(c) *For a triplet $\{U_1, U_2, U_3\}$ of bases written as unitary matrices, the unitary map U_1^* carries it to $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ with $H = \sqrt{6}\,U_1^* U_2$ and $B = \sqrt{6}\,U_1^* U_3$ in $\mathcal{H}$.*

(d) *Let a unitary map W carry a triplet with bases X_1, X_2, X_3 (unitary matrices) onto a triplet with bases Y_1, Y_2, Y_3, up to the order of the bases and the order and phases of the vectors: $WX_i = Y_{\sigma(i)} P_i D_i$ with σ a permutation of $\{1,2,3\}$, permutation matrices P_i and diagonal unitary matrices D_i. Then $\sqrt{6}\,Y_{\sigma(i)}^* Y_{\sigma(j)} = P_i D_i (\sqrt{6}\,X_i^* X_j) D_j^* P_j^{\mathsf{T}} \cong \sqrt{6}\,X_i^* X_j$ for $i \neq j$. Hence each matrix $\sqrt{6}\,V^* V'$ for two different bases V, V' of one triplet is $\cong$ such a matrix for the other; the transition matrices in cyclic order of the second triplet are, in some order, $\cong$ those of the first if σ preserves the cyclic order and $\cong$ their adjoints otherwise. If W is antiunitary, the same holds with $X_i^* X_j$ replaced by its entrywise conjugate.*

Proof. (a) Unitary and antiunitary maps preserve $|\langle u, v\rangle|$, hence orthonormality and unbiasedness. If U is a clock of $W\mathcal{T}$, then $W^{-1}UW$ is unitary, it is not scalar, its cube is $W^{-1}U^3W$, a scalar, and it fixes or permutes lines of the bases of $\mathcal{T}$ exactly as U does on their images.

(b) It suffices to treat the generators. For $H' = DPH$ with D diagonal unitary and P a permutation matrix, the unitary DP maps the lines of the standard basis onto themselves and the columns of $H/\sqrt{6}$ onto those of $H'/\sqrt{6}$. For $H' = HPD$ the identity serves, since the columns of H' are those of H reordered and rephased. For $H' = \overline{H}$ the coordinatewise conjugation serves. For $H' = H^{\mathsf{T}}$ the antiunitary map $v \mapsto \overline{H^* v/\sqrt{6}}$ serves: the unitary

$H^*/\sqrt{6}$ maps the columns of $H/\sqrt{6}$ to the standard basis and the standard basis to the columns of $H^*/\sqrt{6}$, whose conjugates are the columns of $H^{\mathsf{T}}/\sqrt{6}$. The rest follows from (a).

(c) $U_1^*U_1 = I$, and $U_1^*U_2$ and $U_1^*U_3$ are unitary with all entries of modulus $1/\sqrt{6}$, since their bases are unbiased to U_1.

(d) P_iD_i is unitary with inverse $D_i^*P_i^{\mathsf{T}}$, so $Y_{\sigma(i)} = WX_iD_i^*P_i^{\mathsf{T}}$ and $Y_{\sigma(i)}^*Y_{\sigma(j)} = P_iD_iX_i^*W^*WX_jD_j^*P_j^{\mathsf{T}} = P_iD_i(X_i^*X_j)D_j^*P_j^{\mathsf{T}}$; the outer factors are products of permutation and diagonal unitary matrices, so the two matrices are $\cong$. The statement is symmetric, as W^{-1} carries the second triplet onto the first. If σ preserves the cyclic order, $(\sigma(i),\sigma(i+1))$ runs over the cyclic pairs $(k,k+1)$; otherwise over the pairs $(k+1,k)$, and $\sqrt{6}\,Y_k^*Y_{k+1} = (\sqrt{6}\,Y_{k+1}^*Y_k)^*$, while $\cong$ commutes with taking adjoints. For antiunitary $W = W_0\circ\kappa$ (κ coordinatewise conjugation, W_0 unitary), $WX_i = W_0\overline{X_i}$ and the computation gives $\overline{X_i}^*\overline{X_j} = \overline{X_i^*X_j}$. □

3.2. **Covers and the descent lemma.** A *cover* is a finite set $\mathcal{R}$ of subsets of $\mathcal{H}$, called records, whose elements are their *members*, such that every $X \in \mathcal{H}$ is equivalent to a member of some record. A matrix $X \in \mathcal{H}$ *lies in* a record R if it is equivalent to a member of R; records may overlap, and X may lie in several. A *list* for $\mathcal{R}$ is a subset $\mathcal{L} \subseteq \mathcal{R}$ that contains every record with a member H for which $\{I, H/\sqrt{6}\}$ has a third basis. A triplet *passes through* a member H of a record, or is a triplet *through* H, if it is $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ for a third basis $B/\sqrt{6}$ of $\{I, H/\sqrt{6}\}$; so no triplet passes through H exactly when $\{I, H/\sqrt{6}\}$ has no third basis.

The idea of the descent is this. A triplet $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ with H a member of a record R is carried by the unitary $B^*/\sqrt{6}$ to the triplet $\{I, B^*/\sqrt{6}, B^*H/6\}$, whose first transition matrix is B^*. If B^* is known to lie only in records to which R points in a finite directed graph without cycles, repeating the step must end, since such a graph has no infinite path. The conditions are universal over the records in which a matrix lies, so overlaps of records do no harm.

Lemma 3.2 (descent)**.** *Let $\mathcal{R}$ be a cover, $\mathcal{L}$ a list for $\mathcal{R}$, $\mathcal{M} \subseteq \mathcal{L}$, and $E \subseteq \mathcal{M}\times\mathcal{M}$ such that the directed graph $(\mathcal{M}, E)$ has no directed cycle (a loop (R,R) counts as a cycle). Suppose that*

(i) *for every $R \in \mathcal{L}\setminus\mathcal{M}$, every member H of R and every third basis $B/\sqrt{6}$ of $\{I, H/\sqrt{6}\}$, the triplet $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ has a clock, and*

(ii) *for every $R \in \mathcal{M}$, every member H of R and every third basis $B/\sqrt{6}$ of $\{I, H/\sqrt{6}\}$, the triplet $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ has a clock or $(R,R') \in E$ for every $R' \in \mathcal{M}$ in which B^* lies.*

Then every triplet of mutually unbiased bases of $\mathbb{C}^6$ has a clock.

Proof. Every triplet is carried by a unitary map onto some $\{I, W/\sqrt{6}, C/\sqrt{6}\}$ (Lemma 3.1 (c)), and then, since $\mathcal{R}$ is a cover, by a unitary or antiunitary map onto a triplet $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ with H a member of a record R (Lemma 3.1 (b)); as H has a third basis, $R \in \mathcal{L}$. Suppose some triplet has no clock, and call a record *bad* if a unitary or antiunitary map carries a triplet without a clock onto a triplet through one of its members. Then some record is bad. A bad record lies in $\mathcal{L}$, since a triplet passes through one of its members, and so in $\mathcal{M}$ by (i) and Lemma 3.1 (a). As $(\mathcal{M}, E)$ is finite and has no directed cycle, some bad record R has no edge to a bad record. Let a triplet without a clock be carried onto $\mathcal{T} = \{I, H/\sqrt{6}, B/\sqrt{6}\}$ with H a member of R. By Lemma 3.1 (a), $\mathcal{T}$ has no clock, so by (ii) $(R,R') \in E$ for every $R' \in \mathcal{M}$ in which B^* lies. The unitary $B^*/\sqrt{6}$ carries $\mathcal{T}$ to $\{I, B^*/\sqrt{6}, B^*H/6\}$, and, as $\mathcal{R}$ is

a cover, $B^* \in \mathcal{H}$ is equivalent to a member H' of some record R', in which B^* therefore lies; by Lemma 3.1 (b) a unitary or antiunitary map carries $\{I, B^*/\sqrt{6}, B^*H/6\}$ onto a triplet through H'. As a composite of unitary and antiunitary maps is unitary or antiunitary, R' is bad, hence $R' \in \mathcal{M}$, and $(R, R') \in E$: an edge from R to a bad record, a contradiction. □

3.3. **The aligned distance.** Theorem 5.1 (a) and Theorem 7.3 say that a matrix lies in *no* record of a set. They are certified through a distance on phase matrices that does not see the equivalences. Phases are measured in cycles: for a real 6×6 matrix A, the *phase matrix*, let $e(A)$ be the matrix with entries $\mathrm{e}^{2\pi i A_{jk}}$. Phases written in an exponent $\mathrm{e}^{i\varphi}$, from Section 6 on, are in radians. Identify real 6×6 matrices with $\mathbb{R}^{36}$, with the Euclidean norm. Let $L \subset \mathbb{R}^{36}$ be the subspace of matrices $(a_j + b_k)_{j,k}$ and Π the orthogonal projection onto $L^\perp$, the double centering; a matrix in $L^\perp$, one whose row and column sums vanish, is *horizontal* [19, Section 2.2]. Define

$$d(A) = \min_{N \in \mathbb{Z}^{36}} \|\Pi(A - N)\|,$$

the distance from A to the closed subgroup $\mathbb{Z}^{36} + L$ (a minimum, since $\Pi(\mathbb{Z}^{36}) \subseteq \frac{1}{36}\mathbb{Z}^{36}$ is discrete). The group G acts on $\mathbb{R}^{36}$: permutations of rows and columns and transposition permute entries, and conjugation acts as $A \mapsto -A$, so that $e(gA) = g(e(A))$. For phase matrices A, B and a fixed phase matrix Y, the *landmark*, put

$$d_G(A, B) = \min_{g \in G} d(gA - B), \qquad f_Y(A) = d_G(A, Y).$$

So $d_G(A, B)$ is the least, over $g \in G$ and $N \in \mathbb{Z}^{36}$, of $\|\Pi(gA - B - N)\|$. For one g and N this norm is an aligned distance in the sense of [43, Lemma 4.15] and [19, Section 2.4], which write it for matrices dephased at row and column 0, a change by an element of L that Π removes. We call d_G the *aligned distance*. A *phase ball* with center P and radius r is the set of matrices $e(P + \delta) \in \mathcal{H}$ with δ horizontal and $\|\delta\| \le r$; the records of Section 4 are phase balls.

Lemma 3.3.

(a) *d_G is symmetric, satisfies the triangle inequality, and $d_G(gA, B) = d_G(A, B)$ for $g \in G$.*
(b) *If $e(A), e(B) \in \mathcal{H}$ and $e(A) \sim e(B)$, then $d_G(A, B) = 0$.*
(c) *If $e(A) \in \mathcal{H}$ lies in a phase ball with center P and radius r, then $d_G(A, P) \le r$.*
(d) *$|f_Y(A) - f_Y(B)| \le d_G(A, B)$ for every landmark Y.*

Proof. Each $g \in G$ is linear, preserves $\mathbb{Z}^{36}$, L and the norm, and commutes with Π; hence $d(gA) = d(A)$, $d(-A) = d(A)$, $d(A + N + t) = d(A)$ for $N \in \mathbb{Z}^{36}$ and $t \in L$, and $d(A + B) \le d(A) + d(B)$. (a) $d(gA - B) = d(A - g^{-1}B) = d(g^{-1}B - A)$ gives symmetry, and $d(hgA - C) \le d(hgA - hB) + d(hB - C) = d(gA - B) + d(hB - C)$ gives the triangle inequality, and invariance holds because G is a group. (b) If $e(B) = D_1 g(e(A)) D_2$, then $e(B) = e(gA + t)$ for the matrix $t \in L$ of the phases of D_1 and D_2, so $B = gA + t + N$ with $N \in \mathbb{Z}^{36}$, and $d(gA - B) = 0$. (c) If $e(A) \sim e(P + \delta)$ with δ horizontal and $\|\delta\| \le r$, then $d_G(A, P + \delta) = 0$ by (b), and $d_G(P + \delta, P) \le d(\delta) \le \|\Pi\delta\| = \|\delta\| \le r$. (d) follows from (a). □

So a matrix $X = e(A)$ does not lie in the phase ball (P, r) as soon as $d_G(A, P) > r$, or as soon as $|f_Y(A) - f_Y(P)| > r$ for some landmark Y. The second test needs only the distances of A and of the centers to a few landmarks, and Section 5 uses both tests, Section 7 the second.

4. Records: the cover and the list

This section gives the cover of Section 3 explicitly: 8,630 records of one radius (Proposition 4.1), and a list $\mathcal{L}^+$ of 1,565 of them outside which no member has a third basis (Proposition 4.3). The second also provides the graphs that Section 5 uses.

4.1. **Records.** We use the records of [19, Section 2.4], which follow [43, Definitions 4.1 and 4.14]. A *Root certificate* consists of a center $P \in \mathbb{R}^{36}$, a matrix $T \in \mathbb{R}^{36\times k}$, its *frame*, whose k columns approximate tangent directions of the set of phase matrices of complex Hadamard matrices at P, and seven constants. It is *certified* if the inequalities of [19, Section 2.4] (Appendix C (U1)) hold, and then every horizontal δ with $e(P+\delta) \in \mathcal{H}$ lies close to the span of the columns of T, by an explicit quadratic bound in $\|\delta\|$. A *record* is a certified Root certificate with a radius r, and its *members* are the matrices $e(P+\delta) \in \mathcal{H}$ with δ horizontal and $\|\delta\| \le r$, the phase ball of Section 3.3. Apart from the records of the earlier computation named in the proof of Proposition 4.1 (atlas, mid, fine at radius 0.02 and their finer children), all records in this paper have the radius

$$\tau = \mathrm{fl}(0.016) = 1152921504606847 \cdot 2^{-56}, \tag{4.1}$$

the binary64 number [15] nearest to 0.016, the fine radius of [43, Section 5.6], which exceeds 0.016 by less than $4 \cdot 10^{-19}$. Every statement below holds for all members at this radius.

4.2. **The cover.**

Proposition 4.1 (computer-assisted)**.** *There are* 8,630 *records of radius* τ *such that every complex Hadamard matrix of order six is equivalent to a member of one of them: the* 5,934 *fine records of* [19]*, followed by* 2,696 *records made here. Of these records,* 8,629 *have* $k = 4$*, and one, about Tao's matrix* S_6 [50]*, has* $k = 0$*.*

We number the records $0, \dots, 8629$ in this order. The records of [19] keep their numbers $0, \dots, 5933$, and record 5933 is the one with $k = 0$. The defect of Tadej and Życzkowski, an upper bound for the dimension of a smooth family of inequivalent complex Hadamard matrices through a given one, is 4 at F_6 and 0 at S_6 [49].

Proof. By Appendix C (U7), which follows the proof of [19, Theorem 1.1] and checks it from the stored files, every complex Hadamard matrix of order six is equivalent to a member of one of the 46,040 records of the atlas of that computation, and so to a member of one of its 1,406 *mid* records, of radius $\mathrm{fl}(0.07)$. That computation refined the mid records to its fine records at radius 0.02, except for one mid record refined at τ, and excluded every fine record, directly or through finer children: for no member H does $\{I, H/\sqrt{6}\}$ lie in four mutually unbiased bases [19, Section 4.2]. To have one radius throughout, we ran the refinement step of [19, Section 2.6] once more on all 1,406 mid records with children of radius τ, taking the 5,934 fine records, each with radius τ, as the initial store of children and no other covering records. The step completed every mid record and made 2,696 new children. By Appendix C (U6), the solution sets of the children then contain, up to equivalence, those of the mid records.

An independent check, the program of [19, Section 5] for the refinement step (Stage B there; its source is in [20], used unchanged), recomputes in Arb ball arithmetic at 128 bits every relation between a parent and a child, every decision on a box of a parent's tangent grid (of half-size 15 for every parent, read from the refinement's log), every subdivision, every grid and the order in which children were used, and confirms all of them; the stores of children that it builds equal those of the run. The Root certificates of the 2,696 new

records pass the sufficient conditions for certification of [19, Lemma C.23] (Appendix C (U2)), checked in Arb, each at its stored coordinates or after transposition, which suffices by [19, Lemma C.24] (Appendix C (U3)). The Root certificates of the mid records and of the 5,934 fine records passed the same check in [19, Section 5]. □

4.3. **The list.** A third basis of $\{I, H/\sqrt{6}\}$ consists of six pairwise orthogonal unit vectors, each unbiased to the standard basis and to the columns of $H/\sqrt{6}$. Stage C of [19, Section 2.7] encloses all such vectors, for all members H of a record at once, and records which of them may be orthogonal. We use it with one six-clique in place of two.

For a record, write every vector unbiased to the standard basis, up to a factor of modulus one, as $e(\beta)/\sqrt{6}$ with $\beta \in \mathbb{R}^6$ and $\beta_0 = 0$. Stage C divides the range of $(\beta_1, \dots, \beta_5)$ into bins in L layers, of resolutions $12, 24, \dots, M = 12 \cdot 2^{L-1}$ bins per phase as in [43, Section 6.3]; it keeps the bins that pass the necessary tests (SC1)–(SC3) for the record's members, groups the kept bins of the last layer into *vector balls* of radius less than 0.1, and joins two balls by an edge of the graph $\mathcal{O}$ unless a necessary test, (SC4) or (SC5), shows that they contain no two orthogonal vectors. Two vectors in one ball are never orthogonal. These are the tests of [19, Appendix A.7], restated in Section 6.4. For each bin that it assigns to a ball, the program makes the ball's radius s at least $\|h\| + X_M$, as in Lemma 6.8 (c), and it reports the run unresolved unless every radius is below 0.1 [19, Appendix A.7]. For the members of a record, Lemma 6.8 (i) proves the statements made here about the tests: (a) and (b) for the bins, (c) for the balls and (d) for the edges. How the kept bins are grouped into balls is decided by a parameter, the *target radius*, which is 0.042 in [43, Section 6.3]. A *full step* is a run of Stage C with L layers and target radius 0.042. A *core step* keeps the bins of the balls that lie in some six-clique of $\mathcal{O}$, groups them into balls of a smaller target radius, and rebuilds the edges. Our search runs, on each record, the *schedule* of three full steps, with $L = 4, 5, 6$, followed by two core steps at $L = 6$ with target radii 0.021 and 0.0105. It stops at the first step whose graph has no six-clique, and six-cliques are searched exhaustively.

Lemma 4.2. *Let H be a member of a record and $B/\sqrt{6}$ a third basis of $\{I, H/\sqrt{6}\}$, with lines $x_1, \dots, x_6$. After every step of the schedule, each x_j lies in a kept bin, the six bins lie in six distinct balls of that step, and these six balls are pairwise joined by edges of $\mathcal{O}$. In particular, if the search stops at a step whose graph has no six-clique, no member of the record has a third basis.*

Proof. By induction over the steps. The bin tests are necessary conditions for a bin to contain a vector unbiased to the columns of $H/\sqrt{6}$ for some member H, so each x_j lies in a kept bin. Each kept bin lies in a ball, and two of the x_j lie in distinct balls since they are orthogonal. The edge tests are necessary conditions for two balls to contain orthogonal vectors, so the six balls are pairwise joined. They then form a six-clique, so a core step keeps their bins, and the argument repeats. □

Proposition 4.3 (computer-assisted)**.** *Let $\mathcal{L}^+$ be the set of* 1,565 *records listed in the deposit* [21]*. No member of a record of Proposition 4.1 outside $\mathcal{L}^+$ has a third basis. So $\mathcal{L}^+$ is a list for the cover of Proposition 4.1.*

Proof. On each of the 7,065 records outside $\mathcal{L}^+$ the schedule stops at a step whose graph has no six-clique (the deposit gives the number excluded at each step), and Lemma 4.2 applies. The search uses the binary64 interval arithmetic [39] of the Stage C program of [19]. An independent check proves each of the 7,065 exclusions again, at the step where the search stopped, as in the proof of Lemma 4.2, taking the records of Proposition 4.1, with their

certified constants, as given: it regenerates the bins, derives each discarded bin, each ball and each missing edge from (SC1)–(SC5) and Lemma 6.8 in Arb ball arithmetic and exact arithmetic, and decides the clique questions exactly. The code that proposes (Appendix A.2) chooses only which test, direction and ball to use and which balls a core step keeps, so a wrong proposal can only make the check fail. □

Remark 4.4. The record about S_6 is not in $\mathcal{L}^+$. For the work of Sections 5, 6, 7 and 8, the records of $\mathcal{L}^+$ were sorted in floating point by whether their centers are close to a 2-circulant matrix or to the Fourier family. The sorting enters the proofs only through the explicit sets of records it produced, which are deposited and on which every statement is checked. We say that a record of $\mathcal{L}^+$ is of *Fourier type* if it lies neither in the set $\mathcal{K}$ of Section 5 nor in the set $\mathcal{S}$ of Section 7. There are 557 such records (Proposition 9.1).

5. Records near two-circulant matrices

This section proves Theorem 5.1, the property of a set $\mathcal{K}$ of 978 records of the list $\mathcal{L}^+$ that the descent of Section 9 uses. In floating point, these are the records of $\mathcal{L}^+$ whose nearest triplet from the construction has its clock fixing the third basis. For such a triplet $\{I, H/\sqrt{6}, B/\sqrt{6}\}$, Theorem 2.4 applied at the fixed basis shows that B^* is equivalent to a member $F^{\mathsf{T}}(x, y)$ of the Fourier family, while H, the transition between the two bases that the clock moves, is equivalent to a 2-circulant matrix. So one expects B^* to lie far from the records of $\mathcal{K}$, except near the point D where the families meet. Theorem 5.1 replaces this expectation by a weaker exact statement that suffices, for every member of every record of $\mathcal{K}$.

Let $\mathcal{K}$ be the 978 records listed in the deposit [21], and $\mathcal{T}$ the 48 records of $\mathcal{L}^+$ outside $\mathcal{K}$ whose centers lie, in floating point, within 0.05 of the point D (Section 7) in the distance of Section 3.3, also listed there. The 1,026 records of $\mathcal{K} \cup \mathcal{T}$ are the *targets*.

Theorem 5.1 (computer-assisted)**.** *There are sets $N(R) \subseteq \mathcal{K} \cup \mathcal{T}$ for $R \in \mathcal{K}$ and ranks $\rho\colon \mathcal{K} \cup \mathcal{T} \to \{1, \dots, 5\}$, equal to 1 exactly on $\mathcal{T}$, listed in the deposit, such that:*

(a) *for every $R \in \mathcal{K}$, every member H of R and every third basis $B/\sqrt{6}$ of $\{I, H/\sqrt{6}\}$, the matrix B^* lies in no record of $(\mathcal{K} \cup \mathcal{T}) \setminus N(R)$;*

(b) *$\rho(R') < \rho(R)$ for every $R \in \mathcal{K}$ and every $R' \in N(R)$.*

The directed graph with an edge from R to each $R' \in N(R)$ has 231,210 edges.

Sections 5.1, 5.2 and 5.3 give the three ingredients of the proof: Stage C's graph holds every third basis, the graph encloses the transition B^*, and aligned distances are bounded in exact arithmetic. Section 5.4 puts them together.

5.1. The graph of a record.

For $v \in \mathbb{R}^6$ and $s \geq 0$ let

$$\mathcal{B}_{\mathrm{vec}}(v, s) = \Big\{ \lambda\, e(v + y)/\sqrt{6} : \ |\lambda| = 1,\ y \in \mathbb{R}^6,\ \textstyle\sum_j y_j = 0,\ \|y\| \leq s \Big\},$$

where $e(v)$ is the vector with entries $\mathrm{e}^{2\pi i v_j}$, as in [19, Appendix A.1]; a line lies in this *vector ball* if its unit vectors do. Let R be a record of $\mathcal{K}$. The search of Section 4.3 did not stop on R, so its last step, the core step of target radius 0.0105, ended with a graph $\mathcal{G}_R$ that has a six-clique. The vertices of $\mathcal{G}_R$ are vector balls $\mathcal{B}_{\mathrm{vec}}(v_\beta, s_\beta)$, with binary64 centers v_β and radii s_β, and its edges are those of $\mathcal{O}$. Let K_R be the set of balls of $\mathcal{G}_R$ that lie in some six-clique. It is computed from $\mathcal{G}_R$ by an exhaustive search in integer arithmetic. By Lemma 4.2, applied to the last step (for the stored $\mathcal{G}_R$ the conditions of the last three steps are proved again in the proof of Theorem 5.1, under the hypotheses of Appendix A.1):

Lemma 5.2. *Let $R \in \mathcal{K}$, H a member of R and $B/\sqrt{6}$ a third basis of $\{I, H/\sqrt{6}\}$. The six lines of $B/\sqrt{6}$ lie in six distinct balls of K_R that form a six-clique of $\mathcal{G}_R$.*

The six-cliques themselves are not listed. They are covered by boxes. A *box* is a 6-tuple $(P_1, \dots, P_6)$ of pairwise disjoint subsets of K_R, and it *covers* the six-cliques that have one ball in each P_k. A *grouping* is a partition of K_R into groups such that no edge of $\mathcal{G}_R$ joins two balls of the same group. Then a six-clique has its six balls in six distinct groups, and these groups are pairwise joined in the quotient graph, so the boxes formed by the six-cliques of the quotient graph, in any order of their groups, cover every six-clique of $\mathcal{G}_R$. Two operations keep the covering:

(i) *pruning*: remove from a set P_k of a box a ball with no neighbor in some other set P_l, and drop a box with an empty set. The removed ball lies in no clique that the box covers;

(ii) *splitting*: replace a box by the two boxes obtained by dividing one of its sets into two parts. Every clique the box covers is covered by exactly one of the two.

5.2. **An enclosure of the transition.** Write Π_6 for the orthogonal projection of $\mathbb{R}^6$ onto the vectors with vanishing sum.

Lemma 5.3. *Let $(P_1, \dots, P_6)$ be a box of a record R, and let $c_1, \dots, c_6 \in \mathbb{R}^6$ and $\rho_1, \dots, \rho_6 \geq 0$ satisfy*

$$\rho_k \geq \|\Pi_6(v_\beta - c_k - n_\beta)\| + s_\beta \qquad \text{for every } \beta \in P_k,$$

for some $n_\beta \in \mathbb{Z}^6$. Let X_0 be the matrix with rows $-c_1, \dots, -c_6$ and $\sigma = (\sum_k \rho_k^2)^{1/2}$. If the k-th column of B spans a line in a ball of P_k, for $k = 1, \dots, 6$, then every phase matrix A with $e(A) = B^*$ satisfies $d(A - X_0) \leq \sigma$, and so $d_G(A, X_0) \leq \sigma$. The same bound $d_G(A, X_0) \leq \sigma$ holds for every ordering of the columns of B.

Proof. Let $\beta \in P_k$ be the ball of the k-th line. Its unit vector $b_k/\sqrt{6}$, b_k the k-th column of B, is $\lambda_k e(v_\beta + y_k)/\sqrt{6}$ with $|\lambda_k| = 1$, $\sum_j y_{k,j} = 0$ and $\|y_k\| \leq s_\beta$. Put $w = v_\beta - c_k - n_\beta$ and write $w = \Pi_6 w + \mu \mathbf{1}$. Since $e(n_\beta) = \mathbf{1}$,

$$b_k = \lambda_k \mathrm{e}^{2\pi i \mu}\, e(c_k + z_k), \qquad z_k = \Pi_6 w + y_k, \qquad \|z_k\| \leq \|\Pi_6 w\| + s_\beta \leq \rho_k.$$

The k-th row of B^* is $\overline{b_k}^{\mathsf{T}}$, so $A = X_0 - \mathbf{z} + t + N$, where $\mathbf{z}$ has rows z_k, $t \in L$ holds the phases of $\overline{\lambda_k}\mathrm{e}^{-2\pi i\mu}$ in its rows, and $N \in \mathbb{Z}^{36}$. Hence $d(A - X_0) = d(-\mathbf{z}) \leq \|\mathbf{z}\| \leq \sigma$, and $d_G(A, X_0) \leq d(A - X_0)$. Reordering the columns of B permutes the rows of B^*, which does not change $d_G(\cdot, X_0)$ by Lemma 3.3 (a). □

5.3. **Computed aligned distances.** For a phase matrix A let A' be A dephased at row and column 0, that is $A'_{jk} = A_{jk} - A_{j0} - A_{0k} + A_{00}$, and for $N \in \mathbb{Z}^{36}$ put $c_N(A) = \|\Pi(A' - N)\|$.

Lemma 5.4. *$d(A) \leq c_N(A)$ for every $N \in \mathbb{Z}^{36}$. If $d(A) < 1/4$, every entry of A' lies within $2d(A) < 1/2$ of an integer, and for the integer matrix N_0 of these integers $c_{N_0}(A) = d(A)$.*

Proof. Since $A' - A \in L$, $d(A) = d(A') \leq \|\Pi(A' - N)\|$. Let N^* attain $d(A)$ and $\Delta = \Pi(A - N^*) = A - N^* - t$ with $t \in L$, so $\|\Delta\| = d(A)$. Dephasing is linear and vanishes on L, so $\Delta' = A' - N^{*\prime}$, where the dephased matrix $N^{*\prime}$ is an integer matrix. For $j, k \geq 1$, $|\Delta'_{jk}| = |\Delta_{jk} - \Delta_{j0} - \Delta_{0k} + \Delta_{00}| \leq 2(\Delta_{jk}^2 + \Delta_{j0}^2 + \Delta_{0k}^2 + \Delta_{00}^2)^{1/2} \leq 2\|\Delta\|$, and Δ', A', $N^{*\prime}$ vanish in row and column 0. So every entry of A' lies within $2d(A)$ of the corresponding entry of $N_0 := N^{*\prime}$, and $\Pi(A' - N_0) = \Pi\Delta' = \Pi\Delta = \Delta$, since $\Delta' - \Delta \in L$ and $\Delta \in L^\perp$. □

We bound d_G from both sides in exact arithmetic, for Theorem 7.3 and for the transition sets of Section 5.4. In this paragraph $c_N(A)$ is taken with N a matrix of nearest integers to the entries of A' (any choice at ties). Then $d(A) \le c_N(A)$, and $d(A) \ge \min(1/4, c_N(A))$: if $d(A) < 1/4$, Lemma 5.4 shows that $N = N_0$, so $c_N(A) = d(A)$. Moreover $|d(A) - d(A+E)| \le d(E) \le \|E\|$, and $\|gE\| = \|E\|$ for $g \in G$ (proof of Lemma 3.3). For a matrix W vanishing in row and column 0, with row sums R_j, column sums C_k and total S,

$$\|\Pi W\|^2 = \|W\|^2 - \tfrac{1}{6}\textstyle\sum_j R_j^2 - \tfrac{1}{6}\sum_k C_k^2 + \tfrac{1}{36}S^2. \tag{5.1}$$

Let X and Y be phase matrices with rational entries of absolute value at most 4, let $\hat{X}$ and $\hat{Y}$ be X and Y with their entries rounded to multiples of 2^{-24}, and let $\delta \ge \|X - \hat{X}\| + \|Y - \hat{Y}\|$ be rational. A program in the deposit [21] computes, for every $g \in G$, the integer $36 \cdot 4^{24} c_N(g\hat{X} - \hat{Y})^2$ exactly, by (5.1), in 64-bit integer arithmetic in which every intermediate value is below 2^{57} in absolute value, and chooses an integer $T \ge 0$. Every g for which this integer is at least T has $d(gX - Y) \ge \min(1/4, (T/(36 \cdot 4^{24}))^{1/2}) - \delta$; for every other g the program computes $c_N(gX - Y)^2$ in exact rational arithmetic, and $\min(1/4, c_N(gX - Y)) \le d(gX - Y) \le c_N(gX - Y)$. The least of these lower bounds, rounded down to a multiple of 2^{-64}, is a lower bound $f^-(X, Y)$ for $d_G(X, Y)$. If some g has its integer below T, as for every pair (X, Y) used here, the least of the $c_N(gX - Y)$ over these g, rounded up, is an upper bound $f^+(X, Y)$ for $d_G(X, Y)$. With a landmark Y we write $f_Y^{\pm}(X) = f^{\pm}(X, Y)$, bounds for $f_Y(X) = d_G(X, Y)$.

5.4. **Transition sets.**

Proposition 5.5. *Let $(P_1, \dots, P_6)$ be a box of a record R with X_0 and σ as in Lemma 5.3, and let R' be a record with center P' and radius r'. Suppose that for some landmark Y one of the following holds:*

(S1) $f_Y^+(X_0) + \sigma < f_Y^-(P') - r'$;

(S2) $f_Y^-(X_0) - \sigma > f_Y^+(P') + r'$;

(S3) $f^-(X_0, P') > \sigma + r'$, *the lower bound for* $d_G(X_0, P')$.

Then for every member H of R and every third basis $B/\sqrt{6}$ of $\{I, H/\sqrt{6}\}$ whose lines lie in a six-clique covered by the box, B^* does not lie in R'.

Proof. Let A be a phase matrix of B^*. By Lemma 5.3, $d_G(A, X_0) \le \sigma$. If B^* lay in R', then $d_G(A, P') \le r'$ by Lemma 3.3 (c), so $d_G(X_0, P') \le \sigma + r'$ and, by Lemma 3.3 (d), $|f_Y(X_0) - f_Y(P')| \le \sigma + r'$ for every landmark Y. Each of (S1)–(S3) contradicts one of these inequalities, since $f^- \le f \le f^+$. □

The *transition set* $N(R)$ of a record $R \in \mathcal{K}$ is produced by a branch and bound over boxes. The run on R is given a list of targets and a sublist F of *free* targets, which go into $N(R)$ untested. Free targets save work: a target that stays in the pending sets keeps a box from being accepted, so the box is split until leaves are declared.

(1) Compute K_R, a grouping (any partition such that no edge joins two balls of a group, a condition checked exactly), and the initial boxes of the six-cliques of the quotient graph, and prune.

(2) For each box, choose binary64 centers c_k and integer vectors n_β (any choice is admissible) and compute ρ_k and σ in exact rational arithmetic, with square roots rounded up to multiples of 2^{-64}. Every box carries a *pending set* of targets, initially all targets outside F, and for a box obtained by splitting, the pending set of its parent.

(3) Remove from the pending set each target R' for which (S1) or (S2) holds with the one of 33 fixed landmarks that a floating-point comparison picks for it, or, when few targets remain, (S3) holds. These tests are comparisons of rational numbers; the runs make them on binary64 approximations, and the independent check of Theorem 5.1 makes them exactly.

(4) A box with an empty pending set is *accepted*. Otherwise it is split at its set of largest ρ_k and both halves are pruned and treated in the same way, or it is declared a *leaf*; leaves are declared by rules on the depth, the number of boxes and the progress of the pending set.

(5) $N(R)$ is F together with the pending sets of all leaves.

The landmarks are 33 binary64 phase matrices close to members of the Fourier family, listed in the deposit; any matrices would do, and the choice affects only how many targets each test removes.

Proposition 5.6. *Let $R \in \mathcal{K}$, and let $N(R)$ be produced by a run with a list of targets, and suppose that every test by which a target was removed holds exactly. For every member H of R and every third basis $B/\sqrt{6}$ of $\{I, H/\sqrt{6}\}$, if B^* lies in a target R', then $R' \in N(R)$.*

Proof. If R' is free, $R' \in N(R)$. Otherwise, by Lemma 5.2 the lines of $B/\sqrt{6}$ form a six-clique Q of $\mathcal{G}_R$ in K_R. Some initial box covers Q, pruning keeps it covered, and a split box has exactly one half that covers Q. The boxes form a finite tree, so there is a chain of boxes, each covering Q, from an initial box to an accepted box or a leaf. A target removed from the pending set of a box of this chain satisfies one of (S1)–(S3) for that box, so by Proposition 5.5 it does not hold B^*. Hence R' is never removed along the chain. So the chain does not end in an accepted box, and it ends in a leaf whose pending set contains R'; thus $R' \in N(R)$. □

Proof of Theorem 5.1. For each record R of $\mathcal{K}$, $N(R)$ is the union of the sets produced by the branch and bound of Section 5.4 against target lists whose union is $\mathcal{K} \cup \mathcal{T}$; the lists and the sets are in the deposit. In each of them the records of $\mathcal{T}$ among the targets are free, so $\mathcal{T} \subseteq N(R)$.

An independent check, computing its bounds with the program of Section 5.3, verifies the tree of the branch and bound on each record of $\mathcal{K}$: its grouping of K_R, that its initial boxes against each list are those of all the six-cliques of the quotient graph, that each split keeps every clique its box covers in one of its halves, that the radii ρ_k and σ of each box satisfy the hypotheses of Lemma 5.3, that every test by which a target was removed holds with these bounds, and that the pending set of each leaf lies in $N(R)$. It also proves Lemma 5.2 for the stored graph $\mathcal{G}_R$. For this it shows that every pair of balls of $\mathcal{G}_R$ without an edge fails (SC4) or (SC5) for $E = \mathcal{O}$, except one pair on each of records 5465 and 5883, which a second test, in ball arithmetic, proves fails (SC5). So the argument of Proposition 5.6 applies to $N(R)$: a target belongs to one of the lists, and against that list every six-clique is covered by an initial box. This is (a).

In the directed graph of the theorem, on the vertices $\mathcal{K} \cup \mathcal{T}$, the records of $\mathcal{T}$ have no outgoing edges. A program finds that its 1,026 strongly connected components are single vertices and that it has no loops, so it is acyclic. Hence $\rho(R) = 1$ for a record without outgoing edges and $\rho(R) = 1 + \max_{R' \in N(R)} \rho(R')$ otherwise defines ρ, and ρ satisfies (b). The program computes ρ and the counts, and an independent program, by Kahn's elimination [23] of vertices without outgoing edges, rebuilds the edges from the listed sets $N(R)$ and confirms them, the acyclicity, the ranks and the counts; the number of records of each rank is listed in the deposit. Since $\mathcal{T} \subseteq N(R)$ for every $R \in \mathcal{K}$, ρ equals 1 exactly on $\mathcal{T}$. □

The arithmetic of each step, the programs, and the counts of boxes, leaves and separations of every run are in the deposit.

6. Records of Fourier type

This section proves a criterion, Theorem 6.12, that closes a record near the Fourier family: every triplet through a member has the clock Z, up to a relabeling. It rests on a local rigidity statement at the triplets of the family (Proposition 6.2), on certified continuations of the common unbiased lines (Section 6.3), and on Stage C of the search run on small cubes of a record (Section 6.4). Proposition 6.13 applies the criterion to 483 records, four of them through its analogue Theorem 8.9 near F_6.

6.1. **Triplets of the Fourier family in phase coordinates.** Index the rows of a 6×6 matrix by $r=(a,i)\mapsto 3a+i$ and its columns by $c=(h,k)\mapsto 3h+k$, with $a,h\in\{0,1\}$ and $i,k\in\mathbb{Z}/3$. For $d\in\mathbb{T}^3$ let

$$H(d)_{(a,i),(h,k)}=(-1)^{ah}d_i^h\,\omega^{ik},\qquad\text{that is}\qquad H(d)=\begin{pmatrix}F_3 & \Delta F_3\\ F_3 & -\Delta F_3\end{pmatrix},\quad \Delta=\operatorname{diag}(d).$$

Then $H(1,x,y)$ and $F^{\mathsf T}(x,y)$ have the same columns, in another order (column k of $\overline{F_3}$ is column $-k$ of F_3), so they define the same basis. For $s=(\alpha,\beta)\in\mathbb{R}^2$ we write $d(s)=(1,\mathrm{e}^{i\alpha},\mathrm{e}^{i\beta})$. Let $V(d)$ be the set of lines unbiased to the standard basis and to the columns of $H(d)/\sqrt{6}$. For a vector $v=(p;q)$, $p,q\in\mathbb{C}^3$, put $\Theta v=(-\Delta\bar q;\Delta\bar p)$ and let $K(v)$ be the matrix whose column $(0,k)$ is Z^kv and whose column $(1,k)$ is $Z^k\Theta v$. Since $\Theta v=-Jv$ for the map J of (2.4) with $E=\Delta$, Proposition 2.6 shows that for $v\in V(d)$ with unimodular entries, $K(v)/\sqrt{6}$ is a third basis of $\{I,H(d)/\sqrt{6}\}$. The matrix T_3 of Theorem 2.8 (b) satisfies $(MT_3^{-1})_{r,(h,k)}=M_{r,(h,k+1)}$ for every 6×6 matrix M. From $\omega^i\omega^{ik}=\omega^{i(k+1)}$,

$$Z\,H(d)=H(d)\,T_3^{-1},\qquad Z\,K(v)=K(v)\,T_3^{-1}.\tag{6.1}$$

Fix d and $v\in V(d)$, and let $\mathcal{T}=\{I,H/\sqrt{6},K/\sqrt{6}\}$ with $H=H(d)$ and $K=K(v)$. For $x=(\varphi,\psi)\in\mathbb{R}^{36}\times\mathbb{R}^{36}$ let $H_x=H\circ\mathrm{e}^{i\varphi}$ and $K_x=K\circ\mathrm{e}^{i\psi}$ (entrywise), and let $F_{\mathcal{T}}(x)\in\mathbb{R}^{96}$ have as components the real and imaginary parts of $(H_xH_x^*)_{rs}$ and $(K_xK_x^*)_{rs}$ for $r<s$, and $|(H_x^*K_x)_{cc'}|^2-6$ for all c,c'. So $F_{\mathcal{T}}(x)=0$ exactly when $\{I,H_x/\sqrt{6},K_x/\sqrt{6}\}$ is a triplet. Let $L_{\mathrm{triv}}\subset\mathbb{R}^{72}$ be the space of vectors $(\rho_r+\gamma_c,\rho_r+\gamma'_c)$ with $\rho,\gamma,\gamma'\in\mathbb{R}^6$, of dimension 17: a row phase common to H_x and K_x and column phases of each. It is the triplet analogue of the trivial directions removed by dephasing in the defect [48, Definition 3.2]. Let S act on $\mathbb{R}^{72}$ by $(Sx)_{r,(h,k)}=x_{r,(h,k-1)}$ in both halves. By (6.1),

$$Z\,H_x\,T_3=H_{Sx},\qquad Z\,K_x\,T_3=K_{Sx},\tag{6.2}$$

so $F_{\mathcal{T}}\circ S=A\circ F_{\mathcal{T}}$, where the orthogonal map A rotates the plane of the equation $(r,s)=((a,i),(b,j))$ by ω^{i-j} and permutes the moduli by $(c,c')\mapsto(c+1,c'+1)$ in the second index of $c=(h,k)$ and c'. The fixed space of A is spanned by the planes of the equations with $i=j$ and by the moduli that are constant on the 12 orbits. Let Π_1 be the orthogonal projection onto its orthogonal complement, of dimension 72, and $F_1=\Pi_1F_{\mathcal{T}}$. If $Sx=x$ then $F_{\mathcal{T}}(x)=AF_{\mathcal{T}}(x)$, so

$$F_1(x)=0\qquad\text{whenever } Sx=x.\tag{6.3}$$

Finally $\mathbb{R}^{72}=L_{\mathrm{triv}}\oplus Z_0\oplus Y_1$ orthogonally, where Y_1 is the space of x with $\sum_k x_{r,(h,k)}=0$ and $\sum_r x_{r,c}=0$ for all r,h,c in both halves (dimension 40), and Z_0 is the space of x independent of k and orthogonal to L_{triv} (dimension 15).

6.2. **The tube.**

Lemma 6.1. *For $q \in \mathbb{R}^{72}$ with $\|q\| \le \varrho$ and unit vectors $u, v, w \in \mathbb{R}^{72}$, $\|D^3 F_{\mathcal{T}}(q)[u,v,w]\| \le L_3(\varrho)$, where*

$$L_3(\varrho) = \bigl(24 + B(\varrho)^2\bigr)^{1/2}, \qquad B(\varrho) = 12\sqrt{2}(\sqrt{6}+\varrho) + 2(\sqrt{6}+\sqrt{12}\,\varrho)(\sqrt{6}+\varrho+3/\sqrt{2}).$$

Proof. Write $H' = H_q$, $K' = K_q$, $N_2 = \sqrt{6}+\varrho$. Then $\|H'\|_2 \le \|H\|_2 + \|H\circ(\mathrm{e}^{iq^{\varphi}}-1)\|_F \le N_2$, and likewise for K'. For $u \in \mathbb{R}^{72}$ let $P_u = \|u^{\varphi}\|^2$, $Q_u = \|u^{\psi}\|^2$, $p_u = \sqrt{P_u}$, $q_u = \sqrt{Q_u}$, so $P_u + Q_u = 1$.

Hadamard equations. $D^3(H'H'^{*})_{rs}[u,v,w] = -i\sum_c H'_{rc}\overline{H'_{sc}}\,\delta_c^u\delta_c^v\delta_c^w$ with $\delta_c^u = u_{rc} - u_{sc}$, so its modulus is at most $(a^u_{rs}a^v_{rs}a^w_{rs})^{1/2}$, where $a^u_{rs} = \sum_c(\delta_c^u)^2$. Now $\sum_{r<s} a^u_{rs} = \sum_c(6\sum_r u_{rc}^2 - (\sum_r u_{rc})^2) \le 6P_u$ and $a^u_{rs} \le 2P_u$, so $\sum_{r<s}(a^u_{rs})^3 \le 24P_u^3$. With the same for K', Hölder's inequality with exponents $3,3,3$ over the 30 pairs, applied to the numbers $a^u_{rs}, a^v_{rs}, a^w_{rs}$ and their analogues for K', together with $P^3+Q^3 \le (P+Q)^3 = 1$, shows that the squares of these components sum to at most $\prod_{u,v,w}(24(P^3+Q^3))^{1/3} \le 24$.

Moduli. Let $L = H'^{*}K'$ and $m = |L|^2 - 6$ entrywise. Then $D^3 m[u,v,w] = 2\,\mathrm{Re}(\bar{L}\cdot D^3L[u,v,w]) + 2\,\mathrm{Re}\sum \overline{DL[u]}\cdot D^2L[v,w]$, entrywise, the sum over the three ways of singling out one of u,v,w. Here $|L_{cc'}| \le \sqrt{6} + \sum_r |q^{\psi}_{rc'} - q^{\varphi}_{rc}| \le \sqrt{6} + \sqrt{12}\,\varrho$, since $|L_{cc'}(0)| = \sqrt{6}$; $\|DL[u]\|_F = \| -i(H'\circ u^{\varphi})^{*}K' + iH'^{*}(K'\circ u^{\psi})\|_F \le N_2(p_u+q_u) \le \sqrt{2}N_2$; $|D^2L_{cc'}[v,w]| \le (p_v+q_v)(p_w+q_w) \le 2$; and $D^3L[u,v,w]$ is a sum of eight terms $\pm i(H'\circ\prod_{s\in\Sigma}s^{\varphi})^{*}(K'\circ\prod_{s\notin\Sigma}s^{\psi})$ over the subsets $\Sigma \subseteq \{u,v,w\}$, of Frobenius norm at most $(N_2-1)(p_up_vp_w + q_uq_vq_w) + (p_u+q_u)(p_v+q_v)(p_w+q_w) \le N_2 + 3/\sqrt{2}$. Hence $\|D^3m\| \le 2(\sqrt{6}+\sqrt{12}\varrho)(N_2+3/\sqrt{2}) + 6\cdot\sqrt{2}N_2\cdot 2 = B(\varrho)$, and $\|D^3F_{\mathcal{T}}\|^2 \le 24 + B(\varrho)^2$. □

Proposition 6.2. *Let $\sigma, M, M' > 0$ satisfy $\sigma \le \|DF_1(0)y\|$, $M \ge \|D^2F_1(0)[y,y']\|$ and $M' \ge \|D^2F_1(0)[z,y]\|$ for all unit $y, y' \in Y_1$ and $z \in Z_0$. Let $x \in \mathbb{R}^{72}$ with $F_{\mathcal{T}}(x) = 0$, write $x = t + x_0 + y_1$ with $t \in L_{\mathrm{triv}}$, $x_0 \in Z_0$, $y_1 \in Y_1$, and put $a = \|x_0\|$, $b = \|y_1\|$, $\varrho = (a^2+b^2)^{1/2}$. If*

$$\sigma > M'a + \tfrac{M}{2}\,b + \tfrac{L_3(\varrho)}{2}\bigl(a^2 + \tfrac{b^2}{3}\bigr), \tag{6.4}$$

then $y_1 = 0$, and Z is a clock of the triplet $\{I, H_x/\sqrt{6}, K_x/\sqrt{6}\}$.

Proof. (1) A vector of L_{triv} with row phases ρ changes $F_{\mathcal{T}}$ by an orthogonal map (it rotates the plane of the equation (r,s) by $\mathrm{e}^{i(\rho_r-\rho_s)}$, and column phases change nothing), so $F_{\mathcal{T}}(x_0+y_1) = 0$. (2) $Sx_0 = x_0$, so $F_1(x_0) = 0$ by (6.3). (3) Let $p_t = x_0 + ty_1$. Then

$$F_1(x_0+y_1) = DF_1(0)y_1 + \int_0^1\!\!\int_0^1 D^2F_1(\lambda p_t)[p_t, y_1]\,d\lambda\,dt.$$

Write $D^2F_1(\lambda p_t) = D^2F_1(0) + (D^2F_1(\lambda p_t) - D^2F_1(0))$. The first part gives $\|D^2F_1(0)[x_0,y_1] + tD^2F_1(0)[y_1,y_1]\| \le M'ab + tMb^2$, and the second at most $L_3(\varrho)\lambda\|p_t\|^2 b$ by Lemma 6.1 and the mean value inequality, since $\|\lambda p_t\| \le \varrho$ and $\|\Pi_1\| \le 1$. As $\|p_t\|^2 = a^2 + t^2b^2$, integrating gives $\|F_1(x_0+y_1) - DF_1(0)y_1\| \le b(M'a + \frac{M}{2}b + \frac{L_3(\varrho)}{2}(a^2 + \frac{b^2}{3}))$, while $\|DF_1(0)y_1\| \ge \sigma b$. Since $F_1(x_0+y_1) = 0$, (6.4) forces $b = 0$. (4) Let t_0 be the average of t over k and $\xi = x_0 + t_0$, so $S\xi = \xi$ and $x = \xi + (t - t_0)$, where $t - t_0$ consists of column phases only. By (6.2), Z maps the columns of H_ξ and K_ξ to columns of the same matrices, and it fixes the lines of I; column phases do not change lines, so the same holds for H_x and K_x. As Z is not scalar and $Z^3 = I$, it is a clock. □

The condition (6.4) involves only the norms a and b, and holds when both are small. We call the set of vectors $x = t + x_0 + y_1$ as in Proposition 6.2, with $t \in L_{\mathrm{triv}}$ arbitrary and a, b satisfying (6.4) the *tube* about $\mathcal{T}$; by Proposition 6.2, Z is a clock of every triplet $\{I, H_x/\sqrt{6}, K_x/\sqrt{6}\}$ with x in the tube.

At a family triplet the constants are computed as follows, in ball arithmetic: the Jacobian and the 96 Hessians of $F_{\mathcal{T}}$ at 0, from enclosures of H and K; orthonormal bases Q_Y of Y_1 and Q_Z of Z_0; and interval LDL^{T} factorizations with all pivots positive of $G - \sigma^2 I$, $M^2 I - Q_Y^{\mathsf{T}} W Q_Y$ and $M'^2 I - Q_Z^{\mathsf{T}} W Q_Z$, where $G = (\Pi_1 J_{\mathcal{T}} Q_Y)^{\mathsf{T}} (\Pi_1 J_{\mathcal{T}} Q_Y)$ for the Jacobian $J_{\mathcal{T}}$ and $W = \sum_j A_j Q_Y Q_Y^{\mathsf{T}} A_j$ for the Hessians A_j of the components of $\Pi_1 F_{\mathcal{T}}$ in an orthonormal basis. By the Cauchy–Schwarz inequality, $\|D^2 F_1(0)[y,y]\|^2 \le y^{\mathsf{T}} W y$ and $\|D^2 F_1(0)[z,y]\|^2 \le z^{\mathsf{T}} W z$ for unit $y \in Y_1$, $z \in Z_0$. The bound for $y = y'$ gives it for all unit $y, y' \in Y_1$ by polarization: for unit e, the form $b_e = \langle e, D^2 F_1(0)[\cdot,\cdot]\rangle$ satisfies $4b_e(y,y') = b_e(y+y', y+y') - b_e(y-y', y-y')$, so $|b_e(y,y')| \le \frac{M}{4}(\|y+y'\|^2 + \|y-y'\|^2) = M$; take the supremum over e. The family triplets used below are taken at binary64 points s of the family, with v the unique zero of the unbiasedness equations in a box certified by Lemma 6.3; the constants depend on s and v. At a family triplet with $s = (-0.3, 0.6)$, for example, $\sigma = 1.4654$, $M = 12.736$ and $M' = 20.913$ (rounded outward), and (6.4) holds whenever $\varrho < 0.0614$.

The rest of the section turns Proposition 6.2 into a criterion for a record, in three steps. First, every line that can occur in a third basis for a member of the record is enclosed, by Stage C on small cubes of the record (Section 6.4). Second, on cells of these cubes, the enclosed lines are continued uniquely over the parameters as certified zeros of the unbiasedness equations (Sections 6.3 and 6.5). Third, a cell is excluded if no six-tuple of these lines can be pairwise orthogonal, and decided if the triplet of each such six-tuple is shown to lie in the tube about a family triplet. Theorem 6.12 is the criterion that combines the three steps.

6.3. **The lines of a matrix.** For a complex Hadamard matrix H with columns h_c and $\theta \in \mathbb{R}^5$ let $w(\theta) = (1, \mathrm{e}^{i\theta_1}, \dots, \mathrm{e}^{i\theta_5})$, $z_c(\theta) = \langle h_c, w(\theta)\rangle$ and $R_c(\theta; H) = |z_c(\theta)|^2 - 6$. The line of $w(\theta)$ is unbiased to the columns of $H/\sqrt{6}$ if and only if $R_c(\theta; H) = 0$ for $c = 0, \dots, 4$, since $\sum_{c=0}^{5} |z_c|^2 = \|H^* w\|^2 = 36$. Every line unbiased to the standard basis has such a representative, and θ and $\theta + 2\pi m$, $m \in \mathbb{Z}^5$, give the same line.

The next lemma is the Krawczyk test [27, 40] of interval analysis [39], in Rump's form [45, Theorem 13.3], with a parameter.

Lemma 6.3 (parametric Krawczyk test)**.** *Let $X = x_c + [-\epsilon, \epsilon]^n$, let $G(\cdot, p)$ be C^1 near X for p in a set $\mathcal{P}$, let $[G]$ be an interval vector containing $G(x_c, p)$ and $[J]$ an interval matrix containing $\partial_x G(x, p)$ for all $x \in X$ and $p \in \mathcal{P}$, and let Y be a real matrix with*

$$-Y[G] + (I - Y[J])[-\epsilon, \epsilon]^n \subset (-\epsilon, \epsilon)^n$$

in interval arithmetic. Then for every $p \in \mathcal{P}$, $G(\cdot, p)$ has exactly one zero $x(p)$ in X, and $\partial_x G(x(p), p)$ is invertible. The same holds if $Y = Y(p)$ depends on p and the inclusion holds for each p with $[G]$ and $[J]$ replaced by enclosures valid for that p.

Proof. Row i of the inclusion gives $\max |(Y[G])_i| + \epsilon \sum_j \max |(I - Y[J])_{ij}| < \epsilon$, so $\|I - YA\|_\infty < 1$ for every real $A \in [J]$, and Y and A are invertible. For $x \in X$, $G(x,p) - G(x_c,p) = \tilde{A}(x - x_c)$ with $\tilde{A} \in [J]$ (the mean value theorem, row by row), so $\kappa(x) = x - YG(x,p)$ maps X into X and has a fixed point, a zero of $G(\cdot, p)$. Two zeros x, x' in X give $\tilde{A}(x - x') = 0$ for some $\tilde{A} \in [J]$, so $x = x'$. □

Lemmas 6.4 and 6.6 are arguments of Newton–Kantorovich type with an approximate inverse Y, as in [45, Section 13] and in the radii polynomials of [7].

Lemma 6.4 (contraction on a ball)**.** *Let F be a C^1 map from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$, let $x_0 \in \mathbb{R}^n$ and $\epsilon > 0$, let Y be a real $n \times n$ matrix, and let $b, \kappa \geq 0$; vector norms are Euclidean and $\|\cdot\|_2$ is the induced matrix norm. Assume that the closed Euclidean ball $\bar{B} = \bar{B}(x_0, \epsilon)$ lies in the domain of F, that $\|YF(x_0)\| \leq b$, that $\|I - Y\,DF(x)\|_2 \leq \kappa$ for $x \in \bar{B}$, and that $\kappa < 1$ and $b + \kappa\epsilon \leq \epsilon$. Then F has exactly one zero in $\bar{B}$. If moreover $\|I - Y\,DF(x)\|_2 \leq \kappa' < 1$ for every x in a closed ball $\bar{B}(x_0, \epsilon') \supseteq \bar{B}$ in the domain of F, this zero is the only zero of F in $\bar{B}(x_0, \epsilon')$.*

Proof. Y is invertible, since $\|I - Y\,DF(x_0)\|_2 < 1$, so the zeros of F are the fixed points of $T(x) = x - YF(x)$. On a closed ball about x_0 on which $\|I - Y\,DF\|_2 \leq k$, the segment between two points x, x' of the ball lies in it, and $T(x) - T(x') = \int_0^1 (I - Y\,DF(x' + t(x - x'))\,dt\,(x - x')$, so $\|T(x) - T(x')\| \leq k\|x - x'\|$. For $x \in \bar{B}$ this gives $\|T(x) - x_0\| \leq \|T(x) - T(x_0)\| + \|YF(x_0)\| \leq \kappa\epsilon + b \leq \epsilon$, so T is a contraction of $\bar{B}$ into itself and has exactly one fixed point there. Two zeros x, x' in $\bar{B}(x_0, \epsilon')$ satisfy $\|x - x'\| = \|T(x) - T(x')\| \leq \kappa'\|x - x'\|$, so $x = x'$. □

Lemma 6.5. *For unit $x, y, v \in \mathbb{R}^5$, $\|D^3R(\theta; H)[x, y, v]\| \leq 2\sqrt{6}\,(3 + \max_c |z_c(\theta)|)$, where $R = (R_c)_{c=0}^4$.*

Proof. Extend vectors of $\mathbb{R}^5$ by a zero coordinate 0 and put $A_c(x) = \sum_l \overline{H_{lc}} \mathrm{e}^{i\theta_l} x_l$. Then $\partial_x z_c = iA_c(x)$ and $\partial_y A_c(x) = iA_c(x \circ y)$, so

$$D^3R_c[x, y, v] = 2\,\mathrm{Re}\big(-i\overline{A_c(y \circ v)}A_c(x) + i\overline{A_c(y)}A_c(x \circ v)\big) \\ - 2\,\mathrm{Re}\big(-i\overline{A_c(v)}A_c(x \circ y) + i\bar{z}_c A_c(x \circ y \circ v)\big).$$

Now $|A_c(u)| \leq \|u\|_1$, $\sum_c |A_c(u)|^2 \leq \|H^*(w \circ u)\|^2 = 6\|u\|^2$, $\|y \circ v\|_1 \leq 1$ and $\|x \circ y \circ v\| \leq 1$. So each of the first three terms has 2-norm over c at most $2\sqrt{6}$, and the last at most $2\sqrt{6}\max_c |z_c|$. □

Lemma 6.6. *Let θ^* be a zero of $R(\cdot; H)$, Y a real 5×5 matrix, $\kappa \geq \|I - Y\,DR(\theta^*)\|_2$ with $\kappa < 1$, $M_Y \geq \sup_{\|e\|=1} \|Y\,D^2R(\theta^*)[e, e]\|$, and $\varrho > 0$ with*

$$1 - \kappa > \tfrac{M_Y}{2}\varrho + \|Y\|_2 \tfrac{L(\varrho)}{6}\varrho^2, \qquad L(\varrho) = 2\sqrt{6}\,(3 + \sqrt{6} + \sqrt{5}\,\varrho).$$

Then θ^* is the only zero of $R(\cdot; H)$ with $\|\theta - \theta^*\| \leq \varrho$.

Proof. Let $0 < \|e\| = s \leq \varrho$. Then $YR(\theta^* + e) = Y\,DR(\theta^*)e + \int_0^1 (1 - t)\,YD^2R(\theta^* + te)[e, e]\,dt$ and $\|Y\,DR(\theta^*)e\| \geq (1 - \kappa)s$. On the segment, $|z_c| \leq \sqrt{6} + \|te\|_1 \leq \sqrt{6} + \sqrt{5}s$, since $|z_c(\theta^*)| = \sqrt{6}$; so by Lemma 6.5 $\|Y(D^2R(\theta^* + te) - D^2R(\theta^*))[e, e]\| \leq \|Y\|_2 L(\varrho) t s^3$. With $\int_0^1 (1 - t)t\,dt = 1/6$, $\|YR(\theta^* + e)\| \geq s\big((1 - \kappa) - \frac{M_Y}{2}s - \|Y\|_2 \frac{L(\varrho)}{6}s^2\big) > 0$. □

6.4. **Cubes of a record.** Let R be a record with center P, Root certificate $(T, n_0, \iota, C, \omega, \dots)$ with $k = 4$ (here ω is a real constant, not the cube root of unity), and radius r, and put $r_H = (1 + \omega)^{1/2}r$ and $n_R = n_0 + \iota r + Cr^2$. For a member $e(P + \delta)$ let $\eta = T^{\mathsf{T}}\delta$, its *tangent coordinate*; then $\|\eta\| \leq r_H$ and $\|(I - TT^{\mathsf{T}})\delta\| \leq n_R$ [19, Lemma C.36] (Appendix C (U5)). For $c \in \mathbb{R}^4$ and $h > 0$ let $Q(c, h) = c + [-h, h]^4$, and call the members with $\eta \in Q(c, h)$ the members of the *cube* $Q(c, h)$. The closed cubes $Q((2m + 1)h, h)$, $m \in \mathbb{Z}^4$, that meet the ball $\|\eta\| \leq r_H$ contain every member, and the 16 cubes $Q(c \pm h/2, h/2)$ cover $Q(c, h)$. A cube is described more sharply from a Root certificate of its own. Here Π is the projection of $\mathbb{R}^{36}$ onto the horizontal matrices (Section 3.3).

Lemma 6.7. *Let $Q(c,h)$ be a cube of R, and let $(P_s, T_s, n_{0,s}, \iota_s, C_s, \omega_s, \dots)$ be a certified Root certificate with $k=4$; put $n_s(t) = n_{0,s} + \iota_s t + C_s t^2$, $\tau_s = (I-\Pi)(P-P_s)$, $c_0 = c - T^{\mathsf T}\Pi(P_s - P)$, $M = T^{\mathsf T}T_s$ and $g = \|M^{\mathsf T}M - I\|$, and assume $g<1$. Call $x>0$* admissible *if $\|\Pi(P+\delta-P_s)\| \le x$ for every member δ of the cube. Then:*

(a) *$x_0 = (1+\omega)^{1/2} h\sqrt{4} + n_R + \|P + Tc - P_s\|$ is admissible;*

(b) *if x is admissible, so is $x' = ((1+\omega_s)a^2 + n_s(x)^2)^{1/2}$, where $a = (1-g)^{-1/2}(2h + \|c_0\| + (1+\omega)^{1/2} n_s(x))$;*

(c) *if x is admissible, every member of the cube satisfies $P+\delta = P_s + \delta'' + \tau_s$, where δ'' is horizontal, $e(P_s+\delta'') \in \mathcal{H}$, $\|\delta''\| \le x$ and $\delta'' = T_s\eta' + \nu''$ with $\|\eta'\| \le a$ and $\|\nu''\| \le n_s(x)$.*

Proof. Let δ be a member, $\eta = T^{\mathsf T}\delta$, $\nu = (I - TT^{\mathsf T})\delta$, and $\delta'' = \Pi(P+\delta-P_s)$. Since δ is horizontal, $P+\delta-P_s = \delta'' + \tau_s$ with τ_s a sum of row and column phases, so $e(P_s+\delta'')$ differs from $e(P+\delta)$ by diagonal unitary factors and is complex Hadamard. (a) $P+\delta-P_s = T(\eta - c) + \nu + (P + Tc - P_s)$, $\|\eta - c\| \le 2h$ and $\|T\| \le (1+\omega)^{1/2}$ [19, Lemma C.32] (Appendix C (U4)). (b), (c) Let $\|\delta''\| \le x$. The Root certificate at P_s gives $\nu'' = (I - T_sT_s^{\mathsf T})\delta''$ with $\|\nu''\| \le n_s(x)$; with $\eta' = T_s^{\mathsf T}\delta''$, $\delta'' = T_s\eta' + \nu''$, so $T^{\mathsf T}\delta'' = M\eta' + T^{\mathsf T}\nu''$, while $T^{\mathsf T}\delta'' = T^{\mathsf T}\delta + T^{\mathsf T}\Pi(P-P_s) = (\eta - c) + c_0$. By Weyl's inequality the smallest singular value of M is at least $(1-g)^{1/2}$, so $\|\eta'\| \le a$, and the second inequality of the Root certificate [19, (A.3)] (Appendix C (U1)) gives $\|\delta''\|^2 \le (1+\omega_s)a^2 + n_s(x)^2$. □

Starting from x_0 and replacing x by $\min(x, x')$ a few times gives a small admissible x; we call P_s with the resulting bounds a *sub-root* of the cube. Stage C applies to a cube with a sub-root (Lemma 6.8).

The tests of Stage C. We restate, up to notation and scaling, the bin tests (1)–(3), the edge tests and the joint test that the Stage C program of [19] makes [19, Appendix A.7]; the correspondence is listed in the deposit. The data are (P, T, r, r_H, n, K): a real 6×6 matrix P, a matrix $T \in \mathbb{R}^{36\times k}$ with $k \ge 0$ and numbers $r, r_H, n, K \ge 0$; when $k=0$, $\mathbb{R}^0 = \{0\}$, the empty vector has norm 0, and $T\eta$, $B\eta$ and $\|B^{\mathsf T}y\|$ below are 0. A real 6×6 matrix δ is a vector of $\mathbb{R}^{36}$, with the Euclidean norm $\|\delta\|$; the row of T with index (j,i) belongs to the entry (j,i), and $\delta_{:,i}$ is column i of δ. For a complex matrix, $\|\cdot\|$ is the operator norm. We write $\mathbf{1} = (1,\dots,1) \in \mathbb{R}^6$. In the tests and in Lemma 6.8 and its proof, X, L and I are numbers, M is a resolution and s a radius; the identity matrix occurs only in $6I$ and $I-\Pi$, and a subscript s marks a sub-root's data. For a center $u \in \mathbb{R}^6$ put

$$q_{ji} = \frac{e(u_j - P_{ji})}{\sqrt{6}}, \quad S_i = \sum_j q_{ji}, \quad c_i = \frac{|S_i|^2 - 1}{4\pi},$$
$$J_{ij} = -\operatorname{Im}(\overline{S_i}\, q_{ji}), \quad B_{il} = \sum_j J_{ij} T_{(j,i),l},$$

and for $y \in \mathbb{R}^6$ and $X, I \ge 0$ put $G_y = \max_j |\sum_i y_i \overline{S_i} q_{ji}|$, $N_y = (\sum_{i,j} y_i^2 J_{ij}^2)^{1/2}$, $S\circ y = (S_i y_i)_i$ and

$$\mathcal{E}(y; X, I) = nN_y + \pi\Big[\|y\|_\infty (KX + r)^2 + \min\{G_y X^2,\, K\|S\circ y\| I X\} + \tfrac{2}{\sqrt{6}}\|S\circ y\| X r + \tfrac{1}{\sqrt{6}}\|S\circ y\|_\infty r^2\Big].$$

A *bin* of resolution $M \geq 1$ is the set of $\beta \in \mathbb{R}^6$ with $\beta_0 = 0$ and $\beta_j \in [b_j/M, (b_j+1)/M]$ for $j = 1, \dots, 5$, where $b \in \{0, \dots, M-1\}^5$; its midpoint u has $u_0 = 0$, its half side is $\varepsilon = 1/(2M)$, and we put $X_M = (29/6)^{1/2}\varepsilon$, $L_M = 5\varepsilon$ and $I_M = \frac{3}{2}\varepsilon$. A vector *lies in the bin* if it is $\lambda e(\beta)/\sqrt{6}$ with $|\lambda| = 1$ and β in the bin. With the quantities formed at its midpoint, a bin *passes* the bin tests if

(SC1) $\max\{0, 1-\vartheta\}^2 \leq |S_i|^2 \leq (1+\vartheta)^2$ for $0 \leq i \leq 5$, where $\vartheta = 2\pi(L_M/\sqrt{6} + r)$, and $\sum_i (|S_i| - 1)^2 \leq (2\pi(KX_M + r))^2$;

(SC2) for each i, $|c_i| \leq \varepsilon \sum_{j>0} |J_{ij}| + r\|J_{i,:}\| + \rho$, where, with $X = X_M$, $L = L_M$, $I = I_M$ and $S_{\max} = \max_i |S_i|$,

$$\rho = \min\Big\{\pi\Big[(KX + r)^2 + S_{\max}\Big(KIX + \frac{2Xr + r^2}{\sqrt{6}}\Big)\Big],\ \pi\Big[\Big(\frac{L}{\sqrt{6}} + r\Big)^2 + \frac{S_{\max}(X+r)^2}{\sqrt{6}}\Big]\Big\};$$

(SC3) $|y \cdot c| \leq \varepsilon \sum_{j>0} |(J^{\mathsf{T}} y)_j| + r_H \|B^{\mathsf{T}} y\| + \mathcal{E}(y; X_M, I_M)$ for every $y \in \mathbb{R}^6$.

For two vector balls (Section 5.1) with centers u, v and radii s_u, s_v put $Q_j = e(u_j - v_j)$, $S_e = \sum_j Q_j$, $r_e = s_u + s_v$, let $D_{\mathcal{O}} \in \mathbb{R}^{2\times 6}$ have the columns $(-\operatorname{Im} Q_j, \operatorname{Re} Q_j)^{\mathsf{T}}$, let $D_{\mathcal{U}} \in \mathbb{R}^{1\times 6}$ have the entries $(\operatorname{Re} S_e\,(D_{\mathcal{O}})_{1j} + \operatorname{Im} S_e\,(D_{\mathcal{O}})_{2j})/6$, and put $q_{\mathcal{O}} = (\operatorname{Re} S_e, \operatorname{Im} S_e)^{\mathsf{T}}/(2\pi)$, $q_{\mathcal{U}} = (|S_e|^2/6 - 1)/(4\pi)$, $E_{\mathcal{O}} = \pi r_e^2$, $E_{\mathcal{U}} = \pi r_e^2(1 + |S_e|/6)$, $\ell_{\mathcal{O}} = 2$ and $\ell_{\mathcal{U}} = 1$. The pair *passes* the edge tests for orthogonality ($E = \mathcal{O}$) or for unbiasedness ($E = \mathcal{U}$) if

(SC4) $|S_e| \leq 2\pi\sqrt{6}\, r_e$ and $|S_e|^2/(2\pi) \leq 6r_e\|D_{\mathcal{U}}\| + E_{\mathcal{O}}|S_e|$ (for $E = \mathcal{O}$), respectively $\big||S_e| - \sqrt{6}\big| \leq 2\pi\sqrt{6}\, r_e$ and $|q_{\mathcal{U}}| \leq r_e\|D_{\mathcal{U}}\| + E_{\mathcal{U}}$ (for $E = \mathcal{U}$);

(SC5) for all $y_0, y_1 \in \mathbb{R}^6$ and $y_e \in \mathbb{R}^{\ell_E}$,

$$\begin{aligned}|y_0 \cdot c^0 + y_1 \cdot c^1 + y_e \cdot q_E| \leq \sum_{t=0}^{1}\Big[s_t\|(J^t)^{\mathsf{T}} y_t + (-1)^t D_E^{\mathsf{T}} y_e\| + \mathcal{E}_t^{\mathrm{ball}}(y_t)\Big]\\ + r_H\|(B^0)^{\mathsf{T}} y_0 + (B^1)^{\mathsf{T}} y_1\| + E_E\|y_e\|,\end{aligned}$$

where $(v_0, s_0) = (u, s_u)$, $(v_1, s_1) = (v, s_v)$, the quantities $c^t, J^t, B^t, S^t, q^t, G^t, N^t$ are formed at the center v_t, and $\mathcal{E}_t^{\mathrm{ball}}(y)$ is $\mathcal{E}(y; s_t, \cdot)$, formed at v_t, with $\min\{G_y X^2, \dots\}$ replaced by $G_y^t s_t^2$.

Here (SC4) collects the four inequalities of the edge tests, and (SC5) is the joint test. The program checks (SC3) and (SC5) in three directions each: for (SC3) $y = c$ and two solutions of regularized least-squares problems, for (SC5) three such solutions [19, Section 3.7 and Appendix A.7]; any directions would do, since the inequalities that Lemma 6.8 (b) and (d) give hold for every y, so a failure in one direction excludes the bin or the pair of balls.

Lemma 6.8. *Let (P, T, r, r_H, n, K) be as above with $\|e(P)/\sqrt{6}\| \leq K$, and let $\mathcal{D}$ be a set of real 6×6 matrices each of which satisfies $\|\delta\| \leq r$ and $\delta = T\eta + \nu$ for some $\eta \in \mathbb{R}^k$ and $\nu \in \mathbb{R}^{36}$ with $\|\eta\| \leq r_H$ and $\|\nu\| \leq n$. For $\delta \in \mathcal{D}$ let $\mathcal{V}(\delta)$ be the set of unit vectors of $\mathbb{C}^6$ unbiased to the standard basis and to the columns of $e(P+\delta)/\sqrt{6}$. Then:*

(a) *every unit vector unbiased to the standard basis is $\lambda e(\beta)/\sqrt{6}$ with $|\lambda| = 1$, $\beta_0 = 0$ and $\beta_1, \dots, \beta_5 \in [0, 1)$; for each $M \geq 1$ it lies in the bin $b_j = \lfloor M\beta_j \rfloor$ of resolution M, and $\lfloor 2M\beta_j \rfloor \in \{2b_j, 2b_j + 1\}$;*

(b) *if a vector of $\mathcal{V}(\delta)$, $\delta \in \mathcal{D}$, lies in a bin, the bin passes (SC1)–(SC3);*

(c) *if a bin of resolution M with midpoint u is assigned $v \in \mathbb{R}^6$, $m \in \mathbb{Z}^6$ and $s \geq \|h\| + X_M$, where $h = u - v - m - \frac{1}{6}\sum_j (u_j - v_j - m_j)\mathbf{1}$, then $\mathcal{B}_{\mathrm{vec}}(v, s)$ contains every vector that*

lies in the bin; two vectors of one ball $\mathcal{B}_{\mathrm{vec}}(v,s)$ *with* $s<0.1$ *are neither orthogonal nor unbiased;*

(d) *if* $f\in\mathcal{B}_{\mathrm{vec}}(u,s_u)$ *and* $f'\in\mathcal{B}_{\mathrm{vec}}(v,s_v)$ *lie in* $\mathcal{V}(\delta)$ *for the same* $\delta\in\mathcal{D}$ *and are orthogonal, respectively unbiased, the pair of balls passes* (SC4) *and* (SC5) *for* $E=\mathcal{O}$*, respectively* $E=\mathcal{U}$*.*

The hypotheses hold, so (a)–(d) apply,

(i) *for the members* $e(P+\delta)$ *of any record, with any number* $k\ge 0$ *of columns, with its center* P*, its frame* $T\in\mathbb{R}^{36\times k}$*, its radius* r *and the constants* $n_0,\iota,C,\omega,\varphi$ *of its Root certificate, where* $\|F(P)\|\le\varphi$ [19, Sections 2.2 and 2.4]*, with* $r_H=(1+\omega)^{1/2}r$*,* $n=n_0+\iota r+Cr^2$ *(which is* n_R *when* $k=4$*) and* $K=(1+2\pi\sqrt{2}\,\varphi/6)^{1/2}$*, and* $\mathcal{D}$ *the set of horizontal* δ *with* $\|\delta\|\le r$ *and* $e(P+\delta)\in\mathcal{H}$*;*

(ii) *for the members of a cube of the record* R *of this subsection* ($k=4$) *with a sub-root as in Lemma 6.7, written* $e(P_s+\delta')$ *with* $\delta'=P+\delta-P_s$*, with the center* P_s*, the frame* T_s*,* $r_H=a$*,* $n=n_s(x)+\|\tau_s\|$*,* $r=(x^2+\|\tau_s\|^2)^{1/2}$ *and* $K=(1+\|e(P_s)e(P_s)^*-6I\|_F/6)^{1/2}$*, where* x *is the admissible number and* a *the bound of Lemma 6.7, and* $\mathcal{D}$ *the set of these* δ'*.*

Nothing in (a)–(d) uses that δ is horizontal, that $e(P+\delta)$ is complex Hadamard, that $\eta=T^{\mathsf{T}}\delta$, or that ν is orthogonal to the columns of T.

Proof. Elementary bounds. For real t, $|e(t)-1|=2|\sin\pi t|\le 2\pi|t|$, and $R_2(t)=e(t)-1-2\pi it=\int_0^t 2\pi i\,(e(\sigma)-1)\,d\sigma$ satisfies $|R_2(t)|\le\int_0^{|t|}4\pi^2\sigma\,d\sigma=2\pi^2t^2$. Fix a center u, a vector $x\in\mathbb{R}^6$ and a real 6×6 matrix δ, and put $S_i'=\sum_j q_{ji}e(x_j-\delta_{ji})$ and $\Delta_i=S_i'-S_i$. Let Q be the linear map of $\mathbb{C}^6$ with $(Qw)_i=\sum_j q_{ji}w_j$. Then $Q=e(P)^*\,\mathrm{diag}(e(u))/\sqrt{6}$, so $\|Q\|=\|e(P)/\sqrt{6}\|\le K$, and $|q_{ji}|=1/\sqrt{6}$. The inner product of column i of $e(P+\delta)/\sqrt{6}$ with $e(u+x)/\sqrt{6}$ is $\frac{1}{6}\sum_j e(u_j+x_j-P_{ji}-\delta_{ji})=S_i'/\sqrt{6}$, so $\lambda e(u+x)/\sqrt{6}$ ($|\lambda|=1$) is unbiased to this column if and only if $|S_i'|=1$. Each row of J sums to $-\mathrm{Im}(\overline{S_i}S_i)=0$.

(a) and the vectors of a bin. A unit vector f unbiased to the standard basis has $|f_j|=1/\sqrt{6}$, so $f_j=e(\varphi_j)/\sqrt{6}$; take $\lambda=e(\varphi_0)$ and $\beta_j\in[0,1)$ congruent to $\varphi_j-\varphi_0$ modulo 1. Then $M\beta_j\in[b_j,b_j+1)$ with $0\le b_j\le M-1$, and $2M\beta_j\in[2b_j,2b_j+2)$. If β lies in a bin with midpoint u, write $\beta=u+d$, so $d_0=0$ and $|d_j|\le\varepsilon$, and $x=d-\frac{1}{6}(\sum_j d_j)\mathbf{1}$; then $e(\beta)=e(\frac{1}{6}\sum_j d_j)\,e(u+x)$, $\sum_j x_j=0$, and $\|x\|\le X_M$, $\|x\|_1\le L_M$, $\|x\|_\infty\le I_M$. Indeed, $\|x\|^2=\|d\|^2-(\sum_j d_j)^2/6$ and $\|x\|_1$ are convex in each of $d_1,\dots,d_5$, so on the box their maxima are attained at vertices, $|d_j|=\varepsilon$ for $j>0$. At a vertex $\sum_j d_j$ is an odd multiple of ε, so $\|x\|^2\le 5\varepsilon^2-\varepsilon^2/6$; if p coordinates equal ε and $5-p$ equal $-\varepsilon$, then $\|x\|_1=|\mu|+p|\varepsilon-\mu|+(5-p)|\varepsilon+\mu|$ with $\mu=(2p-5)\varepsilon/6$, which is $\frac{10}{6}\varepsilon,4\varepsilon,5\varepsilon,5\varepsilon,4\varepsilon,\frac{10}{6}\varepsilon$ for $p=0,\dots,5$. Finally $|x_j|=|\frac{5}{6}d_j-\frac{1}{6}\sum_{l\ne 0,j}d_l|\le\frac{9}{6}\varepsilon$ for $j>0$, and $|x_0|=\frac{1}{6}|\sum_j d_j|\le\frac{5}{6}\varepsilon$.

Bin test (SC1). Let $\|x\|\le X$, $\|x\|_1\le L$ and $\|\delta\|\le r$. Then $\Delta_i=\sum_j q_{ji}(e(x_j)-1)+\sum_j q_{ji}e(x_j)(e(-\delta_{ji})-1)$. The first sums are the entries of $Q(e(x)-\mathbf{1})$, a vector of norm at most $K\cdot 2\pi\|x\|\le 2\pi KX$, and each has modulus at most $\frac{1}{\sqrt{6}}\sum_j 2\pi|x_j|\le 2\pi L/\sqrt{6}$. The second sum has modulus at most $\frac{2\pi}{\sqrt{6}}\sum_j|\delta_{ji}|\le 2\pi\|\delta_{:,i}\|$, and these numbers form a vector of norm at most $2\pi\|\delta\|\le 2\pi r$. So

$$\|S'-S\|\le 2\pi(KX+r),\qquad |\Delta_i|\le 2\pi(L/\sqrt{6}+r).\tag{6.5}$$

If a vector of $\mathcal{V}(\delta)$ lies in the bin, it is $\lambda e(u+x)/\sqrt{6}$ with x as above, so $|S_i'|=1$ for all i, and $||S_i|-1|\le|\Delta_i|$ gives (SC1) by (6.5) with $X=X_M$, $L=L_M$.

An identity. Suppose $|S_i'| = 1$ and put $w_j = x_j - \delta_{ji}$. Then $\Delta_i = \sum_j q_{ji}(2\pi i w_j + R_2(w_j))$ and $\operatorname{Re}(\overline{S_i}\, 2\pi i q_{ji} w_j) = 2\pi J_{ij} w_j$; expanding $1 = |S_i + \Delta_i|^2$ and dividing by 4π gives

$$c_i + \sum_j J_{ij} x_j - \sum_j J_{ij}\delta_{ji} + \frac{1}{2\pi}\operatorname{Re}\Big(\overline{S_i}\sum_j q_{ji} R_2(w_j)\Big) + \frac{|\Delta_i|^2}{4\pi} = 0. \tag{6.6}$$

Since $e(x_j - \delta_{ji}) = e(x_j)e(-\delta_{ji})$, $R_2(w_j) = R_2(x_j) + R_2(-\delta_{ji}) + (e(x_j)-1)(e(-\delta_{ji})-1)$.

Bin test (SC2). Let x come from a vector of $\mathcal{V}(\delta)$ in the bin, as above, so $\|x\| \le X$, $\|x\|_1 \le L$ and $\|x\|_\infty \le I$. As the row $J_{i,:}$ sums to 0, $\sum_j J_{ij}x_j = \sum_{j>0} J_{ij} d_j$, of modulus at most $\varepsilon \sum_{j>0}|J_{ij}|$, and $|\sum_j J_{ij}\delta_{ji}| \le \|J_{i,:}\|\,\|\delta_{:,i}\| \le r\|J_{i,:}\|$. By (6.6) it remains to bound $\rho_i = |S_i|\,|\sum_j q_{ji}R_2(w_j)|/(2\pi) + |\Delta_i|^2/(4\pi)$ by both numbers of which ρ is the minimum. First, $|\Delta_i| \le \|S' - S\| \le 2\pi(KX + r)$, and $|\sum_j q_{ji}R_2(w_j)| \le \|Q(R_2(x_j))_j\| + \frac{2\pi^2}{\sqrt{6}}\sum_j \delta_{ji}^2 + \frac{4\pi^2}{\sqrt{6}}\sum_j |x_j\delta_{ji}| \le 2\pi^2 KIX + \frac{2\pi^2}{\sqrt{6}}(r^2 + 2Xr)$, using $\|Q(R_2(x_j))_j\| \le K\|(R_2(x_j))_j\| \le K\,2\pi^2(\sum_j x_j^4)^{1/2} \le 2\pi^2 K\|x\|_\infty\|x\|$, $\sum_j \delta_{ji}^2 \le r^2$ and $\sum_j |x_j\delta_{ji}| \le \|x\|\,\|\delta_{:,i}\| \le Xr$; so $\rho_i \le \pi(KX + r)^2 + \pi S_{\max}(KIX + (2Xr + r^2)/\sqrt{6})$. Second, $|\Delta_i| \le 2\pi(L/\sqrt{6} + r)$ by (6.5), and $|\sum_j q_{ji}R_2(w_j)| \le \frac{2\pi^2}{\sqrt{6}}\|x - \delta_{:,i}\|^2 \le \frac{2\pi^2}{\sqrt{6}}(X + r)^2$; so $\rho_i \le \pi(L/\sqrt{6} + r)^2 + \pi S_{\max}(X + r)^2/\sqrt{6}$.

Bin test (SC3). Let $\delta \in \mathcal{D}$, $\delta = T\eta + \nu$ as in the hypothesis, and let $\lambda e(u + x)/\sqrt{6} \in \mathcal{V}(\delta)$ with $\|x\| \le X$. Since $\sum_j J_{ij}(T\eta)_{ji} = \sum_l \sum_j J_{ij}T_{(j,i),l}\eta_l = (B\eta)_i$, we have $\sum_j J_{ij}\delta_{ji} = (B\eta)_i + \sum_j J_{ij}\nu_{ji}$; multiplying (6.6) by y_i and summing over i,

$$y\cdot(c + Jx - B\eta) = \sum_{i,j} y_i J_{ij}\nu_{ji} - \frac{1}{2\pi}\sum_i y_i \operatorname{Re}\Big(\overline{S_i}\sum_j q_{ji}R_2(w_j)\Big) - \sum_i y_i \frac{|\Delta_i|^2}{4\pi}.$$

The first sum has modulus at most $N_y\|\nu\| \le nN_y$ (Cauchy–Schwarz), and the last at most $\|y\|_\infty\|S' - S\|^2/(4\pi) \le \pi\|y\|_\infty(KX + r)^2$ by (6.5). Split $R_2(w_j)$ as above. The part with $R_2(x_j)$ is $\frac{1}{2\pi}\operatorname{Re}\sum_j(\sum_i y_i\overline{S_i}q_{ji})R_2(x_j)$, of modulus at most $\frac{1}{2\pi}G_y\sum_j 2\pi^2 x_j^2 \le \pi G_y X^2$; it is also $\frac{1}{2\pi}\operatorname{Re}\sum_i y_i\overline{S_i}\,(Q(R_2(x_j))_j)_i$, of modulus at most $\frac{1}{2\pi}\|S\circ y\|\,K\,2\pi^2\|x\|_\infty\|x\| \le \pi K\|S\circ y\|IX$ if $\|x\|_\infty \le I$. The part with $R_2(-\delta_{ji})$ is at most $\frac{1}{2\pi}\sum_i |y_iS_i|\frac{2\pi^2}{\sqrt{6}}\|\delta_{:,i}\|^2 \le \frac{\pi}{\sqrt{6}}\|S\circ y\|_\infty r^2$, and the product part at most $\frac{1}{2\pi}\sum_i |y_iS_i|\frac{4\pi^2}{\sqrt{6}}\sum_j |x_j\delta_{ji}| \le \frac{2\pi}{\sqrt{6}}X\sum_i |y_iS_i|\,\|\delta_{:,i}\| \le \frac{2\pi}{\sqrt{6}}\|S\circ y\|Xr$. Hence, for every $y \in \mathbb{R}^6$,

$$|y\cdot(c + Jx - B\eta)| \le \mathcal{E}(y; X, I) \quad \text{if } \|x\|_\infty \le I, \qquad |y\cdot(c + Jx - B\eta)| \le \mathcal{E}^{(1)}(y; X) \quad \text{always}, \tag{6.7}$$

where $\mathcal{E}^{(1)}(y; X)$ is $\mathcal{E}(y; X, \cdot)$ with $\min\{G_yX^2, \dots\}$ replaced by G_yX^2. For a vector in a bin, x is as in (a), so $\sum_j (J^{\mathsf{T}}y)_j = 0$ gives $y\cdot Jx = \sum_{j>0}(J^{\mathsf{T}}y)_j d_j$, of modulus at most $\varepsilon\sum_{j>0}|(J^{\mathsf{T}}y)_j|$, and $|y\cdot B\eta| \le \|B^{\mathsf{T}}y\|\,\|\eta\| \le r_H\|B^{\mathsf{T}}y\|$; with (6.7) for $X = X_M$, $I = I_M$ this is (SC3).

(c). Let $\beta = u + d$ lie in the bin and x be as in (a), and put $\mu = \frac{1}{6}\sum_j(u_j + d_j - v_j - m_j)$. Then $u + d = v + m + (h + x) + \mu\mathbf{1}$, so $\lambda e(\beta)/\sqrt{6} = \lambda e(\mu)\,e(v + h + x)/\sqrt{6}$, and $h + x$ has coordinate sum 0 and norm at most $\|h\| + X_M \le s$. Two vectors $\lambda e(v+x)/\sqrt{6}$, $\lambda' e(v+x')/\sqrt{6}$ of $\mathcal{B}_{\mathrm{vec}}(v, s)$ have an inner product of modulus $|\sum_j e(z_j)|/6$ with $z = x - x'$, $\|z\| \le 2s < 0.2$, and $|\sum_j e(z_j)| \ge 6 - 2\pi\|z\|_1 \ge 6 - 2\pi\sqrt{6}\,\|z\| > 6 - 0.4\pi\sqrt{6} > 6 - 0.4\cdot\frac{22}{7}\cdot\frac{5}{2} = \frac{20}{7} > \sqrt{6}$; orthogonality would need $\sum_j e(z_j) = 0$ and unbiasedness $|\sum_j e(z_j)| = \sqrt{6}$.

(d), test (SC4). Write $f = \lambda e(u+x)/\sqrt{6}$ and $f' = \lambda' e(v+x')/\sqrt{6}$ with $\sum_j x_j = \sum_j x_j' = 0$, $\|x\| \le s_u$, $\|x'\| \le s_v$, and put $z = x - x'$, so $\|z\| \le r_e$. Their inner product has modulus $|S_e'|/6$ with $S_e' = \sum_j Q_j e(z_j)$, and $|S_e' - S_e| \le 2\pi\|z\|_1 \le 2\pi\sqrt{6}\,r_e$. Orthogonality means $S_e' = 0$,

which gives $|S_e| \le 2\pi\sqrt{6}\, r_e$; unbiasedness means $|S'_e| = \sqrt{6}$, which gives $\big||S_e| - \sqrt{6}\big| \le 2\pi\sqrt{6}\, r_e$. Put $R_e = \sum_j Q_j R_2(z_j)$, so $S'_e = S_e + 2\pi i \sum_j Q_j z_j + R_e$ and $|R_e| \le 2\pi^2 \|z\|^2 \le 2\pi^2 r_e^2$. In $\mathbb{R}^2 = \mathbb{C}$ the number iQ_jz_j is $(D_{\mathcal{O}})_{:,j} z_j$. If $S'_e = 0$, dividing by 2π gives $q_{\mathcal{O}} + D_{\mathcal{O}} z = -R_e/(2\pi)$, so

$$\|q_{\mathcal{O}} + D_{\mathcal{O}} z\| \le \pi r_e^2 = E_{\mathcal{O}}; \tag{6.8}$$

taking the inner product with $(\operatorname{Re} S_e, \operatorname{Im} S_e)$, which maps $q_{\mathcal{O}}$ to $|S_e|^2/(2\pi)$ and $D_{\mathcal{O}} z$ to $6D_{\mathcal{U}} z$, gives $|S_e|^2/(2\pi) \le 6|D_{\mathcal{U}} z| + \pi r_e^2 |S_e| \le 6 r_e \|D_{\mathcal{U}}\| + E_{\mathcal{O}} |S_e|$. If $|S'_e|^2 = 6$, then

$$0 = \frac{|S'_e|^2/6 - 1}{4\pi} = q_{\mathcal{U}} + \frac{\operatorname{Re}(\overline{S_e}(S'_e - S_e))}{12\pi} + \frac{|S'_e - S_e|^2}{24\pi},$$

and $\operatorname{Re}(\overline{S_e}\, 2\pi i \sum_j Q_j z_j)/(12\pi) = D_{\mathcal{U}} z$, since $-\operatorname{Im}(\overline{S_e} Q_j) = -\operatorname{Re} S_e \operatorname{Im} Q_j + \operatorname{Im} S_e \operatorname{Re} Q_j = 6(D_{\mathcal{U}})_j$. The other terms are at most $|S_e||R_e|/(12\pi) \le \pi |S_e| r_e^2/6$ and $(2\pi\sqrt{6}\, r_e)^2/(24\pi) = \pi r_e^2$, so

$$|q_{\mathcal{U}} + D_{\mathcal{U}} z| \le E_{\mathcal{U}}, \tag{6.9}$$

and $|q_{\mathcal{U}}| \le |D_{\mathcal{U}} z| + E_{\mathcal{U}} \le r_e \|D_{\mathcal{U}}\| + E_{\mathcal{U}}$.

(d), test (SC5). Take $\delta = T\eta + \nu$ as in the hypothesis, one decomposition for both vectors, and write $x_0 = x$, $x_1 = x'$. By (6.7) at the center v_t, with $X = s_t$, $|y_t \cdot (c^t + J^t x_t - B^t \eta)| \le \mathcal{E}_t^{\text{ball}}(y_t)$ for $t = 0, 1$, and by (6.8), (6.9) $|y_e \cdot (q_E + D_E(x_0 - x_1))| \le E_E \|y_e\|$. The sum of the three quantities inside the absolute values is $y_0 \cdot c^0 + y_1 \cdot c^1 + y_e \cdot q_E + ((J^0)^{\mathsf{T}} y_0 + D_E^{\mathsf{T}} y_e) \cdot x_0 + ((J^1)^{\mathsf{T}} y_1 - D_E^{\mathsf{T}} y_e) \cdot x_1 - ((B^0)^{\mathsf{T}} y_0 + (B^1)^{\mathsf{T}} y_1) \cdot \eta$, and $\|x_t\| \le s_t$, $\|\eta\| \le r_H$; this gives (SC5).

The two cases. (i) For a member, δ is horizontal, $\|\delta\| \le r$ and $e(P + \delta) \in \mathcal{H}$, which is the condition $F(P + \delta) = 0$ of [19, Section 2.2], so $\eta = T^{\mathsf{T}} \delta$ and $\nu = \delta - T\eta$ satisfy $\|\eta\| \le r_H$ and $\|\nu\| \le n$, and $\|e(P)/\sqrt{6}\| \le K$ [19, Lemma C.36 (i), (ii)]. (ii) By Lemma 6.7 (c), $P + \delta = P_s + \delta'$ with $\delta' = \delta'' + \tau_s$, so $e(P + \delta) = e(P_s + \delta')$ is the same matrix, with the same unbiased vectors, and $\delta' = T_s \eta' + (\nu'' + \tau_s)$ with $\|\eta'\| \le a$ and $\|\nu'' + \tau_s\| \le n_s(x) + \|\tau_s\|$. As δ'' is horizontal and $\tau_s = (I - \Pi)(P - P_s)$ lies in the orthogonal complement of the horizontal space (Section 3.3), $\|\delta'\|^2 = \|\delta''\|^2 + \|\tau_s\|^2 \le x^2 + \|\tau_s\|^2$. Finally, for a unit vector w, $\|e(P_s)^* w\|^2 = 6 + w^*(e(P_s) e(P_s)^* - 6I) w \le 6 + \|e(P_s) e(P_s)^* - 6I\|_F$, so $\|e(P_s)/\sqrt{6}\| = \|e(P_s)^*\|/\sqrt{6} \le K$. □

A *cube cover* of R to level L consists of the following. Stage C is first run on R with the schedule of Section 4.3. If it ends with no six-clique, no member has a third basis (Lemma 4.2). Otherwise the cubes of level 1 are the cubes $Q((2m+1)h_1, h_1)$, $h_1 = 2^{-8}$, that meet the ball $\|\eta\| \le r_H$. On each cube, with a sub-root, the bins kept at the last step of the run on R are tested again with the data of the sub-root, as in Lemma 6.8, grouped into balls with target radius 0.042, and joined by edges; a cube whose graph has no six-clique is *excluded*. A cube of level $\ell < L$ that is not excluded is split into those of its 16 half-cubes of level $\ell + 1$ that meet the ball, which start from its own kept bins; a cube of level L that is not excluded is a *candidate*, and its kept bins are grouped into *components*, the connected components of the relation "the bins touch", each with the smallest box of phase vectors containing its bins.

Lemma 6.9. *Let a cube cover of R to level L be given. Every member of R lies in a cube excluded at some level or in a candidate cube. Members of excluded cubes have no third basis, and for a member H of a candidate cube, every line of every third basis of $\{I, H/\sqrt{6}\}$, written $e(u)/\sqrt{6}$ with $u_0 = 0$, has u in the box of a component modulo $\mathbb{Z}^5$.*

Proof. The cubes of each level cover the members of their parents, so by induction every member lies in a cube of every level until one is excluded. By Lemma 4.2 the bins kept at the last step on R contain every line of every third basis of every member. By Lemma 6.8, the argument of Lemma 4.2 applies on a cube with its sub-root: the retested bins containing these lines are kept, they lie in distinct balls pairwise joined, so an excluded cube has no member with a third basis, and the bins handed to the half-cubes, and finally to the components, still contain the lines. □

6.5. **The interface on a cell.** On a cell of a candidate cube, the lines that a third basis can contain are uniquely continued zeros of the unbiasedness equations; a graph records which pairs of them can be orthogonal, and a third basis gives a six-clique of this graph; a cell is excluded if the graph has none, and decided if the triplet of each six-clique is placed in a tube of Proposition 6.2. We first write the members of a cube as a family over parameters, in a frame anchored at a point of the Fourier family.

Fix a cube $Q(c,h)$ of R with a sub-root as in Lemma 6.7, and put $W = T_s M^{-1}$ and $m' \geq \|T^{\mathsf{T}}(I - T_s T_s^{\mathsf{T}})\|$.

Lemma 6.10. *Every member of the cube satisfies $P + \delta = P_s + W(\Delta + c_0) + \varsigma + \tau_s$ with $\Delta = \eta - c \in [-h,h]^4$ and*

$$\|\varsigma\| \leq n_s(x)\big(1 + \|W\| m'\big) + \|W\|\,\|T\|\,\|T_s\|^2\,\omega_s\,x.$$

Proof. In the notation of the proof of Lemma 6.7, $\eta' = M^{-1}(\Delta + c_0 - T^{\mathsf{T}}\nu'')$, so $\delta'' = W(\Delta + c_0) + \varsigma$ with $\varsigma = \nu'' - WT^{\mathsf{T}}\nu''$. From $(I - T_sT_s^{\mathsf{T}}) - (I - T_sT_s^{\mathsf{T}})^2 = T_s(I - T_s^{\mathsf{T}}T_s)T_s^{\mathsf{T}}$ we get $T^{\mathsf{T}}\nu'' = T^{\mathsf{T}}(I - T_sT_s^{\mathsf{T}})\nu'' + T^{\mathsf{T}}T_s(I - T_s^{\mathsf{T}}T_s)T_s^{\mathsf{T}}\delta''$, which gives the bound. □

So the members of the cube, as matrices, form a family $e(P_s + W(\Delta + c_0) + \varsigma)$, up to the fixed row and column phases τ_s, over the *parameters* (Δ, ς) in the product of the cube and a ball, the *slab*. Fix an *anchor*, a binary64 point s_A of the family, and let $H_A = H(d(s_A)) = e(P_A)$. Dephasing $P_s - P_A$ at row and column 0, every member $e(P + \delta) = e(P_s + \delta'' + \tau_s)$ equals $H_A \circ \mathrm{e}^{i\varphi}$ up to row and column phases that are the same for all members of the cube, with $\varphi = 2\pi\big(D_0(P_s - P_A) + \delta''\big)$ affine in (Δ, ς), where $(D_0X)_{jk} = X_{jk} - X_{j0} - X_{0k} + X_{00}$. Row phases move the phase vector of every line by one constant vector, so the lines of a member correspond to the zeros θ of $R(\cdot; H_A \circ \mathrm{e}^{i\varphi})$ by $u = \theta/2\pi + \sigma_0$ modulo $\mathbb{Z}^5$, with $\sigma_0 = \big((P_s + \tau_s)_{l0} - (P_s + \tau_s)_{00}\big)_{l=1}^{5}$ the same for all members (enclosed in ball arithmetic), and its third bases to those of $H_A \circ \mathrm{e}^{i\varphi}$ by a diagonal unitary map, which commutes with Z. We call this description of the members, their lines and their third bases the *anchor frame*.

Let $Q(c_*, h_*)$ be a candidate cube of R, with its components, as in Lemma 6.9. A *cell* of $Q(c_*, h_*)$, when this cube has a sub-root, consists of the cube $Q(c,h) = Q(c_*, h_*)$, the *sub-root cube* of the cell, with its anchor frame as above, and a box $\mathcal{C}$ of parameters Δ, by which we also name the cell; in tangent coordinates the cell's *box* is $c + \mathcal{C}$. The cell *holds* a point $\eta \in \mathbb{R}^4$ if $\eta \in Q(c,h) \cap (c + \mathcal{C})$, and it holds a member of R if it holds the member's tangent coordinate and that coordinate lies in $Q(c_*, h_*)$; for such a member, Lemma 6.10 applied to $Q(c,h)$ gives parameters (Δ, ς) with $\Delta \in \mathcal{C}$ and ς in the slab. A finite set of cells of $Q(c_*, h_*)$ *holds the cube* if every member of R whose tangent coordinate lies in $Q(c_*, h_*)$

is held by one of them. An *interface certificate* on $\mathcal{C}$, for all (Δ, ς) with $\Delta \in \mathcal{C}$ and ς in the slab, consists of a finite set $J_{\mathcal{C}}$ of indices, those of the cell's *branches*, and:

(C1) *branches*: boxes $X_j \subset \mathbb{R}^5$, $j \in J_{\mathcal{C}}$, which may move with the parameters, on which Lemma 6.3, applied to the offset from the center, certifies a unique zero $\theta_j(\Delta, \varsigma)$ of the map $R(\cdot; H_A \circ \mathrm{e}^{i\varphi})$, given by the formulas of Section 6.3 also where $H_A \circ \mathrm{e}^{i\varphi}$ is not complex Hadamard;

(C2) *uniqueness radii* $\varrho_j > 0$, $j \in J_{\mathcal{C}}$, for which Lemma 6.6 holds at $\theta_j(\Delta, \varsigma)$ at every parameter at which $H_A \circ \mathrm{e}^{i\varphi}$ is complex Hadamard, with bounds uniform over the parameters;

(C3) *covering*: each component box of $Q(c_*, h_*)$, transported by $u \mapsto 2\pi(u - \sigma_0)$ and lifted by multiples of 2π, is covered by finitely many boxes each of which either, for some $j \in J_{\mathcal{C}}$, lies within distance ϱ_j of every point of a box that contains $\theta_j(\Delta, \varsigma)$ for all parameters, or contains no zero of $R(\cdot; H_A \circ \mathrm{e}^{i\varphi})$ for any parameter at which $H_A \circ \mathrm{e}^{i\varphi}$ is complex Hadamard;

(C4) *pairs*: a set of pairs $\{j, j'\} \subset J_{\mathcal{C}}$ for which $\langle w(\theta_j), w(\theta_{j'})\rangle \neq 0$ is certified for all parameters, and Γ is the graph on $J_{\mathcal{C}}$ whose edges are the other pairs;

and an *outcome*: *excluded*, if Γ has no six-clique, or *decided*, if for every six-clique $\mathcal{J}$ of Γ there are a family triplet $\{I, H(d(s_1))/\sqrt{6}, K(v_1)/\sqrt{6}\}$, for a binary64 point s_1 of the family and the zero v_1 certified as at the end of Section 6.2, with certified constants, and a bijection μ from $\mathcal{J}$ to the columns of $K(v_1)$, such that, for all parameters, the right side of (6.4) is smaller than σ for the phase vector $x = (\varphi - \varphi_F, \psi)$, its entries taken with some choice of representatives modulo 2π. Here φ_F is the exact phase vector with $H_A \circ \mathrm{e}^{i\varphi_F} = H(d(s_1))$, $(\varphi_F)_{(a,i),(h,k)} = h(\arg d_i(s_1) - \arg d_i(s_A))$, and $\psi_{r,\mu(j)} = \theta_{j,r} - \arg K(v_1)_{r,\mu(j)}$ for $j \in \mathcal{J}$ and $r = 0, \ldots, 5$, with $\theta_{j,0} = 0$, so that $K(v_1) \circ \mathrm{e}^{i\psi}$ has column $\mu(j)$ equal to $w(\theta_j)$. Interface certificates are proposed by the *interface program* and checked by the interface checker of Appendix A.3, and an *interface run* is a run of the two on a cube.

Proposition 6.11. *Let $\mathcal{C}$ be a cell of a candidate cube $Q(c_*, h_*)$, and let H be a member of R held by $\mathcal{C}$, with parameters (Δ, ς).*

(a) *If $\mathcal{C}$ carries branches, uniqueness radii and a covering as in (C1)–(C3), then in the anchor frame every line of every third basis of $\{I, H/\sqrt{6}\}$ is one of the branches $\theta_j(\Delta, \varsigma)$.*

(b) *If $B/\sqrt{6}$ is a third basis of $\{I, H/\sqrt{6}\}$ whose six lines are, in the anchor frame, the branches $\theta_j(\Delta, \varsigma)$, $j \in \mathcal{J}$, for a set $\mathcal{J}$ of six indices, certified as in (C1), and a family triplet and a bijection μ from $\mathcal{J}$ to the columns of $K(v_1)$ satisfy the condition of the outcome decided for $\mathcal{J}$, then Z is a clock of $\{I, H/\sqrt{6}, B/\sqrt{6}\}$.*

(c) *If $\mathcal{C}$ carries an interface certificate (C1)–(C4), the lines of every third basis are six branches forming a six-clique of Γ; so if the outcome is excluded, $\{I, H/\sqrt{6}\}$ has no third basis, and if it is decided, every triplet $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ has the clock Z.*

Part (b) needs no covering, so Proposition 7.1, whose cells carry the covering of (B1) instead of (C3), can cite it.

Proof. Let $Q(c, h)$ be the sub-root cube of $\mathcal{C}$ and η the tangent coordinate of H. As $\eta \in Q(c, h)$, the anchor frame of $Q(c, h)$ applies: $H = D_1(H_A \circ \mathrm{e}^{i\varphi})D_2$ with D_1, D_2 diagonal unitary, the third bases of $\{I, H/\sqrt{6}\}$ are the $D_1 B_0/\sqrt{6}$ for the third bases $B_0/\sqrt{6}$ of $\{I, H_A \circ \mathrm{e}^{i\varphi}/\sqrt{6}\}$, and the common unbiased lines of $\{I, H/\sqrt{6}\}$, written $e(u)/\sqrt{6}$ with $u_0 = 0$, correspond to the zeros θ of $R(\cdot; H_A \circ \mathrm{e}^{i\varphi})$ by $u = \theta/2\pi + \sigma_0$ modulo $\mathbb{Z}^5$.

(a) As $\eta \in Q(c_*, h_*)$, every line of a third basis has, by the conclusion of Lemma 6.9, u in a component box modulo $\mathbb{Z}^5$, so a representative θ in the transported and lifted component box, hence in a covering box of (C3). That box contains a zero, so by (C3) it lies, for some j, within ϱ_j of $\theta_j(\Delta, \varsigma)$, and by (C2) the line is the branch $\theta_j(\Delta, \varsigma)$.

(b) Write $B = D_1 B_0$, and let $\theta_j = \theta_j(\Delta, \varsigma)$ and K' be the matrix with column $\mu(j)$ equal to $w(\theta_j)$ for $j \in \mathcal{J}$. Then $K'/\sqrt{6}$ has the lines of $B_0/\sqrt{6}$, so $\{I, H_A \circ \mathrm{e}^{i\varphi}/\sqrt{6}, K'/\sqrt{6}\}$ is a triplet. By the definitions of φ_F and ψ, $H_A \circ \mathrm{e}^{i\varphi} = H(d(s_1)) \circ \mathrm{e}^{i(\varphi - \varphi_F)}$ and $K' = K(v_1) \circ \mathrm{e}^{i\psi}$; so, for the family triplet $\mathcal{T}$ and $x = (\varphi - \varphi_F, \psi)$, $H_x = H_A \circ \mathrm{e}^{i\varphi}$, $K_x = K'$ and $F_{\mathcal{T}}(x) = 0$, for any representatives of the entries of x. By hypothesis (6.4) holds for x (it suffices that it hold at upper bounds of a and b, since L_3 is increasing), so Z is a clock of $\{I, H_x/\sqrt{6}, K_x/\sqrt{6}\}$ by Proposition 6.2. A clock is defined through lines, so Z is a clock of $\{I, H_A \circ \mathrm{e}^{i\varphi}/\sqrt{6}, B_0/\sqrt{6}\}$, and, as D_1 commutes with Z, of its image $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ under D_1.

(c) Distinct lines of a third basis are orthogonal, so by (a) they are six distinct branches, no two of them in a pair of (C4): a six-clique of Γ. If the outcome is excluded, there is none; if it is decided, (b) applies to this clique. □

6.6. **The criterion.**

Theorem 6.12. *Let R be a record of radius τ with $k = 4$, let π_r, π_c be permutations of $\{0, \dots, 5\}$ with permutation matrices Π_r and Π_c, and let R' be the record with center $(P_{\pi_r(a)\pi_c(b)})_{a,b}$ and its frame permuted in the same way. Suppose that no member of R' has a third basis, or that the conclusion of Lemma 6.9 holds for R' to some level $L \le 3$ with candidate cubes each of which is held by finitely many cells with interface certificates whose outcomes are excluded or decided. Then for every member H of R, every triplet $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ has the clock $\Pi_r^{-1} Z \Pi_r$. If no member of R' has a third basis, or every outcome is excluded, then no member of R has a third basis.*

Proof. The members of R' are the matrices $H' = \Pi_r H \Pi_c^{\mathsf{T}}$ for the members H of R [19, Lemma C.24] (Appendix C (U3)). If $B/\sqrt{6}$ is a third basis of $\{I, H/\sqrt{6}\}$, then $\Pi_r B/\sqrt{6}$ is one of $\{I, H'/\sqrt{6}\}$. If no member of R' has a third basis, there is no such B; otherwise, by the conclusion of Lemma 6.9, H' lies in an excluded cube, and then $\{I, H'/\sqrt{6}\}$ has no third basis, or in a candidate cube, and then one of the cells of the hypothesis holds it; by Proposition 6.11 (c), either no such basis exists, or Z is a clock of $\{I, H'/\sqrt{6}, \Pi_r B/\sqrt{6}\}$, and then $\Pi_r^{-1} Z \Pi_r$ is a clock of $\{I, H/\sqrt{6}, B/\sqrt{6}\}$. □

The computation that realizes the hypotheses is described in Appendix A.3; it uses the dyadic grid $h_1 = 2^{-8}$, whose cube centers are exact binary64 numbers at every level, so that the cubes cover exactly, and it checks in exact rational arithmetic that the cells of a cube hold it.

Proposition 6.13 (computer-assisted)**.** *For each of the 483 records of $\mathcal{L}^+$ listed in the deposit with permutations π_r, π_c and an anchor, the hypotheses of Theorem 6.12 hold, or, for four of them, those of Theorem 8.9 in Section 8. For each of these records, every triplet through a member has a clock.*

Each of the 483 records meets the hypotheses in one of five ways: Stage C on R' ends with no six-clique; the cube cover has no candidate cube; every candidate cube is excluded; some cell is decided (Theorem 6.12); or, for four records, some cell is decided through the chart (Theorem 8.9). In the first three no member has a third basis, and in the last two every triplet through a member has a clock. The number of records of each kind, and the level of each cube cover, are listed in the deposit.

7. The records near the point D

Let $D = H(1, 1, \mathrm{e}^{i\pi/3})$, one of the 24 points of Proposition 2.10. Up to the order of rows and columns, D is the matrix $F^{\mathsf{T}}(\omega, \omega)$, in which the family $F^{\mathsf{T}}(x, y)$ of (1.1) and Szöllősi's family $X(\alpha)$ intersect [29, Section 4]. Indeed, D has the columns of $F^{\mathsf{T}}(1, \mathrm{e}^{i\pi/3})$ (Section 6); replacing y by $-y$ in (2.2) exchanges the rows $(0, 2)$ and $(1, 2)$, and $-\mathrm{e}^{i\pi/3} = \omega^2$; and since $\operatorname{diag}(1, \omega, \omega^2)\overline{F_3} = \overline{F_3}S$ with S as in (2.3), $F^{\mathsf{T}}(\omega x, \omega^2 y) = F^{\mathsf{T}}(x, y)\operatorname{diag}(I_3, S)$, so that $F^{\mathsf{T}}(\omega, \omega) = F^{\mathsf{T}}(1, \omega^2)\operatorname{diag}(I_3, S)$. The pair $\{I, D/\sqrt{6}\}$ has 42 common unbiased lines instead of 48. As $H(1, x, y)$ tends to D along the Fourier family, two of the eight Kramers bases of $\{I, H(1, x, y)/\sqrt{6}\}$ tend to the same Kramers basis of D, and their lines come together in six pairs, each pair formed by lines of these two different Kramers bases (Remark B.8). Near D the branches of Section 6.3 cannot all be certified, and Theorem 6.12 does not close the records. Instead, Theorem 7.3 shows for 30 records that a triplet through a member has a clock or a transition B^* in no record of $\mathcal{K}$ and in no sink of the same or a later tier (Section 7.2); when the triplet has no clock, its proof finds B^* close to F_6. This alternative is suggested by Proposition 2.9 (b): at D the Kramers basis $K(v_0)/\sqrt{6}$ of Remark B.8, in which two Kramers bases merge, makes with $\{I, D/\sqrt{6}\}$ a triplet permutationally unitarily equivalent to $\mathcal{T}_1$ and $\mathcal{T}_2$, whose transition $B^* = K(v_0)^*$ is $\cong F_6$; we do not use this.

7.1. **Routing.** Let $P_{F_6} = (jk/6)_{j,k}$, a phase matrix of F_6 up to the order of rows and columns, and let $f_{F_6}(A) = d_G(A, P_{F_6})$ in the notation of Section 3.3; for a complex Hadamard matrix M we write $f_{F_6}(M)$ for f_{F_6} of any phase matrix of M, which does not depend on the choice by Lemma 3.3. A *reference* is a matrix q with entries in $\frac{1}{12}\mathbb{Z}$, the phases in cycles of the columns of a matrix X_{ref} ($q_{k,r}$ the phase of entry r of column k), together with an element $g \in G$ such that $g(-q) - P_{F_6} \in \mathbb{Z}^{36} + L$, checked in exact arithmetic. Then $-q$ is a phase matrix of X_{ref}^* and $f_{F_6}(-q) = 0$.

For a cell of a candidate cube (Section 6.5) and a threshold $\gamma > 0$, consider:

(B1) *vertices*: the cell carries some of the branches of (C1), each certified as there and with a uniqueness radius as in (C2), and each component box, transported and lifted as in (C3) (rounded outward), is either declared a *box vertex* or covered by finitely many boxes each of which either is *matched* to one carried branch j, that is, lies within distance ϱ_j of every point of a box containing $\theta_j(\Delta, \varsigma)$ for all parameters of the cell, or is *zero-free*: it contains no zero of $R(\cdot; H_A \circ \mathrm{e}^{i\varphi})$ for any parameter at which $H_A \circ \mathrm{e}^{i\varphi}$ is complex Hadamard; each box vertex, with half-widths ρ_r, satisfies $6 - 2\sum_r \rho_r > 0$;

(B2) *pairs*: $\mathcal{U}$ is a set of carried branches such that each covering box of (B1) is matched to a branch of $\mathcal{U}$ or zero-free, and the graph Γ on the box vertices and the branches of $\mathcal{U}$ has an edge between two vertices unless they are certified non-orthogonal: two branches as in (C4), any other pair when $|1 + \sum_{r=1}^{5} \mathrm{e}^{i(c'_r - c_r)}| > \sum_r (\rho_r + \rho'_r)$ for their boxes $c \pm \rho$ and $c' \pm \rho'$ (for a branch, a box containing it over the cell);

(B3) *cliques*: every six-clique of Γ either consists of six branches $\mathcal{J}$ for which a family triplet and a bijection μ satisfy the condition of the outcome decided (Section 6.5), or has routing bound below γ for some bijection, or its six boxes can be bisected repeatedly into a finite tree each of whose leaves has two boxes certified non-orthogonal as in (B2), or a zero-free box, or routing bound below γ for some bijection.

The *routing bound* of six boxes with centers $c_j \in \mathbb{R}^5$ and half-widths ρ_j (in radians, extended by $c_{j0} = \rho_{j0} = 0$) for a bijection $j \mapsto j'$ of $\{1, \dots, 6\}$ is computed as follows. Let $A_c =$

$(q_{j',r} - c_{j,r}/2\pi)_{j,r}$, N its matrix of nearest integers, $E = \Pi(A_c - N)$ and $\hat{\rho} = \rho/2\pi$; then

$$\mathrm{UB} = \min\Big(\|E\| + \|\hat{\rho}\|,\ \big(\|E\|^2 + 2\textstyle\sum_{j,r} |E_{jr}|\hat{\rho}_{jr} + \|\hat{\rho}\|^2\big)^{1/2}\Big),$$

evaluated in ball arithmetic and rounded up.

Proposition 7.1. *If (B1)–(B3) hold on a cell, then for every member H held by the cell and every third basis $B/\sqrt{6}$ of $\{I, H/\sqrt{6}\}$, the triplet $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ has the clock Z, or $f_{F_6}(B^*) < \gamma$.*

Proof. Work in the anchor frame: a member is $H = D_1(H_A \circ \mathrm{e}^{i\varphi})D_2$ with D_1, D_2 diagonal unitary (Section 6.5), the third bases of $\{I, H/\sqrt{6}\}$ are the $D_1 B_0/\sqrt{6}$ for the third bases $B_0/\sqrt{6}$ of $\{I, H_A \circ \mathrm{e}^{i\varphi}/\sqrt{6}\}$, D_1 commutes with Z, and $B_0^* = (D_1 B_0)^* D_1$ differs from $(D_1 B_0)^*$ by column phases; so neither the clock Z nor $f_{F_6}(B^*)$ changes, and we may take $H = H_A \circ \mathrm{e}^{i\varphi}$. Each line of B has a representative in a transported component box (Lemma 6.9). If that box is a box vertex, send the line to it; otherwise the representative lies in a covering box of (B1), which therefore contains a zero, so by (B2) the box is matched to a branch j of $\mathcal{U}$, and by (C2) the line is $\theta_j(\Delta, \varsigma)$: send it to j. This map is injective: two lines sent to one branch are that branch, and two lines $w(\theta), w(\theta')$ in one box vertex satisfy $|\langle w(\theta), w(\theta')\rangle| \geq 6 - \sum_r |\mathrm{e}^{i(\theta'_r - \theta_r)} - 1| \geq 6 - 2\sum_r \rho_r > 0$, so they are not orthogonal. For lines in boxes $c \pm \rho$ and $c' \pm \rho'$, $|\langle w(\theta), w(\theta')\rangle - (1 + \sum_r \mathrm{e}^{i(c'_r - c_r)})| \leq \sum_r (\rho_r + \rho'_r)$, so a certified pair contains no orthogonal lines, and the images form a six-clique. If it is as in the first case of (B3), the triplet has the clock Z by Proposition 6.11 (b). If it has routing bound below γ, fix a bijection $j \mapsto j'$ for which it does, order the lines like the boxes, and let $\theta_{j,r}$ be their angles, $\theta_{j,0} = 0$ and $\theta_{j,r} = c_{j,r} + \delta_{j,r}$ with $|\delta_{j,r}| \leq \rho_{j,r}$. Row j of B^* is the conjugate of column j of B, so a phase matrix of B^* is $-(\theta_{j,r})_{j,r}/2\pi + t$ with $t \in L$, and the difference A between it and the rows of $-q$ permuted by this bijection is $A_c - \delta/2\pi + t$. Hence

$$d(A) \leq \|\Pi(A - N)\| = \|E - \Pi\delta/2\pi\|, \qquad \|E - \Pi\delta/2\pi\|^2 = \|E\|^2 - 2\langle E, \delta/2\pi\rangle + \|\Pi\delta/2\pi\|^2,$$

since $E = \Pi E$; this is at most the square of either expression in UB. A row permutation lies in G, so $f_{F_6}(B^*) \leq d_G(-(\theta_{j,r})_{j,r}/2\pi, -q) + f_{F_6}(-q) \leq d(A) \leq \mathrm{UB} < \gamma$ by Lemma 3.3. In the third case the two halves of a bisected closed box cover it, so the six lines lie in the boxes of a leaf; a leaf of the first two kinds contains no six orthogonal unbiased lines, and on the others the routing bound applies. □

Theorem 7.2. *Let R, π_r, π_c and R' be as in Theorem 6.12, and $\gamma > 0$. Suppose that no member of R' has a third basis, or that the conclusion of Lemma 6.9 holds for R' to some level $L \leq 3$ with candidate cubes each of which is held by finitely many cells on each of which either an interface certificate gives the outcome excluded or decided, or (B1)–(B3) hold. Then for every member H of R and every third basis $B/\sqrt{6}$ of $\{I, H/\sqrt{6}\}$, the triplet $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ has the clock $\Pi_r^{-1} Z \Pi_r$ or $f_{F_6}(B^*) < \gamma$. If, moreover, $\gamma < f_{F_6}(P') - r'$ for every record $\tilde{R}$ of a set $\mathcal{A}$, with center P' and radius r', then in the second case B^* lies in no record of $\mathcal{A}$.*

Proof. The first part follows as Theorem 6.12, with Proposition 7.1 on the cells where it is used; $\Pi_r B$ is a third basis of the relabeled member, and $(\Pi_r B)^* = B^* \Pi_r^{\mathsf{T}}$, so f_{F_6} is the same for both. If B^* lay in $\tilde{R} \in \mathcal{A}$, then $f_{F_6}(P') \leq f_{F_6}(B^*) + r' < \gamma + r'$ by Lemma 3.3 (c), (d). □

7.2. **The sinks.** Let $\mathcal{S}_1 = \{2380, 6889, 7271\}$, $\mathcal{S}_3 = \{7558\}$, and let $\mathcal{S}_2$ be the set of 26 records of $\mathcal{L}^+$ listed in the deposit. These are records whose centers lie, in floating point, close to D. Put $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, whose records we call the *sinks*: in the graph of the descent (Section 9) no edge leads from a record of $\mathcal{S}$ to a record outside $\mathcal{S}$. Let $\lambda = i$ on $\mathcal{S}_i$, the *tier*, and let $\gamma_1 = 17/100$, $\gamma_2 = 19/100$, $\gamma_3 = 39/200$.

Theorem 7.3 (computer-assisted)**.** *With $\mathcal{K}$ as in Section 5, for every $R \in \mathcal{S}_i$, every member H of R and every third basis $B/\sqrt{6}$ of $\{I, H/\sqrt{6}\}$, the triplet $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ has a clock, or B^* lies in no record of $\mathcal{K} \cup \{S \in \mathcal{S} : \lambda(S) \geq i\}$.*

Proof. For each record of $\mathcal{S}_i$, the hypotheses of Theorem 7.2 hold with $\gamma = \gamma_i$ and $L = 3$, with a relabeling and a reference listed in the deposit. For $i = 1, 2, 3$ let $\mathcal{A}_i = \mathcal{K} \cup \{S \in \mathcal{S} : \lambda(S) \geq i\}$. For every record of $\mathcal{A}_i$, with center P' and radius τ, the exact lower bound of Section 5.3 for $f_{F_6}(P') = d_G(P', P_{F_6})$, computed with P_{F_6} reduced modulo 1 (which does not change d_G), exceeds $\gamma_i + \tau$, with least margins above $6 \cdot 10^{-3}$, $5 \cdot 10^{-4}$ and $4 \cdot 10^{-4}$ for $i = 1, 2, 3$. By Theorem 7.2, a triplet through a member of a record of $\mathcal{S}_i$ without a clock has B^* in no record of $\mathcal{A}_i$. □

8. The records near F_6

The 74 records of $\mathcal{L}^+ \setminus (\mathcal{K} \cup \mathcal{S})$ not treated in Proposition 6.13 are closed in Proposition 8.10 cube by cube, as in Section 6, with two more ways to settle part of a record. Let $H_1 = F_2 \otimes F_3$, a matrix equivalent to F_6. At the twelve points of the Fourier family at which $H(d)$ is a row and column permutation of H_1, the family $\{H(d)\}$ meets the family of transposes $\{H(d)^{\mathsf{T}}\}$, and the pair $\{I, H_1/\sqrt{6}\}$ has 16 third bases (Theorem 2.8 (b)); near them the interface certificates of Section 6 do not decide every cell. There we replace the tube of Proposition 6.2 by a chart near H_1: matrices $H_{\mathcal{Q}}(p)$, for p in boxes $\mathcal{Q} \subset \mathbb{R}^4$, that solve 21 fixed linear combinations of the 30 real Hadamard equations, so that they need not be complex Hadamard matrices (Proposition 8.2). Up to row and column phases the two families are two coordinate planes of the chart, and up to equivalence the chart contains every complex Hadamard matrix of the ball of Theorem 8.6 (Lemma 8.5 and the proof of Theorem 8.6). A cell of an interface run may be decided through the chart (Section 8.4), and a cube all of whose members are equivalent to matrices in that ball is settled by Theorem 8.6 (Lemma 8.7).

8.1. **The copies of F_6 in the family.** In the notation of Section 6.1, $H(1,1,1)_{(a,i),(h,k)} = (-1)^{ah}\omega^{ik}$, so $H(1,1,1) = H_1$. For $k \in \mathbb{Z}/3$ let Σ_k be the permutation matrix with $(X\Sigma_k)_{\cdot,(1,m)} = X_{\cdot,(1,m+k)}$ and $(X\Sigma_k)_{\cdot,(0,m)} = X_{\cdot,(0,m)}$, and for $\sigma \in \{0,1\}^2$ let P_σ be the permutation matrix exchanging the rows $(0,i)$ and $(1,i)$ for $i \in \{1,2\}$ with $\sigma_i = 1$.

Lemma 8.1. *For $k \in \mathbb{Z}/3$, $\sigma \in \{0,1\}^2$ and $g = (1, (-1)^{\sigma_1}\omega^k, (-1)^{\sigma_2}\omega^{2k})$, $H(d \cdot g) = P_\sigma H(d) \Sigma_k$ for every $d \in \mathbb{T}^3$ with $d_0 = 1$. In particular $H(\bar{g}) = P_\sigma H_1 \Sigma_k^{-1}$ at the twelve points $\bar{g}$, and these are the points $d(s)$ with*

$$s/\pi \in \{(0,0), (1,0), (0,1), (1,1), (\tfrac{1}{3}, \tfrac{2}{3}), (\tfrac{4}{3}, \tfrac{2}{3}), (\tfrac{1}{3}, \tfrac{5}{3}), (\tfrac{4}{3}, \tfrac{5}{3}), (\tfrac{2}{3}, \tfrac{1}{3}), (\tfrac{5}{3}, \tfrac{1}{3}), (\tfrac{2}{3}, \tfrac{4}{3}), (\tfrac{5}{3}, \tfrac{4}{3})\},$$

pairwise at distance at least $\pi/3$ in the maximum norm on the torus.

Proof. $H(d \cdot g)_{(a,i),(h,m)} = (-1)^{ah} d_i^h g_i^h \omega^{im} = (-1)^{(a+\sigma_i)h} d_i^h \omega^{i(m+kh)}$, which is the entry of $H(d)$ in row $(a + \sigma_i, i)$ and column $(h, m + kh)$. The second statement is the first at $d = \bar{g}$, and the list is $-\arg g$ modulo 2π. □

8.2. **The chart.** Phases are now in radians. For $\varphi \in \mathbb{R}^{36}$ put $\hat{H}(\varphi) = H_1 \circ \mathrm{e}^{i\varphi}$ and let $\mathrm{Had}(\varphi) \in \mathbb{R}^{30}$ consist of the real and imaginary parts of the entries $(\hat{H}(\varphi)\hat{H}(\varphi)^*)_{rs}$, $r < s$; so $\hat{H}(\varphi)$ is a complex Hadamard matrix if and only if $\mathrm{Had}(\varphi) = 0$. For $u, w \in \mathbb{R}^2$ let $Fu, Gw \in \mathbb{R}^{36}$ be the phase vectors with $H_1 \circ \mathrm{e}^{iFu} = H(1, \mathrm{e}^{iu_1}, \mathrm{e}^{iu_2})$ and $H_1 \circ \mathrm{e}^{iGw} = H(1, \mathrm{e}^{iw_1}, \mathrm{e}^{iw_2})^{\mathsf{T}}$; these are the two families through H_1, and Φ_I, Φ_H denote the sets of matrices equivalent to their members. Let $A = \Pi[F \,|\, G] \in \mathbb{Q}^{36\times 4}$, with Π the orthogonal projection of Section 3.3, and let $\Lambda \in \mathbb{R}^{36\times 21}$ and $\Omega \in \mathbb{R}^{30\times 21}$ be two fixed binary64 matrices, given in the deposit, whose columns approximately span a complement of the tangent space of the Hadamard matrices at H_1 and its image under the derivative of Had ($21 = 25 - 4$, with 4 the defect of F_6 [49, Table 1]: the derivative of Had at H_1 has rank 21 on $L^{\perp}$, its kernel there being spanned by the four columns of A, the tangent planes of the two families). The *chart* is

$$\varphi(p, \xi) = Ap + \Lambda\xi, \qquad g(p, \xi) = \Omega^{\mathsf{T}}\mathrm{Had}(\varphi(p, \xi)), \qquad p = (u, w) \in \mathbb{R}^4,\ \xi \in \mathbb{R}^{21}.$$

Lines are written $w(\theta)$, $\theta \in \mathbb{R}^5$, as in Section 6.3. The 48 common unbiased lines of $\{I, H_1/\sqrt{6}\}$ are numbered once and for all, and $B_1, \dots, B_{16}$ are the six-sets of numbers of its 16 third bases, of four kinds, recorded with the numbers in the deposit:

- six *transposed*, invariant under the clock $T_3 = I_2 \otimes S$ of the second family and not under Z;
- six *plain Kramers*, Kramers bases invariant under Z and not under T_3;
- two *crossing*, Kramers bases invariant under Z and T_3;
- two *isolated*, invariant under Z and T_3 and not Kramers bases.

A *chart cell* is a box $\mathcal{Q} \subset \mathbb{R}^4$ of parameters in one of five families, listed in the deposit, whose boxes contain the ones described here (there the half-widths of (Q2), (Q3) and (Q5) are enlarged by 2^{-40}, so that each contains the box described here enlarged by $9.09 \cdot 10^{-13}$, and the origin cubes have centers $\pm h_{\mathrm{o}}$ and half-width $h_{\mathrm{o}} = \mathrm{fl}(0.05)/2$):

(Q1) the 16 *origin cubes* of half-width 0.025 tiling $[-0.05, 0.05]^4$;

(Q2) 548 *narrow cells*, products of a u-box and the w-box $[-0.05, 0.05]^2$, which with the origin cubes cover $[-0.05, 0.05]^4 \cup \{\|u\|_2 \le 0.46,\ \|w\|_\infty \le 0.05\}$;

(Q3) 2,000 *wide cells*, the 500 u-boxes of the narrow cells that meet $\{\|u\|_\infty > 0.05\}$, each times the four quadrants of $[-0.1, 0.1]^2$, covering $\{\|u\|_2 \le 0.46,\ \|u\|_\infty \ge 0.05,\ \|w\|_\infty \le 0.1\}$;

(Q4) 988 *H-cells*, each the product of $[-\frac{1}{16}, \frac{1}{16}]^2$ and a closed square of side $\frac{1}{64}$ of the grid $\frac{1}{64}\mathbb{Z}^2$, so that the u-box contains 0 and the w-box does not. They tile $\{\|u\|_\infty \le \frac{1}{16},\ \frac{3}{64} \le \|w\|_\infty \le \frac{1}{4}\}$;

(Q5) the *ring cells*: of the 2,880 boxes formed by 240 u-boxes of half-width 0.02 times the 12 squares of side 0.1 that tile $\{0.1 \le \|w\|_\infty \le 0.2\}$, 2,848 are ring cells, and each of the other 32 is divided into 4 ring cells, its sub-cells; together they cover $\{0.1 < \|u\|_\infty < 0.3,\ \|u\|_2 \le 0.46,\ 0.1 \le \|w\|_\infty \le 0.2\}$.

That each list covers its region, and that no ring cell meets the plane $u = 0$ or the plane $w = 0$, is checked in exact rational arithmetic.

Proposition 8.2 (computer-assisted)**.** *For every chart cell $\mathcal{Q}$ there are polynomials $P_\xi^{\mathcal{Q}}\colon \mathbb{R}^4 \to \mathbb{R}^{21}$ and $P_{\theta,j}^{\mathcal{Q}}\colon \mathbb{R}^4 \to \mathbb{R}^5$, $j = 1, \dots, 48$, and numbers $\varepsilon^{\mathcal{Q}} < \varepsilon_U^{\mathcal{Q}}$ and $\varepsilon_j^{\mathcal{Q}} < \varepsilon_{U,j}^{\mathcal{Q}}$, computed by the programs in the deposit, such that for every $p \in \mathcal{Q}$, with balls and distances in the Euclidean norm:*

(a) *$g(p,\cdot)$ has exactly one zero $\xi_{\mathcal{Q}}(p)$ in the ball of radius $\varepsilon_U^{\mathcal{Q}}$ about $P_\xi^{\mathcal{Q}}(p)$, and it lies within $\varepsilon^{\mathcal{Q}}$ of $P_\xi^{\mathcal{Q}}(p)$; if $\mathcal{Q}$ is not a ring cell and p lies on the plane $u = 0$ or on the plane $w = 0$, then $\xi_{\mathcal{Q}}(p) = 0$;*

(b) *with $H_{\mathcal{Q}}(p) = \hat{H}(\varphi(p, \xi_{\mathcal{Q}}(p)))$, for each j the map $R(\cdot; H_{\mathcal{Q}}(p))$, given by the formulas of Section 6.3, which make sense for every 6×6 matrix ($H_{\mathcal{Q}}(p)$ need not be a complex Hadamard matrix), has exactly one zero $\theta_j(p)$ in the ball of radius $\varepsilon_{U,j}^{\mathcal{Q}}$ about $P_{\theta,j}^{\mathcal{Q}}(p)$, and it lies within $\varepsilon_j^{\mathcal{Q}}$ of $P_{\theta,j}^{\mathcal{Q}}(p)$; the 48 lines $w(\theta_j(p))$ are pairwise distinct;*

(c) *if six of the lines $w(\theta_j(p))$ are pairwise orthogonal, then $u = 0$ or $w = 0$;*

(d) *if six of the lines $w(\theta_j(p))$ are pairwise orthogonal, then $w = 0$ and the unitary map Z of (2.1) permutes these six lines, or $u = 0$ and the unitary map $U_p = H_{\mathcal{Q}}(p)\, Z\, H_{\mathcal{Q}}(p)^*/6$ (there $\xi_{\mathcal{Q}}(p) = 0$ by (a), as no ring cell meets the plane $u = 0$, so $H_{\mathcal{Q}}(p) = \hat{H}(Ap)$ is a complex Hadamard matrix) permutes them.*

More precisely, on the cells (Q1)–(Q3) the numbers of six pairwise orthogonal lines form one of $B_1, \dots, B_{16}$; on (Q1) and (Q2) the lines of B_k are pairwise orthogonal only if $w = 0$ (B_k plain Kramers), $u = 0$ (transposed), $u = 0$ or $w = 0$ (crossing), or $p = 0$ (isolated); on the H-cells six pairwise orthogonal lines force $u = 0$; on the ring cells no six of the lines are pairwise orthogonal.

Lemma 8.3 (factorised tests)**.** *Let $\mathcal{Q} = U \times W$ with $U, W \subset \mathbb{R}^2$ compact boxes (products of two closed bounded intervals), and let $f\colon \mathcal{Q} \to \mathbb{R}^n$ be C^1, with derivatives continuous up to the boundary.*

(Dw) *If $0 \in W$, $f(u,0) = 0$ for $u \in U$, and every matrix in the convex hull of $\{\partial_w f(q) : q \in \mathcal{Q}\}$ is injective, then $f(u,w) \neq 0$ for $(u,w) \in \mathcal{Q}$ with $w \neq 0$.*

(Du) *The same with the roles of u and w exchanged.*

(Dp) *If $0 \in \mathcal{Q}$, $f(0) = 0$, and every matrix in the convex hull of $\{Df(q) : q \in \mathcal{Q}\}$ is injective, then $f(p) \neq 0$ for $p \in \mathcal{Q} \setminus \{0\}$.*

(B) *If f is C^2, $0 \in U$, $0 \in W$, f vanishes on $\mathcal{Q} \cap \{u = 0\}$ and on $\mathcal{Q} \cap \{w = 0\}$, and every bilinear map Υ in the convex hull of $\{\partial_u \partial_w f(q) : q \in \mathcal{Q}\}$ has $\min_{|\hat{u}|=|\hat{w}|=1} \|\Upsilon[\hat{u}, \hat{w}]\| > 0$ (Euclidean norms), then $f(u,w) \neq 0$ for $(u,w) \in \mathcal{Q}$ with $u \neq 0$ and $w \neq 0$.*

Proof. The convex hulls are of compact sets, so they are compact, and an average of a continuous map from $[0,1]$ or $[0,1]^2$ into a compact set is a limit of convex combinations of its values, so the averages below lie in the hulls. (Dw): the segment from $(u,0)$ to (u,w) lies in $\mathcal{Q}$, so $f(u,w) = f(u,w) - f(u,0) = \big(\int_0^1 \partial_w f(u, \lambda w)\, d\lambda\big) w \neq 0$. (Du) is the same, and (Dp) uses the segment from 0 to p. (B): the rectangle $\{(\lambda u, \mu w) : \lambda, \mu \in [0,1]\}$ lies in $\mathcal{Q}$, and $f(u,w) = f(u,w) - f(u,0) - f(0,w) + f(0,0) = \bar{\Upsilon}[u,w]$ with $\bar{\Upsilon} = \int_0^1\!\int_0^1 \partial_u \partial_w f(\lambda u, \mu w)\, d\lambda\, d\mu$, so $\|f(u,w)\| \geq |u||w| \min_{|\hat{u}|=|\hat{w}|=1} \|\bar{\Upsilon}[\hat{u}, \hat{w}]\| > 0$. □

Proof of Proposition 8.2. The proof is a computation per cell with Taylor models [28]: polynomials of degree at most 6 in p with coefficient enclosures and a remainder bound valid on the cell.

Existence and uniqueness, (a) and (b). Part (a) follows from Lemma 6.4 for $F = g(p,\cdot)$ and $x_0 = P^{\mathcal{Q}}_{\xi}(p)$, with a fixed matrix Y close to the inverse of $\Omega^{\mathsf{T}} D\mathrm{Had}(\tilde{\varphi}(p_c))\Lambda$, where $\tilde{\varphi}(p) = Ap + \Lambda P^{\mathcal{Q}}_{\xi}(p)$ and p_c is the center of $\mathcal{Q}$, with $b = \|Y\|_2 \sup_{\mathcal{Q}} \|g(p, P^{\mathcal{Q}}_{\xi}(p))\|$, and with $\kappa = \|I - Y\Omega^{\mathsf{T}} D\mathrm{Had}(\tilde{\varphi}(p_c))\Lambda\|_2 + 2\sqrt{3}\,\|Y\|_2\|\Omega\|_2\|\Lambda\|_2(\sup_{\mathcal{Q}} \|\tilde{\varphi}(p) - \tilde{\varphi}(p_c)\| + \|\Lambda\|_2\epsilon)$, which bounds $\|I - Y\, D_{\xi} g(p,\xi)\|_2$ for $\|\xi - P^{\mathcal{Q}}_{\xi}(p)\| \le \epsilon$ by the bound $\|D^2\mathrm{Had}(\varphi)[x,y]\| \le 2\sqrt{3}\,\|x\|\|y\|$, valid for every φ (with $H = \hat{H}(\varphi)$, the entry (r,s) of $D^2(HH^*)[x,y]$ is $-\sum_k H_{rk}\overline{H_{sk}}(x_{rk} - x_{sk})(y_{rk} - y_{sk})$, so with x_r, y_r the rows of x, y the square of the left side is at most $\sum_{r<s} \|x_r - x_s\|^2\|y_r - y_s\|^2 \le \max_{r<s} \|x_r - x_s\|^2 \sum_{r<s} \|y_r - y_s\|^2 \le 2\|x\|^2 \cdot 6\|y\|^2$); writing $\kappa(\epsilon)$ for this κ, $\varepsilon^{\mathcal{Q}}$ is an ϵ with $\kappa(\epsilon) < 1$ and $b + \kappa(\epsilon)\epsilon \le \epsilon$, and $\varepsilon^{\mathcal{Q}}_U \ge \varepsilon^{\mathcal{Q}}$ an ϵ' with $\kappa(\epsilon') < 1$; as $\xi_{\mathcal{Q}}(p)$ lies within $\varepsilon^{\mathcal{Q}}$ of $P^{\mathcal{Q}}_{\xi}(p)$, $D_{\xi} g(p, \xi_{\mathcal{Q}}(p))$ is invertible by the Banach lemma. On the planes $u = 0$ and $w = 0$ the matrix $\hat{H}(Ap)$ differs from a member of one of the two families by row and column phases, so it is Hadamard, $g(p,0) = 0$, and the point $\xi = 0$ lies in the uniqueness ball on every cell with $\sup_{\mathcal{Q}} \|P^{\mathcal{Q}}_{\xi}\| \le \varepsilon^{\mathcal{Q}}_U$. This holds on the cells (Q1)–(Q4), where $\sup_{\mathcal{Q}} \|P^{\mathcal{Q}}_{\xi}\|$ is at most 0.0772 on the origin cubes and the narrow cells, 0.1423 on the wide cells and 0.0604 on the H-cells, while $\varepsilon^{\mathcal{Q}}_U \ge 0.2149$; and at the data of Section 8.4 on every chart cell of (R1), since there (R1) gives $\sup_{\mathcal{Q}} \|P^{\mathcal{Q}}_{\xi}\| < \varepsilon^{\mathcal{Q}}_U - \bar{\xi} \le \varepsilon^{\mathcal{Q}}_U$. Part (b) follows from Lemma 6.4 in the same way for the five line equations $R(\cdot; \hat{H}(\varphi(p,\xi)))$, with an approximate inverse $Y(p)$ of $D_{\theta} R$, affine in p with binary64 coefficients, in place of Y, a bound for the second derivative of R in θ and bounds for the change of R and of $D_{\theta} R$ with the phases, which account for ξ ranging over the enclosure of $\xi_{\mathcal{Q}}(p)$; these bounds hold whether or not $\hat{H}(\varphi(p,\xi))$ is Hadamard. Here this step's $\kappa(\varepsilon^{\mathcal{Q}}_j) < 1$ bounds $\|I - Y(p)\, D_{\theta} R(\theta; \hat{H}(\varphi(p,\xi)))\|_2$ for every $p \in \mathcal{Q}$, every ξ with $\|\xi - P^{\mathcal{Q}}_{\xi}(p)\| \le \varepsilon^{\mathcal{Q}}$ and every θ with $\|\theta - P^{\mathcal{Q}}_{\theta,j}(p)\| \le \varepsilon^{\mathcal{Q}}_j$, so that $D_{\theta} R(\theta_j(p); H_{\mathcal{Q}}(p))$ is invertible by the Banach lemma. Distinctness is a lower bound for $6 - |\langle w(\theta_j), w(\theta_{j'})\rangle|$.

Exclusion away from the planes, (c). The graph of the pairs not certified non-orthogonal has no six-cliques other than the B_k. For a clique B_k let $f_k(p) \in \mathbb{R}^{30}$ consist of the real and imaginary parts of the fifteen overlaps of its lines, so that they are pairwise orthogonal at p exactly when $f_k(p) = 0$. Each clique is excluded off the planes named by Lemma 8.3 or by a value test: on the origin cubes by (Dw) for the plain Kramers cliques, (Du) for the transposed, (B) for the crossing and (Dp) for the isolated ones. On the narrow and wide cells the tests are (Dw) for the plain Kramers and crossing cliques and a value test for the others, on the H-cells (Du) for the transposed and crossing cliques and a value test for the others, and on the ring cells, which meet neither plane, a value test for every clique. The w-box of every narrow and wide cell and the u-box of every H-cell contain 0, and the origin cubes are the closed orthants of $[-2h_{\mathrm{o}}, 2h_{\mathrm{o}}]^4$, so the segments and rectangles of Lemma 8.3 for a point of a cell stay in that cell.

The identities on the planes. The tests need $f_k = 0$ on the plane $w = 0$, on the plane $u = 0$, on both, or at 0, and this is certified on each cell. On the plane $w = 0$, $\xi = 0$ and the chart point is $D\,H(1, \mathrm{e}^{iu_1}, \mathrm{e}^{iu_2})\,D'$ with D and D' diagonal unitary. By Section 6.1 the diagonal unitary D^{-1} carries its pair onto $\{I, F^{\mathsf{T}}(\mathrm{e}^{iu_1}, \mathrm{e}^{iu_2})/\sqrt{6}\}$, so the six maps DZ^aD^{-1} and DZ^aJD^{-1}, $a = 0, 1, 2$, with J as in (2.4) for $(x,y) = (\mathrm{e}^{iu_1}, \mathrm{e}^{iu_2})$, carry each common unbiased line v of the chart point's pair to the six lines of a third basis of that pair (Proposition 2.6). For a plain Kramers or crossing clique the computation checks that the images of the enclosure of one of its lines under these six maps lie in the uniqueness balls of its six lines, so these six lines form such a basis, and $f_k(u, 0) = 0$. On the plane

$u = 0$ the chart point is $M = D\,H(1, \mathrm{e}^{iw_1}, \mathrm{e}^{iw_2})^{\mathsf{T}} D'$ with D and D' diagonal unitary. The entrywise conjugate of $H(1, x, y)$ is $F^{\mathsf{T}}(\bar{x}, \bar{y})$, so $M^* = D_1 F^{\mathsf{T}}(\mathrm{e}^{-iw_1}, \mathrm{e}^{-iw_2}) D_2$ with $D_1 = D'^*$ and $D_2 = D^*$. The unitary map $M^*/\sqrt{6}$ carries the common unbiased lines of $\{I, M/\sqrt{6}\}$ onto those of $\{I, M^*/\sqrt{6}\}$, on which the maps $D_1 Z^a D_1^{-1}$ and $D_1 Z^a J D_1^{-1}$, now with J for $(x, y) = (\mathrm{e}^{-iw_1}, \mathrm{e}^{-iw_2})$, act as before, so the six maps $(M/\sqrt{6})\, D_1 Z^a J^b D_1^{-1}\, (M^*/\sqrt{6})$, $a = 0, 1, 2$, $b = 0, 1$, carry each common unbiased line of $\{I, M/\sqrt{6}\}$ to the six lines of a third basis of that pair. For a transposed or crossing clique the computation checks that the images of the enclosure of one of its lines under these six maps lie in the uniqueness balls of its six lines. It compares them after the unitary map $D_1^{-1} M^*/\sqrt{6}$, with the images of the six lines under that map, each vector divided by the phase of its first entry. The vectors compared have entries of modulus one, so a distance η there bounds the distance of the line coordinates θ by $\frac{\pi}{2}(1 + \sqrt{5})\eta$. So these six lines form such a basis, and $f_k(0, w) = 0$. At $p = 0$, where $\hat{H}(0) = H_1$, the six lines of each isolated clique are matched, within their uniqueness balls, with the columns of $Y_2 \otimes (\mathrm{diag}(1, \omega, \omega) F_3)$, $Y_2 = \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$, or of its complex conjugate, which are third bases of $\{I, H_1/\sqrt{6}\}$ after division by $\sqrt{6}$; so $f_k(0) = 0$.

The clock, (d). Let six of the lines $w(\theta_j(p))$ be pairwise orthogonal. No pair of them is certified non-orthogonal, so their numbers form a six-clique, one of the B_k, with $f_k(p) = 0$; so the test applied to B_k on $\mathcal{Q}$ is not a value test, and $\mathcal{Q}$ is not a ring cell. By Lemma 8.3, $f_k(p) = 0$ gives $w = 0$ if this test is (Dw), $u = 0$ if it is (Du), $u = 0$ or $w = 0$ if it is (B), and $p = 0$ if it is (Dp). If it is (Dw), or (B) at a point with $w = 0$, then $w = 0$, and the identity on the plane $w = 0$ was certified for B_k on $\mathcal{Q}$: the six lines are the images of one of them, v, under the six maps $DZ^a J^b D^{-1} = Z^a (DJD^{-1})^b$ (D and Z are diagonal), so Z, which carries the image under $Z^a (DJD^{-1})^b$ to the image under $Z^{a+1} (DJD^{-1})^b$, permutes them. If it is (Du), or (B) at a point with $u = 0$, then $u = 0$, $\xi_{\mathcal{Q}}(p) = 0$, and with $M = H_{\mathcal{Q}}(p)$ the six lines are the images of one of them under the six maps $(M/\sqrt{6}) D_1 Z^a J^b D_1^{-1} (M^*/\sqrt{6}) = U_p^a V^b$, where $V = (M/\sqrt{6}) D_1 J D_1^{-1} (M^*/\sqrt{6})$, since $(M^*/\sqrt{6})(M/\sqrt{6}) = I$ and $D_1 Z D_1^{-1} = Z$; so U_p permutes them. If it is (Dp), then $p = 0$, B_k is isolated, and its six lines are the column lines of $K = Y_2 \otimes (\mathrm{diag}(1, \omega, \omega) F_3)$ or of $\overline{K}$. As $\mathrm{diag}(1, \omega, \omega^2) F_3 = F_3 S^{-1}$ with S as in (2.3), $ZK = K(I_2 \otimes S^{-1})$ and $Z\overline{K} = \overline{Z^{-1}K} = \overline{K}(I_2 \otimes S)$, so Z permutes these lines, and $w = 0$.

Linear models. In the tests each convex hull of Lemma 8.3 is enclosed in a convex set $\{\Psi_0 + \sum_m L_m \Delta_m : \Delta \in \mathcal{Q} - p_c\} + \{E : \|E\|_F \le \varepsilon_{\mathrm{rest}}\}$, where p_c is the center of the cell and Ψ_0 and L_m are binary64 matrices. The f_k are real-analytic near the cell by the implicit function theorem, since the contraction steps of (a) and (b) show that $D_\xi g(p, \xi_{\mathcal{Q}}(p))$ and $D_\theta R(\theta_j(p); H_{\mathcal{Q}}(p))$ are invertible. Their derivatives are enclosed by Taylor models built from Taylor models of the first and second derivatives of $\xi_{\mathcal{Q}}$ and θ_j; the polynomial parts of these are derivatives of $P_\xi^{\mathcal{Q}}$ and $P_{\theta,j}^{\mathcal{Q}}$, and their remainders are bounded through the equations obtained by differentiating $g(p, \xi_{\mathcal{Q}}(p)) = 0$ and $R(\theta_j(p); H_{\mathcal{Q}}(p)) = 0$ implicitly, once or twice. The matrices Ψ_0 and L_m are read from the constant and linear Taylor coefficients of the Taylor models of the derivatives of f_k, and $\varepsilon_{\mathrm{rest}}$ bounds the rest. The matrices of this convex set are injective if $\sigma_{\min}(\Psi_0 + \sum_m L_m \Delta_m) > \varepsilon_{\mathrm{rest}}$ for every Δ, which is certified by bisecting the cell into exact dyadic pieces and the bound $\sigma_{\min}(\Psi_\Delta) \ge \sigma_{\min}(\Psi_{\Delta'}) - \sum_m |\Delta_m - \Delta'_m|\, \|L_m\|_2$, where $\Psi_\Delta = \Psi_0 + \sum_m L_m \Delta_m$. The value test is the same bound for f_k itself: $\|a_0 + \sum_m l_m \Delta_m\| > \varepsilon_{\mathrm{rest}}$ for every Δ shows that f_k has no zero on the cell. On the origin cubes the tests are made in union form: for each clique a reference matrix Ψ_{ref} (for (B), a reference bilinear map Υ_{ref}) and a bound dev for the distance, in the spectral norm (for bilinear maps, in the Frobenius norm,

which bounds $\sup_{|\hat{u}|=|\hat{w}|=1} \|(\Upsilon - \Upsilon_{\mathrm{ref}})[\hat{u},\hat{w}]\|$), of every matrix of every enclosure on every origin cube from it, with dev $< \sigma_{\min}(\Psi_{\mathrm{ref}})$, or, for (B), dev below a lower bound for $\min_{|\hat{u}|=|\hat{w}|=1} \|\Upsilon_{\mathrm{ref}}[\hat{u},\hat{w}]\|$ obtained in exact rational arithmetic on a grid of rational unit vectors $((1-t^2)/(1+t^2), 2t/(1+t^2))$, $t = s/360$, $s = -360, \dots, 360$ (half the circle suffices for each of $\hat{u}$ and $\hat{w}$, since $\Upsilon[-\hat{u},\hat{w}] = \Upsilon[\hat{u},-\hat{w}] = -\Upsilon[\hat{u},\hat{w}]$), with a Lipschitz allowance. The union tests, and the comparisons of the branches at the common corner 0 below, are made in exact rational arithmetic from the models computed on the origin cubes in Arb ball arithmetic.

Compatibility on overlapping origin cubes. On the origin cubes the zeros of any two cubes agree where the cubes meet. For ξ: $\xi_{\mathcal{Q}}(0) = 0$ for every origin cube by (a). For the lines: at the common corner 0 the enclosures of each line from any two cubes lie in each other's uniqueness balls. The intersection of two cubes is a face containing 0, so it is connected, and on it the set where two branches agree is closed and open: open by the uniqueness in (a) for ξ, and then, the chart points being the same, by the uniqueness in (b) for each θ_j. So ξ and the θ_j are single continuous functions on $[-2h_{\mathrm{o}}, 2h_{\mathrm{o}}]^4$, $\hat{H}(0) = H_1$, and the $\theta_j(0)$ are 48 distinct common unbiased lines of $\{I, H_1/\sqrt{6}\}$, hence all of them (Theorem 2.8 (b)).

Arithmetic. On every chart cell the computations are in Arb ball arithmetic, and in them binary64 only proposes: the polynomials, the approximate inverses, the matrices Ψ_0 and L_m read from midpoints, the radii, the integer multiples of 2π, the directions of bisection and the lower bounds of smallest singular values. The directions only choose where to bisect; each other proposal enters the ball computation as an exact number (for a multiple of 2π, its integer), which corrects every rounding of a midpoint and certifies each radius and lower bound; a proposal that is not finite fails the cell or, in a coefficient of $P_\xi^{\mathcal{Q}}$, makes the recorded bound of $\sup \|P_\xi^{\mathcal{Q}}\|$ not finite, and every recorded bound is finite. The bounds are balls of Arb or binary64 numbers; four steps also use routines of the C library or of NumPy (Appendix A.5). □

Corollary 8.4. *Let $p = (u, w) \in \mathcal{Q}$ for a chart cell $\mathcal{Q}$, and let $B/\sqrt{6}$ be a third basis of $\{I, H_{\mathcal{Q}}(p)/\sqrt{6}\}$ whose lines are among the $w(\theta_j(p))$. Then $H_{\mathcal{Q}}(p)$ lies in $\Phi_I \cup \Phi_H$, and the triplet $\{I, H_{\mathcal{Q}}(p)/\sqrt{6}, B/\sqrt{6}\}$ has the clock Z or the clock $U_p = H_{\mathcal{Q}}(p)\, Z\, H_{\mathcal{Q}}(p)^*/6$; hence every triplet that a unitary or antiunitary map carries onto it has a clock.*

Proof. The lines of B are six pairwise orthogonal lines $w(\theta_j(p))$, so by Proposition 8.2 (c) and its last sentence, $\mathcal{Q}$ is not a ring cell, and $u = 0$ or $w = 0$; then $\xi_{\mathcal{Q}}(p) = 0$ by (a), and $H_{\mathcal{Q}}(p) = H_1 \circ \mathrm{e}^{iAp}$. Since Ap differs from $Fu + Gw$ by a matrix of row and column phases, $H_{\mathcal{Q}}(p) = D\, H(1,x,y)\, D'$ with $(x,y) = (\mathrm{e}^{iu_1}, \mathrm{e}^{iu_2})$ if $w = 0$, and $H_{\mathcal{Q}}(p) = D\, H(1,x,y)^{\mathsf{T}} D'$ with $(x,y) = (\mathrm{e}^{iw_1}, \mathrm{e}^{iw_2})$ if $u = 0$, for diagonal unitary D, D'; so $H_{\mathcal{Q}}(p) \in \Phi_I \cup \Phi_H$. By Proposition 8.2 (d), $w = 0$ and Z permutes the lines of B, or $u = 0$ and U_p permutes them. Both maps are unitary, with cube I, and not scalar. In the first case Z fixes the standard lines, and $ZH_{\mathcal{Q}}(p) = DZH(1,x,y)D' = H_{\mathcal{Q}}(p)\, D'^{-1} T_3^{-1} D'$ by (6.1), so Z permutes the column lines of $H_{\mathcal{Q}}(p)$: it is a clock. In the second, $M = H_{\mathcal{Q}}(p)$ has $U_p M = MZ$, so U_p fixes every column line of M; and $M^* = D'^* \overline{H(1,x,y)}\, D^*$ with $Z\overline{H(1,x,y)} = \overline{Z^{-1}H(1,x,y)} = \overline{H(1,x,y)}\, T_3$, so $ZM^* = M^* DT_3 D^*$ and $U_p = (M/\sqrt{6})(M^*/\sqrt{6}) DT_3 D^* = DT_3 D^{-1}$, which permutes the standard lines: U_p is a clock. The last assertion is Lemma 3.1 (a). □

8.3. **The ball around F_6.** For a box $\mathcal{Q} \subset \mathbb{R}^4$ let $d_{\mathcal{Q}} = \min_{q\in\mathcal{Q}} \|q\|$, and put $r = 0.1999101$. For $\varphi \in \mathbb{R}^{36}$ let $(\varpi\varphi)_{(a,i),c} = -\varphi_{c,(a,-i)}$. Since $\overline{(H_1)_{(h,k),(a,i)}} = (H_1)_{(a,-i),(h,k)}$,

$$\big(\hat{H}(\varphi)^*\big)_{(a,i),c} = \hat{H}(\varpi\varphi)_{(a,-i),c},$$

so $\hat{H}(\varpi\varphi)$ is $\hat{H}(\varphi)^*$ with its rows permuted, and $\hat{H}(\varpi\varphi) \sim \hat{H}(\varphi)$ when $\hat{H}(\varphi) \in \mathcal{H}$. The map ϖ permutes the coordinates up to sign, so it is orthogonal, and it maps matrices of row phases to matrices of column phases and conversely, so $\varpi L = L$ and $\varpi\Pi = \Pi\varpi$. From $(Fu)_{(a,i),(h,k)} = h\,u_i$ and $(Gw)_{(a,i),(h,k)} = a\,w_k$ (with $u_0 = w_0 = 0$) one gets $\varpi(Fu + Gw) = Fu^\varpi + Gw^\varpi$ with $u^\varpi = (-w_2, -w_1)$ and $w^\varpi = -u$; so $\varpi Ap = Ap^\varpi$ for $p = (u, w)$ and $p^\varpi = (u^\varpi, w^\varpi)$, and ϖ exchanges the two families.

Lemma 8.5 (placement; computer-assisted)**.** *Let $\psi \in \mathbb{R}^{36}$, with $\hat{H}(\psi)$ a complex Hadamard matrix and $\|\Pi\psi\| \le 0.1999$. Then for $\psi' = \psi$ or $\psi' = \varpi\psi$ there are $p = (u, w) \in \mathbb{R}^4$ and $\xi \in \mathbb{R}^{21}$ such that*

(a) *$\hat{H}(\psi')$ differs from $\hat{H}(\varphi(p, \xi))$ by row and column phases, $g(p, \xi) = 0$, and $\|p\|^2 + \|\xi\|^2 < r^2$;*

(b) *p lies in an origin cube, in a wide cell, or in one of the 2,880 boxes of (Q5) and in a ring cell, the cell of this box or one of its sub-cells.*

Proof. Let $\Xi = [A \,|\, \Pi\Lambda] \in \mathbb{R}^{36\times 25}$. A computation in exact rational arithmetic, with the entries of Λ read as rationals, shows that $\Xi^{\mathsf{T}}\Xi - t_2 I$ is positive definite for $t_2 = 0.9998999999990001$, and that $\varpi\Pi\Lambda = AK_1 + \Pi\Lambda K_2$ with rational matrices K_1, K_2 and $\|K_1\|_F \le 10^{-14}$. So Ξ has rank $25 = \dim L^\perp$, its columns span $L^\perp$, and $\|\Xi x\|^2 \ge t_2\|x\|^2$.

Write $\Pi\psi = Ap + \Pi\Lambda\xi$. Then $\|p\|^2 + \|\xi\|^2 \le \|\Pi\psi\|^2/t_2 \le 0.1999^2/t_2 < r^2$, the last inequality checked exactly, and $\psi - Ap - \Lambda\xi = (I - \Pi)(\psi - \Lambda\xi) \in L$, so $\hat{H}(\psi)$ and $\hat{H}(\varphi(p, \xi))$ differ by row and column phases. Hence $\hat{H}(\varphi(p, \xi))$ is a complex Hadamard matrix, $\mathrm{Had}(\varphi(p, \xi)) = 0$ and $g(p, \xi) = 0$, which is (a) with $\psi' = \psi$. Put $a = \|u\|_\infty$ and $b = \|w\|_\infty$; a, b, $\|u\|$ and $\|w\|$ are at most $\|p\| < 0.2$.

If $a \le 0.05$ and $b \le 0.05$, p lies in $[-0.05, 0.05]^4$, which the origin cubes tile. If $a \ge 0.05$ and $b \le 0.1$, p lies in the region of (Q3), hence in a wide cell. If $a > 0.1$ and $b > 0.1$, then $a < 0.3$ and $b < 0.2$, so p lies in the region of (Q5); the 2,880 boxes of (Q5) cover this region, and the four sub-cells of each of the 32 divided boxes cover that box (exact checks), which gives (b). In the remaining cases, $a < 0.05 < b \le 0.1$ or $a \le 0.1 < b$, take $\psi' = \varpi\psi$. Then $\hat{H}(\psi')$ is a complex Hadamard matrix, $\|\Pi\psi'\| = \|\varpi\Pi\psi\| = \|\Pi\psi\|$ and

$$\Pi\psi' = \varpi Ap + \varpi\Pi\Lambda\xi = A(p^\varpi + K_1\xi) + \Pi\Lambda(K_2\xi),$$

so the argument above gives (a) for ψ' with $p' = p^\varpi + K_1\xi$ and $\xi' = K_2\xi$ in place of p and ξ. The point p^ϖ has $\|u^\varpi\|_\infty = b \in (0.05, 0.2)$, $\|u^\varpi\| = \|w\| < 0.46$ and $\|w^\varpi\|_\infty = a \le 0.1$, so it lies in a wide cell as described in (Q3); and $\|p' - p^\varpi\|_\infty \le \|K_1\|_F\|\xi\| < 10^{-14}$, so p' lies in that cell as given in the deposit, which contains the box described in (Q3) enlarged by $9.09 \cdot 10^{-13}$ in each coordinate (an exact check). □

Theorem 8.6 (computer-assisted)**.** *Let H be a complex Hadamard matrix equivalent to $H_1 \circ \mathrm{e}^{i\psi}$ with $\|\Pi\psi\| \le 0.1999$. Then $\{I, H/\sqrt{6}\}$ has exactly 48 common unbiased lines, and if some triplet of mutually unbiased bases contains a pair mapped onto $\{I, H/\sqrt{6}\}$ by a unitary or antiunitary map, then H lies in $\Phi_I \cup \Phi_H$ and every such triplet has a clock.*

Proof. The idea is this. On each cell of a list that covers the chart points of Lemma 8.5, the common zeros of the line equations are nonsingular, so their number is locally constant in p, hence constant on the cell. The certified zeros $\tilde{\xi}_{\mathcal{Q}}$ of g on two meeting cells agree, and the graph of meeting cells is connected, so the count 48 at H_1 holds on every cell. The chart of Proposition 8.2 then identifies these 48 lines, and Corollary 8.4 gives the clock.

By Lemma 3.1 (b) the statement holds for H if it holds for a matrix equivalent to H, and $H \sim \hat{H}(\psi) \sim \hat{H}(\varpi\psi)$. So by Lemma 8.5 we may assume $H = \hat{H}(\varphi(p, \xi))$ with p and ξ as there.

Completeness. The list L_{20} in the deposit consists of all the origin cubes, wide cells and boxes of (Q5) with $d_{\mathcal{Q}} \le r$, as an exact check confirms. On each cell $\mathcal{Q}$ of L_{20}, with center p_c and half-widths $h_1, \dots, h_4$, a search proposes a polynomial $\tilde{P}_\xi^{\mathcal{Q}}$ in $p - p_c$ of degree at most 6, with binary64 coefficients, and a tiling of $[-\pi, \pi]^5$ by boxes, and two programs independent of the search check them in ball arithmetic. The first certifies the first assertion of Proposition 8.2 (a) for $\tilde{P}_\xi^{\mathcal{Q}}$ in place of $P_\xi^{\mathcal{Q}}$, by Lemma 6.4, with binary64 numbers $\tilde{\varepsilon}^{\mathcal{Q}} < \tilde{\varepsilon}_U^{\mathcal{Q}}$ of its own choice, and gives a binary64 number $\tilde{s}^{\mathcal{Q}}$; exact rational arithmetic confirms that the cell the two programs read is $\mathcal{Q}$, that $\tilde{\varepsilon}^{\mathcal{Q}} < \tilde{\varepsilon}_U^{\mathcal{Q}}$, that $\tilde{s}^{\mathcal{Q}} > 0$ and that $\sum_i \big(\sum_e |a_{i,e}| h^e\big)^2 \le (\tilde{s}^{\mathcal{Q}})^2$, where $a_{i,e}$ is the coefficient of $(p - p_c)^e$, $e \in \{0, \dots, 6\}^4$, in the i-th component of $\tilde{P}_\xi^{\mathcal{Q}}$ and $h^e = \prod_m h_m^{e_m}$, so that $\sup_{\mathcal{Q}} \|\tilde{P}_\xi^{\mathcal{Q}}\| \le \tilde{s}^{\mathcal{Q}}$. This defines a zero $\tilde{\xi}_{\mathcal{Q}}(p)$ of $g(p, \cdot)$ and $\tilde{H}_{\mathcal{Q}}(p) = \hat{H}(\varphi(p, \tilde{\xi}_{\mathcal{Q}}(p)))$ for every p in every cell of L_{20}, and $\tilde{\xi}_{\mathcal{Q}}$ is continuous on $\mathcal{Q}$, since its graph, which by the uniqueness is $\{(p, \xi) : p \in \mathcal{Q},\ g(p, \xi) = 0,\ \|\xi - \tilde{P}_\xi^{\mathcal{Q}}(p)\| \le \tilde{\varepsilon}^{\mathcal{Q}}\}$, is closed and bounded. The second program checks in exact arithmetic that the boxes of the tiling tile $[-\pi, \pi]^5$, and certifies in ball arithmetic, for all $p \in \mathcal{Q}$ at once, that each of them, or each box of a bisection of it, contains no zero of $R(\cdot; \tilde{H}_{\mathcal{Q}}(p))$ or has $\partial_\theta R$ nonsingular; exact checks confirm that the boxes it tests are all the boxes of the tiling. In the two programs binary64 only proposes: approximate inverses, used in Lemma 6.4, which holds for every real Y, and in tests of exclusion and of nonsingularity valid for every real matrix; the three bounds of spectral norms in the contraction constant on each cell, from NumPy's singular value decomposition, certified in ball arithmetic up to their sign, and each of these 3,456 bounds is positive (Appendix A.5); and the numbers $\tilde{\varepsilon}^{\mathcal{Q}}$, $\tilde{\varepsilon}_U^{\mathcal{Q}}$ and $\tilde{s}^{\mathcal{Q}}$ of the first program, certified by the exact comparisons above. Hence no common zero of the five functions $R_c(\cdot; \tilde{H}_{\mathcal{Q}}(p))$ is singular, for any $p \in \mathcal{Q}$. So these zeros are isolated, hence finitely many modulo $2\pi\mathbb{Z}^5$. Since $R(\theta; \tilde{H}_{\mathcal{Q}}(p))$ and its derivative in θ are continuous in (θ, p), the implicit function theorem, with continuous dependence on the parameter, continues each of them to exactly one nearby zero for all p' near p; and as $\mathbb{R}^5/2\pi\mathbb{Z}^5$ is compact, every zero for p' near p is near a zero for p. So their number $N_{\mathcal{Q}}(p)$ is locally constant, hence constant on the connected box $\mathcal{Q}$. If two cells $\mathcal{Q}, \mathcal{Q}'$ of L_{20} meet and $\tilde{s}^{\mathcal{Q}} + \tilde{\varepsilon}^{\mathcal{Q}} + \tilde{s}^{\mathcal{Q}'} < \tilde{\varepsilon}_U^{\mathcal{Q}'}$, then at a common point p we have $\|\tilde{\xi}_{\mathcal{Q}}(p) - \tilde{P}_\xi^{\mathcal{Q}'}(p)\| < \tilde{\varepsilon}_U^{\mathcal{Q}'}$, so $\tilde{\xi}_{\mathcal{Q}}(p) = \tilde{\xi}_{\mathcal{Q}'}(p)$ and $N_{\mathcal{Q}} = N_{\mathcal{Q}'}$. This test holds, exactly and in both orders, for all 24,872 pairs of cells of L_{20} that meet (found by an exact test), and the graph of these pairs is connected. On an origin cube $g(0, 0) = 0$ and, as checked, $\|\tilde{P}_\xi^{\mathcal{Q}}(0)\| < \tilde{\varepsilon}_U^{\mathcal{Q}}$, so $\tilde{\xi}_{\mathcal{Q}}(0) = 0$, $\tilde{H}_{\mathcal{Q}}(0) = H_1$, and $N_{\mathcal{Q}}(0) = 48$ by Theorem 2.8 (b)

and Section 6.3, as $H_1 = H(1,1,1)$ has the columns of $F^{\mathsf{T}}(1,1) = F_6$ (Section 6.1). Hence $N_{\mathcal{Q}} = 48$ on every cell of L_{20}.

The chart point. By Lemma 8.5 (b), p lies in an origin cube or a wide cell $\mathcal{Q}_c = \mathcal{Q}_l$, or in a box $\mathcal{Q}_c$ of (Q5) and a ring cell $\mathcal{Q}_l$; both contain p, so $d_{\mathcal{Q}_c}, d_{\mathcal{Q}_l} \le \|p\| < r$ and $\mathcal{Q}_c$ belongs to L_{20}. An exact check confirms that

$$\tilde{s}^{\mathcal{Q}} + (r^2 - d_{\mathcal{Q}}^2)^{1/2} < \tilde{\varepsilon}_U^{\mathcal{Q}}$$

for every cell $\mathcal{Q}$ of L_{20}, and the same inequality without tildes for the origin cubes, the wide cells of L_{20}, the undivided ring cells with $d_{\mathcal{Q}} \le r$ and the sub-cells of the divided boxes, with the data of Proposition 8.2 computed in ball arithmetic on these cells and with $s^{\mathcal{Q}}$ the upper bound of $\sup_{\mathcal{Q}} \|P_\xi^{\mathcal{Q}}\|$ certified on $\mathcal{Q}$. Since $\|\xi\| < (r^2 - \|p\|^2)^{1/2}$, this gives $\|\xi - \tilde{P}_\xi^{\mathcal{Q}_c}(p)\| < \tilde{\varepsilon}_U^{\mathcal{Q}_c}$ and $\|\xi - P_\xi^{\mathcal{Q}_l}(p)\| < \varepsilon_U^{\mathcal{Q}_l}$, and as $g(p,\xi) = 0$, $\xi = \tilde{\xi}_{\mathcal{Q}_c}(p) = \xi_{\mathcal{Q}_l}(p)$. So $H = \tilde{H}_{\mathcal{Q}_c}(p) = H_{\mathcal{Q}_l}(p)$.

The lines. As H is a complex Hadamard matrix, its common unbiased lines with I are the zeros of $R(\cdot; H)$ modulo $2\pi\mathbb{Z}^5$ (Section 6.3); there are $N_{\mathcal{Q}_c}(p) = 48$ of them, and the 48 distinct lines $w(\theta_j(p))$ of Proposition 8.2 (b) on $\mathcal{Q}_l$ are all of them. If a triplet contains a pair mapped onto $\{I, H/\sqrt{6}\}$ by a unitary or antiunitary map, the image of its third basis is a third basis of $\{I, H/\sqrt{6}\}$, so it consists of six pairwise orthogonal lines $w(\theta_j(p))$, and Corollary 8.4, applied to $H = H_{\mathcal{Q}_l}(p)$ and this basis, gives $H \in \Phi_I \cup \Phi_H$ and a clock of the triplet. □

Lemma 8.7. *Let R' be a record with center P', frame T' and $k = 4$, and let $\nu = n_{R'}$ as in Section 6.4, so that every member $e(P' + \delta)$ satisfies $\|(I - T'T'^{\mathsf{T}})\delta\| \le \nu$. Let s be one of the points of Lemma 8.1, $\vartheta = \arg H(d(s))$ (entrywise, in $\frac{2\pi}{12}\mathbb{Z}$), $N \in \mathbb{Z}^{36}$, $\rho > 0$ with $\frac{113}{710}\rho > \nu$, and $\mathcal{C} \subset \mathbb{R}^4$ a closed box. If*

$$\|\Pi(P' - \vartheta/2\pi - N) + T'\eta_v\| \le \tfrac{113}{710}\rho - \nu$$

at the 16 vertices η_v of $\mathcal{C}$, then every member $H = e(P' + \delta)$ of R' with $T'^{\mathsf{T}}\delta \in \mathcal{C}$ is equivalent to $H_1 \circ \mathrm{e}^{i\psi}$ with $\|\Pi\psi\| < \rho$.

Proof. The left side is convex in η, so the inequality holds on $\mathcal{C}$. Let $\eta = T'^{\mathsf{T}}\delta \in \mathcal{C}$; since δ is horizontal, $\Pi(P' + \delta - \vartheta/2\pi - N) = \Pi(P' - \vartheta/2\pi - N) + T'\eta + (I - T'T'^{\mathsf{T}})\delta$, of norm at most $\frac{113}{710}\rho < \rho/2\pi$ (as $355/113 > \pi$). With $\psi' = 2\pi(P' + \delta - N) - \vartheta$, $H = H(d(s)) \circ \mathrm{e}^{i\psi'}$ and $\|\Pi\psi'\| < \rho$. By Lemma 8.1, $H(d(s)) = P_\sigma H_1 \Sigma_k^{-1}$, so $H = P_\sigma(H_1 \circ \mathrm{e}^{i\psi})\Sigma_k^{-1}$, where ψ is ψ' with rows and columns permuted, and $\|\Pi\psi\| = \|\Pi\psi'\|$. □

8.4. **Cells decided through the chart.** Let $\mathcal{C}$ be a cell of a candidate cube of a record (Section 6.5) carrying (C1)–(C3), with anchor s_A, members $H_A \circ \mathrm{e}^{i\varphi}$ up to row and column phases for the parameters of the cell, and certified branches θ_i^G, $i \in J_{\mathcal{C}}$, with uniqueness radii ϱ_i; by Proposition 6.11 (a), every line of every third basis of a member held by $\mathcal{C}$ is one of the branches. Choose k, σ as in Lemma 8.1 and put $s' = s_A + \arg g$. By Lemma 8.1,

$$P_\sigma(H_A \circ \mathrm{e}^{i\varphi})\Sigma_k = H_1 \circ \mathrm{e}^{i\psi}, \qquad \psi = \varphi_F(s') + P_\sigma\varphi\Sigma_k,$$

where $\varphi_F(s)_{(a,i),(h,m)} = h\, s_i$ (with $s_0 = 0$) and $P_\sigma\varphi\Sigma_k$ is φ with rows and columns permuted. Write $\psi = Ap + \Lambda\xi + t$ with t in the space L of row and column phases. Since $[A \,|\, \Lambda]$ together with a basis of L is invertible, (p, ξ, t) is an affine function of the parameters. Here Proposition 8.2 is applied, on the chart cells of (R1), to the chart data that the interface checker computes in ball arithmetic for each such cell (Appendix A.3). Suppose:

(R1) the box enclosing p over the cell is covered by closed pieces, each in a chart cell $\mathcal{Q}$ of Proposition 8.2, and $\bar{\xi} + \sup_{\mathcal{Q}} \|P^{\mathcal{Q}}_{\xi}\| < \varepsilon^{\mathcal{Q}}_{U}$ for an upper bound $\bar{\xi}$ of $\|\xi\|$ over the cell;

(R2) on each piece, for every $i \in J_{\mathcal{C}}$ there is j such that every point of a box X'_j lies within ϱ_i of every point of a box that contains the branch θ^G_i for all parameters of the cell, where X'_j contains the images, under the line map of P_σ^{-1} and the shift by the row phases of t, of all points within $\varepsilon^{\mathcal{Q}}_j$ of $P^{\mathcal{Q}}_{\theta,j}(p)$, for p in the piece and all parameters.

Proposition 8.8. *If* (R1) *and* (R2) *hold, every triplet* $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ *with* H *a member held by the cell has a clock.*

Proof. Let H be such a member, with chart data (p, ξ, t), and let p lie in a piece inside $\mathcal{Q}$. (i) $H_1 \circ \mathrm{e}^{i\psi}$ is Hadamard, hence so is $\hat{H}(Ap+\Lambda\xi)$, which differs from it by row and column phases, so $g(p, \xi) = 0$, and $\|\xi - P^{\mathcal{Q}}_{\xi}(p)\| < \varepsilon^{\mathcal{Q}}_{U}$ by (R1), so $\xi = \xi_{\mathcal{Q}}(p)$ and $\hat{H}(Ap + \Lambda\xi) = H_{\mathcal{Q}}(p)$. (ii) A vector x is unbiased to the columns of $D_1 H_{\mathcal{Q}}(p) D_2$, for diagonal unitary D_1, D_2, if and only if $D_1^* x$ is unbiased to those of $H_{\mathcal{Q}}(p)$, and $|\langle P_\sigma v, P_\sigma x\rangle| = |\langle v, x\rangle|$. So the images θ'_j of the lines $\theta_j(p)$ under the shift and P_σ^{-1} are lines unbiased to I and $H_A \circ \mathrm{e}^{i\varphi}$, and $\theta'_j \in X'_j$ by Proposition 8.2 (b). By (R2) and the uniqueness radii, each branch θ^G_i at these parameters is one of the θ'_j. (iii) The lines of B are, in the anchor frame, six branches, hence six of the θ'_j, hence the images of six pairwise orthogonal lines $w(\theta_j(p))$; these form a third basis $B'/\sqrt{6}$ of $\{I, H_{\mathcal{Q}}(p)/\sqrt{6}\}$. The unitary map that undoes the shift by the row phases of t and P_σ^{-1} of (ii) is a product of a permutation matrix and a diagonal unitary matrix. It carries the triplet $\{I, H/\sqrt{6}, B/\sqrt{6}\}$, written in the anchor frame, onto $\{I, H_{\mathcal{Q}}(p)/\sqrt{6}, B'/\sqrt{6}\}$. That triplet has a clock by Corollary 8.4, and so has $\{I, H/\sqrt{6}, B/\sqrt{6}\}$ (Lemma 3.1 (a)). □

8.5. The criterion near F_6.

Theorem 8.9. *Let* R, π_r, π_c *and* R' *be as in Theorem 6.12. Suppose that the conclusion of Lemma 6.9 holds for* R' *to some level* $L \le 3$ *with candidate cubes each of which* (i) *is held by finitely many cells on each of which an interface certificate gives the outcome excluded or decided, or* (C1)–(C3) *hold together with* (R1) *and* (R2)*; or* (ii) *is a cube* $\mathcal{C}$ *for which Lemma 8.7 applies with* $\rho \le 0.1999$. *Then every triplet through a member of* R *has a clock.*

Proof. As in Theorem 6.12, with Proposition 6.11 (c) on the cells with an outcome and Proposition 8.8 on the cells decided through the chart; in case (ii) Lemma 8.7 with Theorem 8.6. □

8.6. The records near F_6.

Proposition 8.10 (computer-assisted)**.** *For each of the* 74 *records of* $\mathcal{L}^+$ *listed in the deposit with permutations* π_r, π_c*, the hypotheses of Theorem 8.9 hold. For each of these records, every triplet through a member has a clock.*

9. Proof of Theorem 1.2 and its consequences

Proposition 9.1 (computer-assisted)**.** *The sets $\mathcal{K}$ of Section 5 and $\mathcal{S}$ of Section 7 are disjoint subsets of $\mathcal{L}^+$, and $\mathcal{L}^+ \setminus (\mathcal{K} \cup \mathcal{S})$ consists of 557 records: the 483 records of Proposition 6.13 and the 74 records of Proposition 8.10, two disjoint lists.*

Proof. Exact comparison of the deposited lists of record numbers. □

Proof of Theorem 1.2. Let $\mathcal{R}$ be the cover of Proposition 4.1 and $\mathcal{L} = \mathcal{L}^+$, a list for $\mathcal{R}$ by Proposition 4.3. Apply Lemma 3.2 with $\mathcal{M} = \mathcal{K} \cup \mathcal{S}$, where $\mathcal{K}$ and $\mathcal{S}$ are disjoint subsets of $\mathcal{L}$ by Proposition 9.1, and with E consisting of the pairs (R, R') with $R \in \mathcal{K}$ and $R' \in N(R) \cap \mathcal{K}$ (Theorem 5.1), the pairs (R, S) with $R \in \mathcal{K}$ and $S \in \mathcal{S}$, and the pairs (S, S') with $S, S' \in \mathcal{S}$ and $\lambda(S') < \lambda(S)$ (Section 7). No edge leads from $\mathcal{S}$ to $\mathcal{K}$, λ decreases along the edges within $\mathcal{S}$ and ρ along those within $\mathcal{K}$ (Theorem 5.1 (b)), so $(\mathcal{M}, E)$ has no directed cycle. By Proposition 9.1 every record of $\mathcal{L} \setminus \mathcal{M}$ is one of the records of Propositions 6.13 and 8.10 (Table 1), which gives (i). For $R \in \mathcal{K}$, every record of $\mathcal{K}$ in which B^* lies is in $N(R)$ by Theorem 5.1 (a), and $(R, S) \in E$ for every $S \in \mathcal{S}$; for $R \in \mathcal{S}$, by Theorem 7.3 the triplet has a clock or every record of $\mathcal{M}$ in which B^* lies is some $S' \in \mathcal{S}$ with $\lambda(S') < \lambda(R)$. This gives (ii). So every triplet of mutually unbiased bases of $\mathbb{C}^6$ has a clock (Lemma 3.2), and by Theorem 2.4 it is permutationally unitarily equivalent to a triplet of the construction. □

Table 1. How the 1,565 records of $\mathcal{L}^+$ are treated.

Records	Statement	Number
two-circulant records $\mathcal{K}$	Theorem 5.1	978
records near D, $\mathcal{S}$	Theorem 7.3	30
Fourier type, closed in Section 6	Proposition 6.13	483
Fourier type, closed in Section 8	Proposition 8.10	74
all		1,565

9.1. **Pairs that lie in a triplet.** Equivalence in the usual sense, $\cong$, is that of Section 2.4. A 2-circulant matrix is a 2×2 array of circulant 3×3 blocks.

Lemma 9.2. *Let $H = \left(\begin{smallmatrix} A & C \\ A' & C' \end{smallmatrix}\right)$ be a 2-circulant complex Hadamard matrix of order six, and a_k, c_k, a'_k, c'_k the eigenvalues of A, C, A', C' on $\chi_k = (\omega^{kl})_l$, $k \in \mathbb{Z}/3$.*

(a) *$|a_k|^2 + |c_k|^2 = 6$ and $(a'_k, c'_k) = \varepsilon_k(-\overline{c_k}, \overline{a_k})$ with $|\varepsilon_k| = 1$; no a_k and no c_k vanishes.*

(b) *Multiplication on the left and on the right by matrices $\operatorname{diag}(\lambda S^m, \lambda' S^{m'})$ ($\lambda, \lambda' \in \mathbb{T}$, $m, m' \in \mathbb{Z}/3$), which commute with $T_3 = I_2 \otimes S$, carries H to a matrix (1.3) with $x, y, x', y' \in \mathbb{T}$ satisfying (1.4) if $\varepsilon_k = \varepsilon_0 \omega^{nk}$ for some $n \in \mathbb{Z}/3$, and otherwise to $F_2 \otimes A_0$ with A_0 a circulant complex Hadamard matrix of order three.*

(c) *A matrix (1.3) with $x, y, x', y' \in \mathbb{T}$ is complex Hadamard if and only if (1.4) holds; and a complex Hadamard matrix $\left(\begin{smallmatrix} B_1 & B_2 \\ B_2^* & -B_1^* \end{smallmatrix}\right)$ with B_1, B_2 circulant 3×3 matrices, the form [29, (4.2)], is carried to such a matrix, which then satisfies (1.4), by multiplication on the left by $\operatorname{diag}(I_3, \overline{\mu\nu} I_3)$ and on the right by $\operatorname{diag}(\mu I_3, \nu I_3)$ for some $\mu, \nu \in \mathbb{T}$.*

(d) *Every circulant complex Hadamard matrix of order three is $\cong F_3$.*

(e) *H is $\cong F_6$ or $\cong$ a matrix (1.3) with $x, y, x', y' \in \mathbb{T}$ satisfying (1.4).*

Proof. A circulant 3×3 matrix with first row (m_0,m_1,m_2) has the eigenvector $\chi_k=(\omega^{kl})_l$, column k of F_3, with the eigenvalue $\hat m_k=\sum_l m_l\omega^{kl}$; for S this eigenvalue is ω^k. One computes $\sum_k|\hat m_k|^2=3\sum_l|m_l|^2$ and $\sum_k\hat m_k\overline{\hat m_{k+1}}=3\sum_l|m_l|^2\omega^{-l}$, so if the entries have modulus one,

$$\textstyle\sum_k|\hat m_k|^2=9\qquad\text{and}\qquad\sum_k\hat m_k\overline{\hat m_{k+1}}=0.\tag{9.1}$$

Conjugation by the unitary matrix $I_2\otimes F_3/\sqrt3$ turns the four blocks of H into the diagonal matrices of their eigenvalues a_k,c_k,a'_k,c'_k, so $HH^*=6I$ says that each $\left(\begin{smallmatrix}a_k&c_k\\a'_k&c'_k\end{smallmatrix}\right)$ is $\sqrt6$ times a unitary matrix: $|a_k|^2+|c_k|^2=6$ and $(a'_k,c'_k)=\varepsilon_k(-\overline{c_k},\overline{a_k})$ with $|\varepsilon_k|=1$. By (9.1) for A, C, C' and A', the vectors $s=(\overline{a_k}a_{k+1})_k$ and $t=(\overline{c_k}c_{k+1})_k$ satisfy

$$\textstyle\sum_k s_k=\sum_k t_k=\sum_k\varepsilon_k\overline{\varepsilon_{k+1}}\,s_k=\sum_k\varepsilon_k\overline{\varepsilon_{k+1}}\,t_k=0.$$

No a_k vanishes: if $a_k=0$, then $s_{k-1}=s_k=0$, so $s_{k+1}=0$, a second eigenvalue of A vanishes, and the third, a_l, has $|a_l|^2=9$ by (9.1), against $|a_l|^2+|c_l|^2=6$. Likewise no c_k vanishes. This proves (a).

Suppose first that $\varepsilon_k=\varepsilon_0\omega^{nk}$ for some $n\in\mathbb{Z}/3$. As C^* and A^* are circulant with the eigenvalues $\overline{c_k}$ and $\overline{a_k}$, $A'=-\varepsilon_0S^nC^*$ and $C'=\varepsilon_0S^nA^*$, and multiplying the last three rows by the scalar multiple $-\overline{\varepsilon_0}S^{-n}$ of a permutation matrix gives $\left(\begin{smallmatrix}A&C\\C^*&-A^*\end{smallmatrix}\right)$. For $\mu,\nu\in\mathbb{T}$, multiplying this on the left by $\operatorname{diag}(I_3,\overline{\mu\nu}I_3)$ and on the right by $\operatorname{diag}(\mu I_3,\nu I_3)$ replaces A and C by μA and νC. So we may take the first entries of A and C to be 1, and then $A=\operatorname{circ}(1,\bar x,\overline{xy})$ and $C=\operatorname{circ}(1,\overline{x'},\overline{x'y'})$ for some $x,y,x',y'\in\mathbb{T}$: this is (1.3). For every such matrix, as circulant matrices commute, the product with its adjoint has vanishing off-diagonal blocks and both diagonal blocks equal to AA^*+CC^*, a Hermitian circulant matrix with diagonal 6 and entry $(0,1)$ equal to $\overline{x+y+\overline{xy}+x'+y'+\overline{x'y'}}$. So a matrix (1.3) is complex Hadamard exactly when (1.4) holds, and here it holds.

Otherwise the numbers $\varepsilon_k\overline{\varepsilon_{k+1}}$ are not all equal (equal values would be a cube root of unity $\bar\gamma$ with $\varepsilon_k=\varepsilon_0\gamma^k$), so s and t are orthogonal, in the bilinear sense, to two independent vectors of $\mathbb{C}^3$, and are linearly dependent. As no a_k vanishes, $s\neq0$, so t is a multiple of s: with $r_k=c_k/a_k$, the number $\overline{r_k}r_{k+1}=t_k/s_k$ does not depend on k. Then $|r_0r_1|=|r_1r_2|=|r_2r_0|$, all $|r_k|$ are equal, all $|a_k|$ are equal by $|a_k|^2(1+|r_k|^2)=6$, so $|a_k|^2=3=|c_k|^2$ by (9.1) and $|r_k|=1$. Hence r_{k+1}/r_k is a constant cube root of unity ω^n, $r_k=r_0\omega^{nk}$ and $C=r_0S^nA$. Multiplying the last three columns by $\overline{r_0}S^{-n}$ turns C into A and C' into a circulant matrix A'' with the eigenvalues $\overline{r_0}\omega^{-nk}\varepsilon_k\overline{a_k}$; as $\overline{c_k}=\overline{r_0}\omega^{-nk}\overline{a_k}$, these are $-a'_k$, so $A'=-A''$. The matrix $\left(\begin{smallmatrix}A&A\\-A''&A''\end{smallmatrix}\right)$ is complex Hadamard, so $AA^*=A''A''^*=3I$.

A circulant complex Hadamard matrix of order three with first row (m_0,m_1,m_2) has $\sum_l m_l\overline{m_{l-1}}=0$, three numbers of modulus one with sum 0 and product 1, which are therefore $1,\omega,\omega^2$ in some order. The six orders give the first rows of the scalar multiples of $S^n\operatorname{circ}(1,\omega,\omega)$ and of $S^n\overline{\operatorname{circ}(1,\omega,\omega)}$, $n\in\mathbb{Z}/3$. If A were a multiple of one of these two kinds and A'' of the other, then $A''A$ would be a multiple of a power $S^{n''}$, as $\operatorname{circ}(1,\omega,\omega)\overline{\operatorname{circ}(1,\omega,\omega)}=3I$; its eigenvalues $3\overline{r_0}\omega^{-nk}\varepsilon_k$ would be a constant times $\omega^{n''k}$, and ε_k would be $\varepsilon_0\omega^{(n+n'')k}$, the first case. So $A''=\lambda'S^{n'}A$ for some $\lambda'\in\mathbb{T}$ and n', and multiplying the last three rows by $-\overline{\lambda'}S^{-n'}$ gives $\left(\begin{smallmatrix}A&A\\A&-A\end{smallmatrix}\right)=F_2\otimes A$. Since $\operatorname{circ}(1,\omega,\omega)=\operatorname{diag}(1,\omega,\omega)F_3\operatorname{diag}(1,\omega,\omega)$, each of the matrices $\lambda S^n\operatorname{circ}(1,\omega,\omega)$ and $\lambda S^n\overline{\operatorname{circ}(1,\omega,\omega)}$ is $\cong F_3$ or $\cong\overline{F_3}$, so each is $\cong F_3$, as $\overline{F_3}$ is F_3 with its columns permuted (column k of F_3 is column $-k$ of $\overline{F_3}$); this proves (d). So $F_2\otimes A\cong F_2\otimes\overline{F_3}$, which is $F^{\mathsf T}(1,1)=F_6$ by (2.2); with (b) this proves (e). ∎

Proof of Corollary 1.5. By Lemma 3.1 (b), if $H' \sim H$, then $\{I, H/\sqrt{6}\}$ has a third basis exactly when $\{I, H'/\sqrt{6}\}$ has one. So for the "if" direction it suffices to find a triplet containing $\{I, H/\sqrt{6}\}$ when H is $F^{\mathsf{T}}(x,y)$, $F(x,y)$ or (1.3) with (1.4). For $F^{\mathsf{T}}(x,y)$ we follow [46, Corollary 6.2]: by [14, Theorem 2] there are diagonal unitary matrices Δ and Δ' such that $\Delta F^{\mathsf{T}}(x,y)^*\Delta'/\sqrt{6}$ has all row sums equal to 1. With $\mathbf{1} = (1,\dots,1)$, the unit vector $v = \Delta'\mathbf{1}/\sqrt{6}$ has entries of modulus $1/\sqrt{6}$, and so has $F^{\mathsf{T}}(x,y)^*v/\sqrt{6} = \Delta^{-1}\mathbf{1}/\sqrt{6}$; so v is unbiased to the standard basis and to the columns of $F^{\mathsf{T}}(x,y)/\sqrt{6}$, and Proposition 2.6 completes it to a third basis. As $F(x,y) = F^{\mathsf{T}}(x,y)^{\mathsf{T}} \sim F^{\mathsf{T}}(x,y)$, Lemma 3.1 (b) gives the same for $F(x,y)$. The matrix (1.3) is 2-circulant, and complex Hadamard by (1.4) (Lemma 9.2 (c)), so by [47, Proposition 4.2], after [56, Proposition 5.6], it is $\sqrt{6}\,U_1^*U_2$ with unitary matrices U_1 and U_2 whose entries have modulus $1/\sqrt{6}$. Then $\{I, U_1, U_2\}$ is a triplet, as $U_1^*U_2$ has entries of modulus $1/\sqrt{6}$, and the unitary map U_1^* carries it to a triplet containing $\{I, H/\sqrt{6}\}$.

For the "only if" direction let $\{I, H/\sqrt{6}\}$ lie in a triplet $\mathcal{T}$. By Theorem 1.2, $\mathcal{T}$ is permutationally unitarily equivalent to $\mathcal{T}' = \{I, H_1/\sqrt{6}, H_2/\sqrt{6}\}$ with $H_r = D_rF^{\mathsf{T}}(x_r,y_r)$, so by Lemma 3.1 (d) the matrix $H = \sqrt{6}\,I^*(H/\sqrt{6})$ is $\cong$ a matrix $\sqrt{6}\,U^*V$ for two different bases U and V of $\mathcal{T}'$, that is, $\cong$ one of H_1, H_2, $H_1^*H_2/\sqrt{6}$ and their adjoints. Here $\overline{H_r} \cong F^{\mathsf{T}}(x_r,y_r)$ and $H_r^* \cong F^{\mathsf{T}}(x_r,y_r)^*$, which is $F(\bar{x}_r,\bar{y}_r)$ with its rows permuted, since $\overline{F^{\mathsf{T}}(x,y)}$ is $F^{\mathsf{T}}(\bar{x},\bar{y})$ with its columns permuted (proof of Lemma B.5). Finally $H_1^*H_2/\sqrt{6}$ is a 2-circulant complex Hadamard matrix by Theorem 2.4 (a), and so is its adjoint, so Lemma 9.2 applies to both, with $F_6 = F^{\mathsf{T}}(1,1)$. □

Dephased (first row and first column all ones), the matrices (1.3) are Szöllősi's matrices [47, (11)], with x', y' for his u, v, denoted $X_6^{(2)}$ in [36, (7.14)]: for $\alpha = x+y+\overline{xy}$ the numbers x, y, $\overline{xy}$ are the roots of $z^3 - \alpha z^2 + \bar{\alpha}z - 1$, and x', y', $\overline{x'y'}$ those of the same polynomial with $-\alpha$ for α, as in [47, (14)]; every choice of two roots of each polynomial occurs. Up to the order of rows and columns, the matrices $F_6^{(2)}$ of [36, (7.15)] are the matrices $F^{\mathsf{T}}(x,y)$, the transposes of those of [48, Section 5.6.1]. So Corollary 1.5 is [36, Conjecture 8.1], with $X_6^{(2)}$ read as the set of all these matrices. Szöllősi states, from a computer calculation, that each of them is equivalent to the member of his family $X(\alpha)$ at the same α or to its transpose [47, Section 2].

Remark 9.3 (partly computer-assisted). Let Ω be a connected real-analytic manifold, $H\colon \Omega \to \mathcal{H}$ a real-analytic map, E the set of $m \in \Omega$ for which $\{I, H(m)/\sqrt{6}\}$ lies in a triplet, and E' the set of m for which $H(m)$ is $\cong$ one of the matrices of Corollary 1.5. Then (a) E' is closed, $E' \subseteq E$, and $E' = \Omega$ or E' lies in the zero set of a real-analytic function $g\colon \Omega \to \mathbb{C}$ (equivalently of $|g|^2$) that is not identically zero, so that E' has empty interior and Lebesgue measure zero in any chart; (b) by Theorem 1.2, $E = E'$.

Indeed, two 6×6 matrices X and Y whose entries have modulus one satisfy $Y = D_1XD_2$ for diagonal unitary D_1, D_2 if and only if $\widetilde{X} = \widetilde{Y}$, where $\widetilde{X}_{ij} = X_{ij}X_{00}/(X_{i0}X_{0j})$, since $\widetilde{D_1XD_2} = \widetilde{X}$ and $X = \operatorname{diag}(X_{i0}/X_{00})\,\widetilde{X}\,\operatorname{diag}(X_{0j})$. The entries of $F^{\mathsf{T}}(x,y)$, of $F(x,y)$ and of the matrices (1.3) are constants of modulus one times Laurent monomials in the parameters, so for each of the three families the matrices $\widetilde{X}$ of its members with parameters in $\mathbb{T}$ form a constant times the image of a torus under a continuous homomorphism: a coset of a closed subgroup of $\mathbb{T}^{36}$, by Pontryagin duality the set of $z \in \mathbb{T}^{36}$ with $z^{n_s} = c_s$, $s = 1,\dots,r$, for some $n_s \in \mathbb{Z}^{36}$ and $c_s \in \mathbb{T}$. A matrix (1.3) with the same $\widetilde{X}$ as a complex Hadamard matrix is complex Hadamard, so it satisfies (1.4) (Lemma 9.2 (c)). As PDP^{-1}

is diagonal for a permutation matrix P and a diagonal matrix D, the set E' is the union, over the three families and the permutation matrices P_1, P_2, of the $3 \cdot 720^2$ sets on which $P_1\widetilde{H(m)}P_2$ lies in the family's coset, each the common zero set of the real-analytic functions $m \mapsto P_1\widetilde{H(m)}P_2{}^{n_s} - c_s$, so closed. If for one of these sets all these functions vanish identically, $E' = \Omega$. Otherwise choose for each set one of them, g_i, that does not; then E' lies in the zero set of $g = \prod_i g_i$, the union of the zero sets of the g_i, each closed, with empty interior by the identity theorem and of measure zero [38]. Finally, $E' \subseteq E$ is the "if" direction of Corollary 1.5, with Lemma 3.1 (b), and the "only if" direction, by Theorem 1.2, gives $E \subseteq E'$.

For Karlsson's family $K_6^{(2)}(x_1, x_2)$, displayed in [11, Section V.A], the square roots in the entries are of $1/(1 \pm \sin x_1 \sin x_2) - 1/4$, at least $1/4$ on $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : |\sin x_1 \sin x_2| \neq 1\}$, the plane without a discrete set, which is connected; so the family is real-analytic on Ω, and complex Hadamard there ([24] for $-\pi/2 < x_1, x_2 < \pi/2$, hence on Ω by the identity theorem). The Haagerup set of $K_6^{(2)}(1, 1/20)$ has at least 90 elements, by a computation in the deposit [21] with integer intervals, which an independent program repeats in ball arithmetic. That of $F^{\mathsf{T}}(x, y)$, and so that of its transpose $F(x, y)$, has at most 40, since by (2.2) it lies in $\{\pm 1, \omega, \omega^2\} \cup \{\pm\omega^k q : k \in \{0, 1, 2\},\ q \in \{x^{\pm 1}, y^{\pm 1}, (x/y)^{\pm 1}\}\}$; that of a matrix (1.3) has at most 50, since, written out from its entries, it lies in $\{\pm 1\} \cup \{\gamma^{\pm 1}, \gamma'^{\pm 1}\} \cup \{\pm(\beta/\beta')^{\pm 1}\}$ with $\gamma \in \{x^2 y, \bar{x}y, \bar{x}\bar{y}^2\}$, $\beta \in \{xy, \bar{x}, \bar{y}\}$ and γ', β' in the same sets with x', y' in place of x, y. So $K_6^{(2)}(1, 1/20)$ is $\cong$ none of the matrices of Corollary 1.5, and $E' \neq \Omega$. Hence, by Theorem 1.2, a parameter drawn from a distribution with a density lies in a triplet with probability zero.

Goyeneche found triplets numerically through $\{I, H/\sqrt{6}\}$ for H in parts of Karlsson's family $K_6^{(2)}$ [24], among others [11, Section V and Table I]; McNulty and Weigert note that their conjecture then seems to require $K_6^{(2)}$ to meet $X_6^{(2)}$ in ways not yet known [36, Section 8.1]. By Theorem 1.2, each such H whose pair lies in a triplet is $\cong$ one of the matrices of Corollary 1.5. For $K_6^{(2)}$, by the above, the parameters $(x_1, x_2) \in \Omega$ whose pair lies in a triplet form a closed set of measure zero, so the two-dimensional regions of [11, Fig. 1] consist, up to a set of measure zero, of approximate triplets.

9.2. **The two-parameter family and the isolated triplet.** The "in particular" sentence of the Budapest conjecture [29, Conjecture 1] names two cases, a two-parameter family of triplets described in [47] and an isolated triplet whose three transition matrices are $\cong F_6$; read as in Remark 9.11, the first is property P and the second permutational unitary equivalence to $\mathcal{T}_0$. By Theorem 1.2, every triplet lies in exactly one of the two cases (Theorem 9.8); the class of $\mathcal{T}_0$, which contains the exceptional triplet of [29, Section 4] (Proposition 9.9), is an isolated point of the space of classes (Lemma 9.10); and the triplets with property P are parametrized up to equivalence, continuously, by Szöllősi's complex parameter α (Lemma 9.7; which values of α give equivalent triplets is open, Section 9.4) and include a real-analytic disc of dimension two of pairwise inequivalent triplets (Theorem 9.8 (d)).

We write a triplet as an ordered triple (X_1, X_2, X_3) of bases, each taken up to the order and phases of its vectors. The set $\mathfrak{T}$ of triplets is then a closed subset of $(U(6)/M)^3$, where M is the group of products of a permutation matrix and a diagonal unitary matrix, and is compact. Unitary maps and permutations of the three bases act on $\mathfrak{T}$ by homeomorphisms; for a set $\mathcal{S}$ of triplets, $[\mathcal{S}]$ denotes the set of triplets permutationally unitarily equivalent to a member of $\mathcal{S}$. With χ_k as in Lemma 9.2, $S\chi_k = \omega^k \chi_k$, and $T_3 = I_2 \otimes S$ has the eigenvalue

ω^k on the plane of the vectors $(\lambda\chi_k;\mu\chi_k)$. For a 2-circulant complex Hadamard matrix H we use the eigenvalues a_k, c_k, a'_k, c'_k of its blocks on χ_k and the numbers ε_k of Lemma 9.2 (a), by which no a_k and no c_k vanishes.

Lemma 9.4. *Let H be a 2-circulant complex Hadamard matrix of order six. Exactly one third basis of $\{I, H/\sqrt{6}\}$ consists of eigenvectors of T_3. We call it K_H; its vectors are $(\chi_k; \pm\kappa_k\chi_k)/\sqrt{6}$, $k \in \mathbb{Z}/3$, with $\kappa_k = i\varepsilon_k\overline{a_kc_k}/|a_kc_k|$.*

Proof. A basis of eigenvectors of T_3 has two vectors in each eigenspace. A unit vector $v = (\lambda\chi_k;\mu\chi_k)$ is unbiased to the standard basis exactly when $|\lambda| = |\mu| = 1/\sqrt{6}$; write $\mu = z\lambda$ with $|z| = 1$. Then $H^*v = ((\overline{a_k} + \overline{a'_k}z)\lambda\chi_k; (\overline{c_k} + \overline{c'_k}z)\lambda\chi_k)$ and $\|H^*v\|^2 = 6$, so v is unbiased to the columns of $H/\sqrt{6}$, that is, H^*v has entries of modulus one, exactly when $|\overline{a_k} + \overline{a'_k}z| = |\overline{c_k} + \overline{c'_k}z|$. Since $|a_k|^2 + |a'_k|^2 = |c_k|^2 + |c'_k|^2 = 6$ and $(a'_k, c'_k) = \varepsilon_k(-\overline{c_k}, \overline{a_k})$, the difference of the squares of the two sides is $2\operatorname{Re}((a_k\overline{a'_k} - c_k\overline{c'_k})z) = -4\operatorname{Re}(\overline{\varepsilon_k}a_kc_kz)$. As $a_kc_k \neq 0$, it vanishes exactly for $z = \pm\kappa_k$, and the two vectors $(\chi_k; \pm\kappa_k\chi_k)/\sqrt{6}$ are orthogonal. □

Lemma 9.5.

(a) *A triplet has a clock if and only if it is permutationally unitarily equivalent to $(I, H/\sqrt{6}, K_H)$ for a 2-circulant complex Hadamard matrix H.*

(b) *If H is a 2-circulant complex Hadamard matrix, $M_1, M_2 \in M$ commute with T_3 and $H' = M_1HM_2$, then H' is a 2-circulant complex Hadamard matrix and $(I, H/\sqrt{6}, K_H)$ and $(I, H'/\sqrt{6}, K_{H'})$ are permutationally unitarily equivalent.*

Proof. (a) T_3 is a clock of $(I, H/\sqrt{6}, K_H)$: $T_3^3 = I$, T_3 is not scalar, it fixes the lines of K_H, permutes the standard lines and maps column j of H to HT_3e_j, a column of H. Permutational unitary equivalence carries clocks to clocks (the end of the proof of Theorem 2.4), so every triplet permutationally unitarily equivalent to $(I, H/\sqrt{6}, K_H)$ has a clock. Conversely, by Theorem 2.4 a triplet with a clock is permutationally unitarily equivalent to $(I, H_1/\sqrt{6}, H_2/\sqrt{6})$ with $H_r = D_rF^{\mathsf{T}}(x_r, y_r)$ and $H = H_1^*H_2/\sqrt{6}$ 2-circulant. By (2.3), $ZH_1 = H_1T_3$, so $T_3H_1^* = H_1^*Z$ and the columns of $H_1^*/\sqrt{6}$ are eigenvectors of T_3. The unitary map $H_1^*/\sqrt{6}$ carries the triplet to $(H_1^*/\sqrt{6}, I, H/\sqrt{6})$, whose first basis is therefore K_H (Lemma 9.4).

(b) H' commutes with T_3 as M_1, H and M_2 do. The unitary map M_1 carries $(I, H/\sqrt{6}, K_H)$ to $(M_1, H'M_2^{-1}/\sqrt{6}, M_1K_H)$. Its first two bases have the lines of I and of $H'/\sqrt{6}$, and M_1K_H consists of eigenvectors of T_3, as M_1 commutes with T_3; so it is $K_{H'}$. □

Let $\mathcal{V}$ be the set of $p = (x, y, x', y') \in \mathbb{T}^4$ satisfying (1.4), let $S(p)$ be the matrix (1.3), complex Hadamard by Lemma 9.2 (c), and let $\Gamma(p) = (I, S(p)/\sqrt{6}, K_{S(p)})$ and $\mathcal{Z} = \Gamma(\mathcal{V})$. *Szöllősi's triplets* [47, Proposition 4.2 and Theorem 4.3] are the triplets (I, Z_1, Z_2) with

$$Z_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} F_3 & XF_3 \\ F_3 & -XF_3 \end{pmatrix}, \qquad Z_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} UF_3 & UYF_3 \\ VF_3 & -VYF_3 \end{pmatrix},$$

with X, Y, U, V diagonal unitary, such that $Z_1^*Z_2 = N/\sqrt{6}$ with N a complex Hadamard matrix of the form [29, (4.2)]. For $\alpha \in \mathbb{C}$ let $f_\alpha(z) = z^3 - \alpha z^2 + \bar{\alpha}z - 1$, and let $\mathbb{D}$ be the set of α for which all roots of f_α and of $f_{-\alpha}$ have modulus one. For $p \in \mathcal{V}$ and $\alpha(p) = x + y + \overline{xy}$, the numbers $x, y, \overline{xy}$ are the roots of $f_{\alpha(p)}$ and $x', y', \overline{x'y'}$ those of $f_{-\alpha(p)}$ (as recalled in Section 9.1); so $\alpha(\mathcal{V}) = \mathbb{D}$, the points of $\mathcal{V}$ over α being given by the orderings of these roots, and $\mathbb{D}$ is the region of [47, Section 2]. The *cube* of a triplet (X_1, X_2, X_3), the Hadamard

cube of [29, (3.3)], has the entries $C_{jkl} = 6^{3/2}(X_1^*X_2)_{jk}(X_2^*X_3)_{kl}(X_3^*X_1)_{lj}$, and we put $P_n = \sum_{j,k,l} \mathrm{Re}(C_{jkl}^n)$. A unitary map or the rephasing of a vector does not change C, permuting the vectors of a basis or permuting the bases cyclically permutes its entries, and exchanging two bases replaces each entry by the conjugate of another, so P_n is invariant under permutational unitary equivalence.

For $p \in \mathcal{V}$ the blocks of $S(p)$ give $(a_k', c_k') = (\overline{c_k}, -\overline{a_k})$, so $\varepsilon_k = -1$ and $\overline{\kappa_k} = ig_k/|g_k|$ with $g_k = a_k c_k$, which does not vanish. As the g_k are polynomials in x, y, x', y' and their conjugates and $\mathcal{V}$ is compact, this formula defines Γ, as a triple of matrices, real-analytically on a neighborhood of $\mathcal{V}$ in $\mathbb{T}^4$; in particular Γ is continuous on $\mathcal{V}$.

Lemma 9.6. *Let $w(\theta) = 2\mathrm{e}^{i\theta} + \mathrm{e}^{-2i\theta}$ and let $\mathcal{X} = w(\mathbb{R})$, the deltoid.*

(a) *$\mathcal{X} = \{r(\psi)\mathrm{e}^{i\psi} : \psi \in \mathbb{R}\}$ for a continuous 2π-periodic function r with values in $[1, 3]$.*

(b) *f_α has a repeated root if and only if $\alpha \in \mathcal{X}$.*

(c) *All roots of f_α have modulus one if and only if $\alpha \in \widehat{\mathcal{X}} = \{\rho\mathrm{e}^{i\psi} : 0 \le \rho \le r(\psi)\}$, whose interior is $\{\rho\mathrm{e}^{i\psi} : 0 \le \rho < r(\psi)\}$.*

(d) *$\mathbb{D} = \widehat{\mathcal{X}} \cap (-\widehat{\mathcal{X}})$ is the closure of its interior $\{\rho\mathrm{e}^{i\psi} : 0 \le \rho < \min(r(\psi), r(\psi+\pi))\}$, which is star-shaped about 0 and contains the open unit disc. For α in the interior, f_α and $f_{-\alpha}$ have three simple roots of modulus one.*

Part (c) is stated, without proof, in [47, Fact 2]; for (b) compare [47, Fact 1 and (15)].

Proof. (a) $w(\theta) = \mathrm{e}^{i\theta}(2 + \mathrm{e}^{-3i\theta})$, so $|w(\theta)|^2 = 5 + 4\cos 3\theta \in [1, 9]$, and as $\mathrm{Re}(2 + \mathrm{e}^{-3i\theta}) \ge 1$, $h(\theta) = \theta + \mathrm{Arg}(2 + \mathrm{e}^{-3i\theta})$ is a continuous argument of w. Its derivative $\mathrm{Im}(w'/w) = 2(1 - \cos 3\theta)/(5 + 4\cos 3\theta)$ is positive off $(2\pi/3)\mathbb{Z}$, so h is an increasing homeomorphism of $\mathbb{R}$ with $h(\theta + 2\pi) = h(\theta) + 2\pi$; put $r(\psi) = |w(h^{-1}(\psi))|$.

(b) For $u \in \mathbb{T}$ the roots $u, u, \bar{u}^2$ have sum $2u + \bar{u}^2$, second symmetric function $u^2 + 2\bar{u}$ and product 1, so $(z - u)^2(z - \bar{u}^2) = f_{2u+\bar{u}^2}$, and $2u + \bar{u}^2 = w(\theta)$ for $u = \mathrm{e}^{i\theta}$. Conversely, $z^3\overline{f_\alpha(1/\bar{z})} = -f_\alpha(z)$, so $z \mapsto 1/\bar{z}$ permutes the roots of f_α with their multiplicities. A repeated root off $\mathbb{T}$ would give two repeated roots, so a repeated root u lies on $\mathbb{T}$, the third root is $\bar{u}^2$ and $\alpha = 2u + \bar{u}^2 \in \mathcal{X}$.

(c) Let $\mathcal{U}$ be the set of α for which all roots of f_α have modulus one; it is the image of $\mathbb{T}^2$ under $(x, y) \mapsto x + y + \overline{xy}$, as the roots have product 1 (Section 9.1). By (a), $\mathbb{C} \setminus \mathcal{X}$ is the union of the disjoint connected open sets $\mathcal{X}_{\mathrm{in}} = \{\rho < r(\psi)\} \ni 0$ and $\mathcal{X}_{\mathrm{out}} = \{\rho > r(\psi)\}$, which contains the circle $|z| = 4$. Off $\mathcal{X}$ the roots are simple, by (b), and the number of roots of modulus one is locally constant: a simple root $u \in \mathbb{T}$ of f_{α_1} continues to a root $u(\alpha)$ for α near α_1, and $1/\overline{u(\alpha)}$ is a root near u, hence equal to $u(\alpha)$; a root off $\mathbb{T}$ stays off $\mathbb{T}$. This number is 3 at 0, as $f_0 = z^3 - 1$, and less than 3 at 4, since three numbers of modulus one do not have the sum 4. On $\mathcal{X}$ the roots $u, u, \bar{u}^2$ have modulus one. So $\mathcal{U} = \mathcal{X}_{\mathrm{in}} \cup \mathcal{X} = \widehat{\mathcal{X}}$, and its interior is $\mathcal{X}_{\mathrm{in}}$, as every point of $\mathcal{X}$ is a limit of points of $\mathcal{X}_{\mathrm{out}}$.

(d) By (c), $\mathbb{D} = \widehat{\mathcal{X}} \cap (-\widehat{\mathcal{X}})$, whose interior is $\mathcal{X}_{\mathrm{in}} \cap (-\mathcal{X}_{\mathrm{in}})$, the set of $\rho\mathrm{e}^{i\psi}$ with $\rho < m(\psi) = \min(r(\psi), r(\psi + \pi))$. It is star-shaped and contains the open unit disc, as $m \ge 1$, and a point $\rho\mathrm{e}^{i\psi}$ of $\mathbb{D}$ is the limit of the points $t\rho\mathrm{e}^{i\psi}$, $t < 1$, of the interior. For α in the interior, $\pm\alpha \in \mathcal{X}_{\mathrm{in}}$, and the roots of $f_{\pm\alpha}$ are simple by (b) and of modulus one by (c). □

Let $\mathfrak{Q}$ be the set of classes of triplets under permutational unitary equivalence, with the quotient topology, and let $q\colon \mathfrak{T} \to \mathfrak{Q}$ be the quotient map. Since P_n is an invariant, it defines a continuous function $\bar{P}_n$ on $\mathfrak{Q}$ with $P_n = \bar{P}_n \circ q$.

Lemma 9.7. *If $p, p' \in \mathcal{V}$ and $\alpha(p) = \alpha(p')$, then $\Gamma(p)$ and $\Gamma(p')$ are permutationally unitarily equivalent. Hence $q \circ \Gamma = \gamma \circ \alpha$ on $\mathcal{V}$ for a continuous map $\gamma\colon \mathbb{D} \to \mathfrak{Q}$, and $\gamma(\mathbb{D}) = q([\mathcal{Z}])$.*

Proof. Write $S(p) = \left(\begin{smallmatrix} A & C \\ C^* & -A^* \end{smallmatrix}\right)$. For a circulant matrix with first row (v_0, v_1, v_2) of numbers of modulus one, call $(v_0/v_1, v_1/v_2, v_2/v_0)$ its ratio triple; those of A and C are $(x, y, \overline{xy})$ and $(x', y', \overline{x'y'})$, and a circulant matrix with first entry 1 is determined by its ratio triple. The points of $\mathcal{V}$ over α correspond to the orderings of the roots of f_α and of $f_{-\alpha}$, and the symmetric group on three letters is generated by the cyclic permutations and one reversal. So it suffices to show that the class of $\Gamma(p)$ does not change when one of the two ratio triples is permuted cyclically or reversed.

Cyclic permutations. Let Q be a power of the shift S. Then AQ is circulant, with the first row of A permuted cyclically, and as Q commutes with A and C, $\operatorname{diag}(I, Q^*) S(p) \operatorname{diag}(Q, I) = \left(\begin{smallmatrix} AQ & C \\ C^* & -(AQ)^* \end{smallmatrix}\right)$. If λ is the first entry of AQ, multiplying on the left by $\operatorname{diag}(\bar\lambda I, I)$ and on the right by $\operatorname{diag}(I, \lambda I)$ gives $S(p')$, where the ratio triple of A is permuted cyclically and that of C is kept. In the same way, with λ now the first entry of CQ, $\operatorname{diag}(I, \lambda I) \operatorname{diag}(I, Q^*) S(p) \operatorname{diag}(I, Q) \operatorname{diag}(I, \bar\lambda I) = S(p')$ with the ratio triple of C permuted. These multipliers lie in M and commute with T_3, so Lemma 9.5 (b) shows that the class of $\Gamma(p)$ is kept.

Reversing the first triple. $S(p)^* = \left(\begin{smallmatrix} A^* & C \\ C^* & -A \end{smallmatrix}\right)$, and A^* is circulant with the first row $(1, xy, x)$, whose ratio triple $(\overline{xy}, y, x)$ is the reverse of that of A; so $S(p)^* = S(p'')$ with $p'' \in \mathcal{V}$. The unitary map $S(p)^*/\sqrt{6}$ carries $\Gamma(p)$ to $(S(p'')/\sqrt{6}, I, S(p)^* K_{S(p)}/\sqrt{6})$, whose last basis consists of eigenvectors of T_3, since $S(p)$ commutes with T_3; by Lemma 9.4 it is $K_{S(p'')}$, so the image is $\Gamma(p'')$ with its bases permuted.

Reversing both triples. Let R be the permutation matrix of $l \mapsto -l$ and $\tilde R = I_2 \otimes R$. Then conjugation by R transposes every circulant matrix, and $\tilde R T_3 \tilde R = T_3^{-1}$. Hence $\tilde R \overline{S(p)}\, \tilde R = \left(\begin{smallmatrix} A^* & C^* \\ C & -A \end{smallmatrix}\right) = S(p''')$, where both ratio triples are reversed. The unitary map $\tilde R$ carries the complex conjugate $\overline{\Gamma(p)}$ to a triplet whose first two bases are those of I and $S(p''')/\sqrt{6}$ and whose third consists of eigenvectors of T_3, as T_3 is real; by Lemma 9.4 it is $\Gamma(p''')$. Further, with $\hat R = \left(\begin{smallmatrix} 0 & R \\ R & 0 \end{smallmatrix}\right)$, $\hat R\, \overline{S(p)}\, \hat R = \left(\begin{smallmatrix} -A & C \\ C^* & A^* \end{smallmatrix}\right) = \operatorname{diag}(-I, I) S(p) \operatorname{diag}(I, -I)$, so the unitary matrix $U_0 = \left(\begin{smallmatrix} 0 & -R \\ R & 0 \end{smallmatrix}\right)$ satisfies $U_0 \overline{S(p)} = S(p) \operatorname{diag}(I, -I) \hat R$; as $U_0 T_3 = T_3^{-1} U_0$, Lemma 9.4 shows that U_0 carries $\overline{\Gamma(p)}$ onto $\Gamma(p)$. So $\Gamma(p''') = \tilde R U_0^{-1} \Gamma(p)$, and with the previous case, reversing only the second triple keeps the class.

Finally, $\alpha\colon \mathcal{V} \to \mathbb{D}$ is continuous and onto, and closed, as $\mathcal{V}$ is compact; so it is a quotient map, and $q \circ \Gamma$, which is continuous and constant on its fibers, factors through a continuous map γ. Its image is $q(\Gamma(\mathcal{V})) = q([\mathcal{Z}])$. □

Theorem 9.8 (computer-assisted)**.**

(a) *A triplet has a clock if and only if it is permutationally unitarily equivalent to a member of $\mathcal{Z}$ or to $\mathcal{T}_0$, and it is not equivalent to both. $[\mathcal{Z}]$ is the set of triplets permutationally unitarily equivalent to Szöllősi's triplets.*

(b) *Every member of $\mathcal{Z}$ has property P, and a triplet with a clock has property P if and only if it lies in $[\mathcal{Z}]$. The transition matrices of $\Gamma(p)$ in cyclic order are $S(p)$, which is $\ncong F_6$, a matrix $\cong F(x'', y'')$ for some $x'', y'' \in \mathbb{T}$, and $\sqrt{6} K_{S(p)}^* \cong F^{\mathsf T}(\tilde x, \tilde y)$ with $\tilde x = g_1|g_0|/(|g_1|g_0)$ and $\tilde y = g_2|g_0|/(|g_2|g_0)$, where $g_k = a_k c_k$ for $S(p)$.*

(c) *$[\mathcal{T}_0]$, the set of triplets whose three transition matrices are $\cong F_6$, is compact, and it is open in the set of triplets with a clock.*

(d) $[\mathcal{Z}]$ *is compact and does not meet* $[\mathcal{T}_0]$*. It contains a real-analytic disc of dimension two whose members are pairwise not permutationally unitarily equivalent, and every member of* $[\mathcal{Z}]$ *is a limit in* $\mathfrak{T}$ *of members of* $[\mathcal{Z}]$ *not permutationally unitarily equivalent to it.*

By Theorem 1.2 every triplet has a clock, so every triplet lies in exactly one of $[\mathcal{Z}]$ *and* $[\mathcal{T}_0]$*, and the triplets with property P are those of* $[\mathcal{Z}]$*.*

Proof. (b). Let $K = K_{S(p)}$, so that $\overline{\kappa_k} = ig_k/|g_k|$ (the paragraph before Lemma 9.6). Ordering the rows of K^* as $(+,k)$, then $(-,k)$, $\sqrt{6}K^* = \left(\begin{smallmatrix} \overline{F_3} & X\overline{F_3} \\ \overline{F_3} & -X\overline{F_3} \end{smallmatrix}\right)$ with $X = \operatorname{diag}(\overline{\kappa_k})$, which is $F^{\mathsf{T}}(\tilde{x},\tilde{y})\operatorname{diag}(I_3, \overline{\kappa_0}I_3)$ by (2.2). Let $T_3K = K\Lambda$ with Λ diagonal; then $K^*T_3 = \Lambda K^*$, and as $S(p)$ commutes with T_3, the basis $L = K^*S(p)/\sqrt{6}$ satisfies $LT_3 = \Lambda L$. A row vector r with $rS = \omega^k r$, for the shift S, is a multiple of row k of $\overline{F_3}$, and L is unbiased to I, so $\sqrt{6}L = \left(\begin{smallmatrix} U\overline{F_3} & U_1\overline{F_3} \\ V\overline{F_3} & V_1\overline{F_3} \end{smallmatrix}\right)$ with U, U_1, V, V_1 diagonal unitary. The two column blocks of L are orthogonal, so $\overline{U}U_1 + \overline{V}V_1 = 0$, that is, $U_1 = UY$ and $V_1 = -VY$ with $Y = \overline{U}U_1$; and $\sqrt{6}L = \operatorname{diag}(U,V)F^{\mathsf{T}}(y_1/y_0, y_2/y_0)\operatorname{diag}(I_3, y_0I_3)$ for $Y = \operatorname{diag}(y_k)$. The transition matrices of $\Gamma(p)$ in cyclic order are $S(p)$, $S(p)^*K = \sqrt{6}L^*$ and $\sqrt{6}K^*$. So the second is $\cong F^{\mathsf{T}}(y_1/y_0, y_2/y_0)^* \cong F(\overline{y_1/y_0}, \overline{y_2/y_0})$ (proof of Proposition 2.9 (d)), the third is $\cong F^{\mathsf{T}}(\tilde{x},\tilde{y})$, and $S(p) \ncong F_6$ by Proposition 2.9 (c). As $S(p)$ is a complex Hadamard matrix of the form [29, (4.2)], $\Gamma(p)$ has property P.

(a). By Lemma 9.5 (a) a triplet with a clock is permutationally unitarily equivalent to $(I, H/\sqrt{6}, K_H)$ with H 2-circulant. By Lemma 9.2 (b), multiplication on both sides by matrices $\operatorname{diag}(D, D')$ with D and D' scalar multiples of powers of the shift S, which commute with T_3, carries H to $S(p)$ for some $p \in \mathcal{V}$ if the ε_k are $\varepsilon_0\omega^{nk}$, and otherwise to $F_2 \otimes A$ with A a circulant complex Hadamard matrix of order three. In the first case the triplet is equivalent to $\Gamma(p)$ by Lemma 9.5 (b). In the second, $a'_k = c_k = a_k = -c'_k$, so $\varepsilon_k = -a_k/\overline{a_k}$, $\kappa_k = -i$, and $K_{F_2\otimes A}$ consists of the columns of $(Y_2 \otimes F_3)/\sqrt{6}$ with $Y_2 = \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$. The transition matrices of $(I, (F_2\otimes A)/\sqrt{6}, (Y_2\otimes F_3)/\sqrt{6})$ are $F_2 \otimes A$, $(F_2Y_2/\sqrt{2}) \otimes (A^*F_3/\sqrt{3})$ and $Y_2^* \otimes F_3^*$. Every complex Hadamard matrix of order two is $\cong F_2$, and A, $A^*F_3/\sqrt{3} = F_3\operatorname{diag}(\overline{a_k})/\sqrt{3}$ and $F_3^* = \overline{F_3}$ are $\cong F_3$ (A by Lemma 9.2 (d)); so all three are $\cong F_2 \otimes F_3 \cong F_6$. By Proposition 2.9 (c) such a triplet does not have property P, so by Proposition 2.9 (d) it is permutationally unitarily equivalent to $\mathcal{T}_0$. Conversely the members of $\mathcal{Z}$ have the clock T_3 (Lemma 9.5 (a)), and $\mathcal{T}_0$ has a clock by Theorem 2.8 (b): its third basis is invariant under Z, which fixes the standard lines and permutes the columns of $F_6 = F^{\mathsf{T}}(1,1)$ by (2.3), or under T_3, which permutes the standard lines and fixes the column lines of F_6, and that matrix is a clock of $\mathcal{T}_0$. No triplet is equivalent to both, by (b), Proposition 2.9 (c) and the invariance of property P (Proposition 2.9 (d)). For the same reasons a triplet with a clock, which lies in $[\mathcal{Z}]$ or in $[\mathcal{T}_0]$, has property P exactly when it lies in $[\mathcal{Z}]$; this is the second assertion of (b).

For the last sentence of (a): the unitary map K^* carries $\Gamma(p)$ to (K^*, L, I). As $\overline{F_3} = F_3R$ with R the permutation matrix of $l \mapsto -l$, the matrices $K^*(I_2 \otimes R)$ and $L(I_2 \otimes R)$, which have the same column lines as K^* and L, are Szöllősi's Z_1 and Z_2, with $Z_1^*Z_2 = (I_2 \otimes R)S(p)(I_2 \otimes R)/\sqrt{6}$ of the form [29, (4.2)], since conjugation by R keeps circulant matrices circulant. Conversely, if (I, Z_1, Z_2) is one of Szöllősi's triplets with $Z_1^*Z_2 = N/\sqrt{6}$, the columns of Z_1^* are the vectors $(\overline{\chi_k}; \pm\overline{\xi_k}\,\overline{\chi_k})/\sqrt{6}$ with $X = \operatorname{diag}(\xi_k)$, eigenvectors of T_3 since $\overline{\chi_k} = \chi_{-k}$. So Z_1^* carries the triplet to $(K_N, I, N/\sqrt{6})$ (Lemma 9.4), and multiplying N on both sides by diagonal matrices that are scalar on each block, as in Lemma 9.2 (c), gives a matrix $S(p)$; so the triplet lies in $[\mathcal{Z}]$ by Lemma 9.5 (b).

(c). By Proposition 2.9 (a), (c) and (d), $[\mathcal{T}_0]$ is the set of triplets whose three transition matrices are $\cong F_6$. It is the orbit of $\mathcal{T}_0$ under the compact group $U(6) \times \mathfrak{S}_3$, hence compact. By (a), the set of triplets with a clock is the disjoint union of $[\mathcal{T}_0]$ and $[\mathcal{Z}]$, and $[\mathcal{Z}]$ is closed by (d); so $[\mathcal{T}_0]$ is open in it.

(d). By the paragraph before Lemma 9.6, Γ is real-analytic on a neighborhood of $\mathcal{V}$ in $\mathbb{T}^4$. So $[\mathcal{Z}]$, the image of the compact set $U(6) \times \mathfrak{S}_3 \times \mathcal{V}$ under $(W, \pi, p) \mapsto W\pi(\Gamma(p))$, with π permuting the three bases, is compact; by (a) it does not meet $[\mathcal{T}_0]$.

By Lemma 9.7, $\Phi = (P_2, P_3) \circ \Gamma = \varphi \circ \alpha$ on $\mathcal{V}$ with $\varphi = (\bar{P}_2, \bar{P}_3) \circ \gamma$, continuous on $\mathbb{D}$. Over $\operatorname{int}\mathbb{D}$ the roots of f_α and $f_{-\alpha}$ are simple and of modulus one (Lemma 9.6 (d)), hence locally real-analytic functions of α; so through every point of $\mathcal{V}$ over $\operatorname{int}\mathbb{D}$ there is a real-analytic local section s of α with values in $\mathcal{V}$, and $\varphi = \Phi \circ s$ is real-analytic on $\operatorname{int}\mathbb{D}$. Let $\alpha_0 = 5/16 + 3i/16$. As $|\alpha_0| < 1$, $\alpha_0 \in \operatorname{int}\mathbb{D}$ (Lemma 9.6 (d)), and the orderings of the three simple roots of f_{α_0} and of $f_{-\alpha_0}$ give 36 points of $\mathcal{V}$ over α_0. A computation in the deposit [21], in Arb's ball arithmetic, encloses these roots in disjoint balls, each containing exactly one root (Arb's root isolation), and at each of the 36 points encloses, by forward-mode automatic differentiation, the Jacobian matrix J of $(P_2 \circ \Gamma, P_3 \circ \Gamma, \operatorname{Re}\ell, \operatorname{Im}\ell)$, where $\ell(p)$ is the left side of (1.4), with respect to the arguments of $\bar{x}$, $\overline{xy}$, $\overline{x'}$ and $\overline{x'y'}$. In all 36 cases $\det J$ lies in $[0.59, 0.76]$ or $[-0.76, -0.59]$. We use one of these points, p_0. So $d\ell$ has rank two at p_0 and, near p_0, $\mathcal{V}$ is a real-analytic surface with tangent plane $\ker d\ell$, on which $d\Phi$ has rank two; as $d\Phi = d\varphi \circ d\alpha$ there, $d\varphi(\alpha_0)$ has rank two. By the inverse function theorem φ maps a disc $\Delta \ni \alpha_0$ diffeomorphically onto an open set, and we take Δ inside the domain of a local section s with $s(\alpha_0) = p_0$. Then $\Phi = \varphi \circ \alpha$ is injective on the disc $s(\Delta)$ in $\mathcal{V}$, on which it has rank two, and as P_2 and P_3 are invariants, the members of $\Gamma(s(\Delta))$, a real-analytic disc of dimension two, are pairwise not equivalent. As $\operatorname{int}\mathbb{D}$ is connected (star-shaped, Lemma 9.6 (d)) and the real-analytic function $\det d\varphi$ does not vanish at α_0, the identity theorem shows that φ is constant on no nonempty open subset of $\operatorname{int}\mathbb{D}$. Let $p \in \mathcal{V}$ and let $\mathcal{N}$ be a neighborhood of p in $\mathcal{V}$. The point $\alpha(p)$ is a limit of points of $\operatorname{int}\mathbb{D}$ (Lemma 9.6 (d)), and by Rouché's theorem there is a neighborhood $\mathcal{N}'$ of $\alpha(p)$ such that for every $\alpha' \in \mathcal{N}' \cap \operatorname{int}\mathbb{D}$ the roots of $f_{\alpha'}$ and $f_{-\alpha'}$ can be ordered to give a point of $\mathcal{N}$ over α'. As φ is not constant on the nonempty open set $\mathcal{N}' \cap \operatorname{int}\mathbb{D}$, it takes there a value $\varphi(\alpha') \neq \varphi(\alpha(p))$, and the point $p' \in \mathcal{N}$ over α' has $\Phi(p') \neq \Phi(p)$, so $\Gamma(p')$ is not equivalent to $\Gamma(p)$. Since Γ is continuous and the maps of permutational unitary equivalence are homeomorphisms of $\mathfrak{T}$, every member of $[\mathcal{Z}]$ is a limit of members of $[\mathcal{Z}]$ not equivalent to it.

For the last sentence, Theorem 1.2 and Theorem 2.4 give every triplet a clock; then (a) and (b) give it. □

Proposition 9.9 (computer-assisted)**.** *Let $H_0 = F_2 \otimes F_3$ and $K_0 = Y_2 \otimes (D_0 F_3)$ with $Y_2 = \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$ and $D_0 = \operatorname{diag}(1, \omega, \omega)$, the coordinates of $\mathbb{C}^2 \otimes \mathbb{C}^3$ ordered by $(s, t) \mapsto 3s + t$. The triplet $\mathcal{T}_{\text{exc}} = (I, H_0/\sqrt{6}, K_0/\sqrt{6})$ is $\mathcal{T}_3$, and its complex conjugate is $\mathcal{T}_0$. Every triplet (1.2) with $H_1^* H_2/\sqrt{6} \cong F_2 \otimes \operatorname{circ}(\omega, 1, 1)$, as is the exceptional triplet of* [29, Section 4]*, is permutationally unitarily equivalent to $\mathcal{T}_0$.*

Proof. By (2.2), $F_6 = F_2 \otimes \overline{F_3} = \overline{H_0}$, and $\overline{F_3} = F_3 R$, so H_0 has the column lines of F_6. K_0 has entries of modulus one and $K_0^* K_0 = 6I$, and $F_6^* K_0 = (F_2 Y_2) \otimes (F_3 D_0 F_3)$, where the entries of $F_2 Y_2$ have modulus $\sqrt{2}$ and those of $F_3 D_0 F_3$ are $1 + 2\omega$ and $2 + \omega^2$, of modulus $\sqrt{3}$. So $\mathcal{T}_{\text{exc}}$ is a triplet containing $\{I, F_6/\sqrt{6}\}$, one of the $\mathcal{T}_b$. Its transition matrices in cyclic order are $F_2 \otimes F_3$, $(F_2 Y_2/\sqrt{2}) \otimes (F_3^* D_0 F_3/\sqrt{3})$ and $Y_2^* \otimes (\overline{F_3}\,\overline{D_0})$. Each is the tensor product of a complex Hadamard matrix of order two, which is $\cong F_2$, and one of order three that is $\cong F_3$ ($F_3^* D_0 F_3/\sqrt{3}$ is circulant; Lemma 9.2 (d)), so each is $\cong F_2 \otimes F_3 \cong F_6$. By

Proposition 2.9 (a), $\mathcal{T}_{\text{exc}}$ is $\mathcal{T}_0$ or $\mathcal{T}_3$. A computation in the deposit [21], in exact arithmetic in $\mathbb{Q}(\zeta_{12}, 12^{1/4})$ with the deposit's 16 third bases B_b, finds that $B_b^* K_0$ has exactly one nonzero entry in each row and each column for $b = 3$ and for no other b, and $B_b^* \overline{K_0}$ for $b = 0$ only. So $\mathcal{T}_{\text{exc}} = \mathcal{T}_3$ and $\overline{\mathcal{T}_{\text{exc}}} = \mathcal{T}_0$.

The matrix $A = \operatorname{circ}(\omega, 1, 1)$ is a circulant complex Hadamard matrix, as $1 + \omega + \omega^2 = 0$, so $F_2 \otimes A \cong F_6$ (Lemma 9.2 (d)). By the proof of Lemma 9.5 (a), a triplet (1.2) with $N = H_1^* H_2/\sqrt{6} \cong F_2 \otimes A$ is carried by the unitary map $H_1^*/\sqrt{6}$ to $(K_N, I, N/\sqrt{6})$. By Proposition 2.9 (c), N falls in the second case of the proof of Theorem 9.8 (a), so this triplet is permutationally unitarily equivalent to $\mathcal{T}_0$. Equality $H_1^* H_2/\sqrt{6} = F_2 \otimes A$ cannot hold: by (2.2) the two diagonal blocks of $H_1^* H_2$ have equal row sums, as $\operatorname{diag}(1, x_r, y_r)$ has first entry 1, and those of $F_2 \otimes A$ are $\pm(2 + \omega) \neq 0$. So here [29, (4.4)] is read up to the phases of the vectors of the two bases; for example, $\mathcal{T}_{\text{exc}}$ has the lines of the triplet (1.2) with $H_1 = F_6$ and $H_2 = \operatorname{diag}(D_0, iD_0)F_6$. □

The matrices $F_2/\sqrt{2}$, $F_3/\sqrt{3}$ and $D_0 F_3/\sqrt{3}$ are the matrices F_2, H_x and H_y of [34, (12), (16) and (17)], and the columns of $Y_2/\sqrt{2}$ are the vectors $|0_y\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}$ and $|1_y\rangle = (|0\rangle - i|1\rangle)/\sqrt{2}$ of [34, (11) and (42)]. So, in the coordinates of Proposition 9.9, $\mathcal{T}_{\text{exc}}$ is, basis for basis, the first of the two triplets of [34, Theorem 3], so it is permutationally unitarily equivalent to the exceptional triplet of [29, Section 4]. A computation in the deposit [21], in exact arithmetic in $\mathbb{Q}(\zeta_{12}, 12^{1/4})$, finds that the second, with this $|0_y\rangle$, has the sets of lines of the triplet $\mathcal{T}_2$ of Proposition 2.9; so it has property P (Section 2.4), is permutationally unitarily equivalent to one of Szöllősi's triplets [47, Theorem 4.3] (Theorem 9.8 (a)), and is not permutationally unitarily equivalent to the first, even by an antiunitary map.

The isolated triplet [5, (30)] of Cárdenes Wuttig and Tindall is equivalent to the exceptional triplet of [29, Section 4], as they state [5, p. 1]: in the coordinates of Proposition 9.9, it is, up to the order of the vectors in each basis, the complex conjugate $\overline{\mathcal{T}_0}$ of $\mathcal{T}_{\text{exc}}$, which that proposition shows to be permutationally unitarily equivalent to the exceptional triplet. Each triplet of their family [5, (31)] is permutationally unitarily equivalent to a triplet of the construction (1.2), since their symmetry [5, (32)] is a clock (Theorem 1.3).

Lemma 9.10 (computer-assisted)**.** *Every triplet close enough to $\mathcal{T}_0$ in $\mathfrak{T}$ is permutationally unitarily equivalent to $\mathcal{T}_0$. Hence, without Theorem 1.2, $[\mathcal{T}_0]$ is open and closed in $\mathfrak{T}$, and $q(\mathcal{T}_0)$ is an isolated point of $\mathfrak{Q}$.*

Proof. By Proposition 9.9 and Proposition 2.9 (a), $\mathcal{T}_{\text{exc}} = \mathcal{T}_3$ is permutationally unitarily equivalent to $\mathcal{T}_0$, so it suffices to prove the first assertion at $\mathcal{T}_{\text{exc}}$. Write 6×6 matrices with entries of modulus one as $H = (e^{i\theta_{jk}})$ and $K = (e^{i\phi_{jk}})$, with $(\theta, \phi) \in (\mathbb{R}/2\pi\mathbb{Z})^{72}$. Then $(I, H/\sqrt{6}, K/\sqrt{6})$ is a triplet exactly when the 96 real equations

$$\operatorname{Re}(HH^*)_{jk} = \operatorname{Im}(HH^*)_{jk} = \operatorname{Re}(KK^*)_{jk} = \operatorname{Im}(KK^*)_{jk} = 0 \quad (j < k), \qquad |(H^*K)_{kl}|^2 = 6$$

hold. The torus $\mathbb{T}^{18}$ acts on the solutions by $(D_1, D_2, D_3) \cdot (H, K) = (D_1 H D_2^{-1}, D_1 K D_3^{-1})$, with the D_r diagonal unitary; the stabilizer of every point is the circle of the triples of equal scalars, so the orbits are compact submanifolds of dimension 17, and their tangent spaces lie in the kernel of the Jacobian matrix J of the equations.

Let $\zeta = e^{2\pi i/24}$. The derivatives of $(HH^*)_{jk}$ and $(KK^*)_{jk}$ for the ordered pairs $j \neq k$, and of $|(H^*K)_{kl}|^2$, with respect to θ and ϕ form a 96×72 matrix $J_{\mathbb{C}}$ whose rows span over $\mathbb{C}$ the same space as those of J, since the rows for (j, k) and (k, j) are complex conjugates; so $J_{\mathbb{C}}$ and J have the same rank. At (H_0, K_0) the entries of H_0, K_0 and $H_0^* K_0$ lie in $\mathbb{Z}[\zeta]$ (see the proof of Proposition 9.9; $\sqrt{6} = (\zeta^3 + \zeta^{-3})(\zeta^2 + \zeta^{-2})$), and hence so do those of $J_{\mathbb{C}}$.

A computation in the deposit [21] builds H_0 and K_0 exactly in $\mathbb{Q}(\zeta)$ and finds that $J_{\mathbb{C}}$ has rank 55 over $\mathbb{Q}(\zeta)$. So J has rank 55, and its kernel, of dimension 17, is the tangent space of the orbit: the linearized equations at $\mathcal{T}_{\text{exc}}$ have only the solutions tangent to the orbit. The triplet has defect zero in the sense of [48, Definition 3.2], [49] adapted to triplets, and the argument below is that of [48, Lemma 3.3] for an isolated matrix.

Choose 55 of the equations whose differentials at (H_0, K_0) are independent. By the implicit function theorem their common zeros near (H_0, K_0) form a submanifold $\mathcal{M}$ of dimension 17, which contains every nearby solution and a neighborhood of (H_0, K_0) in its orbit; this neighborhood is open in $\mathcal{M}$. So there is a neighborhood N of (H_0, K_0) in which every solution lies in the orbit of (H_0, K_0).

Let (X_1, X_2, X_3) be a triplet close to $\mathcal{T}_{\text{exc}}$. As $U(6) \to U(6)/M$ has local sections, the bases are the columns of unitary matrices U_1, U_2, U_3 close to I, $H_0/\sqrt{6}$ and $K_0/\sqrt{6}$, and the unitary map U_1^{-1} carries the triplet to $(I, H/\sqrt{6}, K/\sqrt{6})$ with $H = \sqrt{6}U_1^{-1}U_2$ and $K = \sqrt{6}U_1^{-1}U_3$ close to H_0 and K_0; their entries have modulus one, as their columns are unbiased to the standard basis. For a triplet close enough, $(H, K) \in N$, so $(H, K) = (D_1 H_0 D_2^{-1}, D_1 K_0 D_3^{-1})$, and the unitary map D_1^{-1} carries $(I, H/\sqrt{6}, K/\sqrt{6})$ onto $\mathcal{T}_{\text{exc}}$.

So $[\mathcal{T}_0]$ contains a neighborhood of $\mathcal{T}_{\text{exc}}$, and hence, applying the homeomorphisms of permutational unitary equivalence, a neighborhood of each of its members: it is open. It is compact by Theorem 9.8 (c), hence closed, and $q^{-1}(q(\mathcal{T}_0)) = [\mathcal{T}_0]$ is open. □

Remark 9.11. We read the "in particular" sentence of [29, Conjecture 1] with "belongs to" as "is ≅ a member of", and $X(\alpha)$, as [29, Section 4] describes it after [47], as the complex Hadamard matrices ≅ one of the form [29, (4.2)], not as Szöllősi's parametrized matrices [47, (11)]. The first case of the "in particular" sentence, triplets whose three transition matrices lie, one each, in $X(\alpha)$, $F(x, y)$ and $F^{\mathsf{T}}(x, y)$, is then property P, and the triplets described in [47] are those of $[\mathcal{Z}]$ (Theorem 9.8 (a)); the triplets of the isolated case are those of the class of $\mathcal{T}_0$. With the bases written as the rows of matrices X_j, the matrices $\sqrt{6}X_1X_2^*$, $\sqrt{6}X_2X_3^*$, $\sqrt{6}X_3X_1^*$ of [29, Conjecture 1] are the transition matrices in cyclic order or their complex conjugates, which changes neither property P nor the property ≅ F_6 (Proposition 2.9 (d)); with the bases as the columns of matrices X_j, the matrices $\sqrt{6}X_jX_k^*$ are not the transition matrices, and we do not treat that reading.

9.3. **The second conjecture of Matolcsi, Matszangosz, Varga and Weiner.** For a 6×6 matrix M with nonzero entries and $\gamma \in \mathbb{Z}^6$ let $g_M(\gamma) = \sum_k \prod_j M_{jk}^{\gamma_j}$ and $G_M(\gamma) = |g_M(\gamma)|^2$, as in [29, (4.7)]; let $\gamma_1 = (1, 1, 1, -1, -1, -1)$ and $\gamma_2 = 3\gamma_1$, and let $\mathfrak{S}_6$ act on $\mathbb{Z}^6$ by permuting the coordinates. In this subsection a *transition matrix* of a triplet is a matrix $\sqrt{6}\,U^*V$, for two different bases U and V of the triplet, or its entrywise conjugate. This covers the transition matrices in cyclic order of Section 2.4 and, with either convention for the inner product, the matrix that [29, (2.1)] assigns to U and V: it is $\sqrt{6}\,U^*V$ for the inner product of [29], antilinear in its first argument [29, (3.2) and p. 8], and its entrywise conjugate for the other. With the vectors of the bases as the rows of matrices Y_j instead, the products $\sqrt{6}Y_j^*Y_k$ are the matrices $\sqrt{6}\,U_jU_k^*$ or their entrywise conjugates; these need not be complex Hadamard and can have zero entries, so g need not be defined for them.

On [29, p. 15], Matolcsi, Matszangosz, Varga and Weiner turn to properties of the Hadamard cubes [29, Section 3] of the triplets of their construction, (1.2) here. They conjecture that for every triplet (X_1, X_2, X_3) the three matrices $H = \sqrt{6}X_1^*X_2, \sqrt{6}X_2^*X_3, \sqrt{6}X_3^*X_1$ satisfy $G_H(\pi\gamma_1) = 0$ and $G_H(\pi\gamma_2)\,G_{H^*}(\pi\gamma_2) = 0$ for all $\pi \in \mathfrak{S}_6$ [29, Conjecture 2], and, with

$\widetilde{G}_H(\gamma) = \sum_{\pi\in\mathfrak{S}_6} G_H(\pi\gamma)$, they state the identity

$$\sum_H \Big(\widetilde{G}_H(\gamma_1) + \widetilde{G}_H(\gamma_2)\,\widetilde{G}_{H^*}(\gamma_2)\Big) = 0,$$

summed over the same three matrices [29, (4.8) and (4.9)]. As $G_H \geq 0$, this identity says that $G_H(\pi\gamma_1) = 0$ for all H and π and that, for each H, $G_H(\pi\gamma_2) = 0$ for all π or $G_{H^*}(\pi\gamma_2) = 0$ for all π; the conjecture asks for the latter only for each π separately, which is weaker. For a single complex Hadamard matrix H of order six, $g_H(\pi\gamma_1) = 0$ for all π whenever H is not equivalent in the usual sense to Tao's matrix S_6 [42, Theorem 1.1], as Matolcsi, Ruzsa and Weiner had conjectured [30, Conjecture 2.3]; see also [5, Proposition S.8] for a different proof. Maxwell and Brierley [33, Theorem 4.1] had proved it for the matrices of Karlsson's three-parameter family [25]. For complex Hadamard matrices M of order six whose four 3×3 blocks are circulant, the equations $g_M(\gamma) = g_{M^*}(\gamma) = 0$ for $\gamma = \gamma_1$ and $\gamma = \gamma_2$, which for those of them in Proposition 9.12 (b) are the case $\pi = 1$ there, were posted with a proof in the public repository `mio-qwq/math` on October 8, 2026 [37, `direct-circulant-character.md`], and a triplet in which one transition matrix H satisfies $g_H(\gamma_2) = 0 \neq g_{H^*}(\gamma_2)$, as in Remark 9.13, on October 9, 2026 [37, `fourier-character-asymmetry.md`]. The statement of Proposition 9.12 (b) was posted with a proof in the public repository `dbccpage/theory_of_finite_quantum_mechanics` on October 18, 2026 [8, `SZOLLOSI_2CIRCULANT_PROOF.md`].

Since the pair formed by the standard basis and $S_6/\sqrt{6}$ lies in no triplet [3, Section 4.1], no transition matrix of a triplet is $\cong S_6$: if $\sqrt{6}\,U^*V$ or its entrywise conjugate were $\cong S_6$, Lemma 3.1 (c) and (b), applied to the triplet or to its complex conjugate, would carry it to a triplet containing $\{I, S_6/\sqrt{6}\}$. Hence [42, Theorem 1.1], whose matrix T is S_6 with rows and columns 3 and 5 (numbered from 0) exchanged [42, Section 4], or [5, Proposition S.8], gives the condition on γ_1 in [29, Conjecture 2] for every triplet of mutually unbiased bases of $\mathbb{C}^6$, without Theorem 1.2; [3, Section 4.1] and [42, Theorem 1.1] are computer-assisted.

Proposition 9.12.

(a) *For all $x, y \in \mathbb{T}$ and $\pi \in \mathfrak{S}_6$, $g_{F^\intercal(x,y)}(\pi\gamma_1) = g_{F(x,y)}(\pi\gamma_1) = g_{F(x,y)}(\pi\gamma_2) = 0$, and $g_{F^\intercal(x,y)}(\pi\gamma_2)$ is 0 or one of $3(1-x^{\pm 6})$, $3(1-y^{\pm 6})$, $3(1-x^{\pm 6}y^{\mp 6})$.*

(b) *For every matrix $M = \left(\begin{smallmatrix} A & B \\ B^* & -A^* \end{smallmatrix}\right)$ with A, B circulant with entries in $\mathbb{T}$ (the form [29, (4.2)], which includes (1.3)) and all π, $g_M(\pi\gamma_1) = g_M(\pi\gamma_2) = g_{M^\intercal}(\pi\gamma_1) = g_{M^\intercal}(\pi\gamma_2) = 0$.*

(c) *Every transition matrix H of a triplet with a clock satisfies $G_H(\pi\gamma_1) = G_{H^*}(\pi\gamma_1) = 0$ for all π, and $G_H(\pi\gamma_2) = 0$ for all π or $G_{H^*}(\pi\gamma_2) = 0$ for all π.*

So [29, Conjecture 2] *and the identity* [29, (4.9)] *hold for every triplet with a clock, for every order of its bases and with either convention in* [29, (2.1)]*, and so, by Theorem 1.2, for every triplet of mutually unbiased bases of $\mathbb{C}^6$.*

Proof. Since $\sum_j \gamma_j = 0$, multiplying a column of M by a nonzero number does not change $g_M(\gamma)$, multiplying a row multiplies it by a nonzero number, permuting the columns does not change it, and permuting the rows replaces γ by a permutation of it; and $g_{\overline{M}} = \overline{g_M}$. So the condition that $g_M(\pi\gamma) = 0$ for all π does not change under $\cong$ or under entrywise conjugation, and neither does the same condition for M^{T}, as $(P_1 D_1 M D_2 P_2)^{\mathsf{T}} = P_2^{\mathsf{T}} D_2 M^{\mathsf{T}} D_1 P_1^{\mathsf{T}}$ and $\overline{M}^{\mathsf{T}} = \overline{M^{\mathsf{T}}}$. As $g_{M^*} = \overline{g_{M^\intercal}}$, we have $G_{H^*} = |g_{H^\intercal}|^2$. So the conditions in (c) are conditions on g_H and $g_{H^\intercal}$; they do not change when H is replaced by H^*, by its entrywise conjugate or

by a matrix $\cong H$. Let $m \in \{1,3\}$ and $\gamma = \pi(m\gamma_1)$; index rows and columns as $(a,b) \mapsto 3a+b$, $a \in \{0,1\}$, $b \in \mathbb{Z}/3$.

(a) By (1.1), $F^{\mathsf{T}}(x,y)_{(a,b),(c,k)} = (-1)^{ac} e_b^c \omega^{-bk}$ with $e = (1,x,y)$. Let $n_b = \gamma_{(0,b)} + \gamma_{(1,b)}$ and $N = \sum_b b n_b$; as $\sum_b \gamma_{(1,b)}$ is odd, the product over the rows for column (c,k) is $(-1)^c (x^{n_1} y^{n_2})^c \omega^{-kN}$, and $g_{F^{\mathsf{T}}(x,y)}(\gamma) = (1 - x^{n_1} y^{n_2}) \sum_k \omega^{-kN}$. Either $n = 0$, or n is a permutation of $(2m, -2m, 0)$ and N is one of $\pm 2m$, $\pm 4m$. For $m = 1$ this gives 0; for $m = 3$ it gives 0 or $3(1 - x^{n_1} y^{n_2})$ with $x^{n_1} y^{n_2} \in \{x^{\pm 6}, y^{\pm 6}, x^{\pm 6} y^{\mp 6}\}$. For $F(x,y) = F^{\mathsf{T}}(x,y)^{\mathsf{T}}$, the sum $g_{F(x,y)}(\gamma)$ runs over the rows of $F^{\mathsf{T}}(x,y)$, with the exponents γ on its columns. The product along row (a,b) is $(-1)^{as} e_b^s \omega^{-bN'}$ with $s = \sum_k \gamma_{(1,k)}$, which is odd, and $N' = \sum_{c,k} k \gamma_{(c,k)}$, so the terms of the rows $(0,b)$ and $(1,b)$ cancel.

(b) With $A = \operatorname{circ}(\alpha)$ and $B = \operatorname{circ}(\beta)$, where $\operatorname{circ}(w)_{jk} = w_{k-j}$, the entries of M are α_{k-j}, β_{k-j}, β_{j-k}^{-1} and $-\alpha_{j-k}^{-1}$ in the four blocks. Write $u_j = \gamma_{(0,j)}$ and $v_j = \gamma_{(1,j)}$. As $\sum_j v_j$ is odd, $g_M(\gamma) = \sum_k P_k - \sum_k Q_k$ with $P_k = \prod_j \alpha_{k-j}^{u_j} \beta_{j-k}^{-v_j}$ and $Q_k = \prod_j \beta_{k-j}^{u_j} \alpha_{j-k}^{-v_j}$. If u is constant, $P_k = Q_k$. Otherwise, for some $\varepsilon = \pm 1$, $u_j = \varepsilon m$ except $u_p = -\varepsilon m$ and $v_j = -\varepsilon m$ except $v_q = \varepsilon m$, and with $c = (\alpha_0 \alpha_1 \alpha_2 \beta_0 \beta_1 \beta_2)^{\varepsilon m}$, $\sum_k P_k = c \sum_i (\alpha_i \beta_{q-p-i})^{-2\varepsilon m} = \sum_k Q_k$ (put $i = k-p$, respectively $i = q-k$). Finally M^{T} has the same form, with α_{-i} and β_i^{-1} in place of α_i and β_i.

(c) By Theorem 2.4 (b), a triplet with a clock is permutationally unitarily equivalent to a triplet $\{I, H_1/\sqrt{6}, H_2/\sqrt{6}\}$ with $H_r = D_r F^{\mathsf{T}}(x_r, y_r)$, so by Lemma 3.1 (d) each matrix $\sqrt{6}\, U^* V$ for two different bases U and V of the first triplet is $\cong$ one of H_1, H_2, $H_1^* H_2/\sqrt{6}$ and their adjoints. By the first paragraph it suffices to prove (c) for H_1, H_2 and $H_1^* H_2/\sqrt{6}$. As $H_r \cong F^{\mathsf{T}}(x_r, y_r)$ and $H_r^{\mathsf{T}} \cong F(x_r, y_r)$, (a) gives (c) for H_r, with the second alternative. The matrix $H_1^* H_2/\sqrt{6}$ is a 2-circulant complex Hadamard matrix by Theorem 2.4 (a), hence by Lemma 9.2 $\cong F_6 = F^{\mathsf{T}}(1,1)$ or $\cong$ a matrix (1.3). In both cases g of the matrix and of its transpose vanishes at all $\pi\gamma_1$ and $\pi\gamma_2$, by (a) with $x = y = 1$ and by (b); this gives (c) for $H_1^* H_2/\sqrt{6}$.

Let (X_1, X_2, X_3) be a triplet with a clock, in any order. By (c), applied to $\sqrt{6} X_1^* X_2$, $\sqrt{6} X_2^* X_3$, $\sqrt{6} X_3^* X_1$ or to their entrywise conjugates, each of these matrices H satisfies $G_H(\pi\gamma_1) = 0$ for all π, and $\widetilde{G}_H(\gamma_2) = 0$ or $\widetilde{G}_{H^*}(\gamma_2) = 0$; this gives [29, Conjecture 2] and the identity [29, (4.9)]. By Theorem 1.2 and Theorem 2.4, every triplet has a clock. □

Remark 9.13. The condition on γ_2 in [29, Conjecture 2] cannot be strengthened to $G_H(\pi\gamma_2) = 0$ for all three matrices H and all π, nor to $G_{H^*}(\pi\gamma_2) = 0$ for all of them. Let $x, y \in \mathbb{T}$ with $(x^6, y^6) \neq (1,1)$. By the "if" direction of Corollary 1.5, which holds without Theorem 1.2, the pair $\{I, F^{\mathsf{T}}(x,y)/\sqrt{6}\}$ lies in a triplet (X_1, X_2, X_3) with $X_1 = I$ and $X_2 = F^{\mathsf{T}}(x,y)/\sqrt{6}$, and $H = \sqrt{6} X_1^* X_2 = F^{\mathsf{T}}(x,y)$. By Proposition 9.12 (a), $G_{H^*}(\pi\gamma_2) = |g_{F(x,y)}(\pi\gamma_2)|^2 = 0$ for all π, while the vectors $\gamma = \pi\gamma_2$ with $n = (-6,6,0)$ and $n = (-6,0,6)$ in its proof give $g_H(\gamma) = 3(1 - x^6)$ and $g_H(\gamma) = 3(1 - y^6)$, which are not both 0. With the first two bases exchanged, $H = F^{\mathsf{T}}(x,y)^*$, and the roles of H and H^* are exchanged.

9.4. **Two questions.** We end with two questions. First, Theorem 1.2 rests on computation, and its reformulation (Theorem 2.4) asks for a reason: is there a proof without computation that every triplet of mutually unbiased bases of $\mathbb{C}^6$ has a clock? The rigidity proved here (Proposition 6.2) is local, near the Fourier family. Second, which values of Szöllősi's parameter α give permutationally unitarily equivalent triplets? An answer would show whether the triplets of the first case, up to equivalence, form a two-parameter family in a stronger sense than that of Lemma 9.7.

Appendix A. Computations

A.1. **Hypotheses.** The proof assumes that the computers and the software that ran its programs worked as specified, and that the deposit [21] documents those runs: the processors and operating systems of the servers, the compilers, CPython 3.12, NumPy and the systems of exact and ball arithmetic (Python's integers and `fractions`, python-flint 0.8.0 with Arb and FLINT, PARI/GP, and SageMath with Singular) did what they are documented to do; binary64 arithmetic, where the proof uses it, rounded to nearest, ties to even; and each run ran the program that the deposit's README names for it, with the arguments that the README gives, and other processes acted on it only as the README states. We assume these in the full form that the README states, clause by clause, which is the form the proofs use: a clause on exact arithmetic and CPython, (F) on Arb, FLINT and the compiled programs, (H1) on binary64 arithmetic, (H-np) on NumPy, and (Runs) and (St) on the runs and their files. Three further hypotheses are about the values that the runs computed:

(K): At the leaves and branches that the deposit's README lists for (K), the binary64 upper bound that the interface checker (Appendix A.3) computes for the largest ratio, over the rows of the inclusion in its Krawczyk test (Lemma 6.3), of the left side to the right side is at most $1 - 2^{-51}$.

(H_T): In the cases that the deposit's README lists for H_T, no binary64 entry of the matrix T of a record or of a sub-root equals -2^{35}: the Stage C checker (Appendix A.3) scales these entries to 64-bit integers, which sends -2^{35} to -2^{63}, the one value at which its test for overflow is not exact.

(O): In the computations in which (H1) states that binary64 arithmetic rounds to nearest, no binary64 operation overflows, except as (H1) states; and no arithmetic on 64-bit integers to which (H-np) applies overflows.

A statement marked computer-assisted assumes these hypotheses, and so does every consequence of one. The interface checker accepts a branch only when, in ball arithmetic, every row of its inclusion holds, so the ratio in (K) is below 1 before it is rounded up to binary64; (K) asks in addition for the margin 2^{-51}, because one bound in the left side can fall short by a factor $1 + 2^{-52}$ at a rounding tie. The files of the sub-roots in (H_T) are not in the deposit; the output of a scan of them (`stagec-checker/post/c4209/ht4209.out`) finds every entry below 0.34 in modulus.

A.2. **Independent checks.** An *independent check* is a program, written without the code of the run, that proves the same finite statement again from the stored inputs and outputs. For the refinements and Stage A of Proposition 4.1, the exclusions of Proposition 4.3, Theorem 5.1, the Stage C checker and the interface checker (Appendix A.3), it also takes as untrusted input the output of *the code that proposes* (a copy of the run's code or, for Stage A, a program built on it), which may run in the checker's own process, and proves it again. Every computer-assisted statement has an independent check except Proposition 8.2, the tests of Lemma 8.7 on 31 records, and the computation after the proof of Proposition 9.9 on the second triplet of [34, Theorem 3]; in Proposition 8.10 the covers and coverage of two records are checked only by the exact programs written with the runs. Of these exceptions, Proposition 8.2, the tests of Lemma 8.7 and the covers and coverage of the two records of Proposition 8.10 enter the proof of Theorem 1.2; the computation on the second triplet does not. The README of the deposit gives, for each computer-assisted statement, the arithmetic that its proof uses and its independent checks.

A.3. **The runs for the criteria.** Theorems 6.12, 7.2 and 8.9 are realized as follows. On the relabeled record R', the *sub-root driver* runs Stage C with the schedule of Section 4.3 and then, as in Section 6.4, on the cubes of the grid of half-width $h_1 = 2^{-8}$ and their dyadic subdivisions down to level $L \leq 3$, each with its sub-root. If it leaves nothing unresolved, it proposes either that the whole record has no six-clique, or a list of candidate cubes, each with a sub-root, every other cube being excluded by Stage C. The *Stage C checker* proves in ball and exact arithmetic that no member of the record has a third basis, or that the conclusion of Lemma 6.9 holds for the list, and the *Root checker* certifies the record and every sub-root. A relabeled record keeps the constants ω, r and r_H of its record; for record 5020 of Proposition 6.13 and seven records of Proposition 8.10, relabeled in another way, a program checks by exact comparison of binary64 values that the result is a record R' as in Theorem 6.12.

On a candidate cube $Q(c_*, h_*)$ the interface program proposes and the *interface checker* decides, in one line of output for the cube; this line of output, which the proof uses, is the cube's *accepted line*. The checker reads the cube and its sub-root, takes the branches and the data of each cell from the program, and tests them in ball arithmetic against the conditions of Section 6.5; before its first test, the checker confirms that the constant 2^{-51} of its chart computations is exact. A cell (x, e, ρ) has box $x + e + [-\rho, \rho]^4$ (exact sum), where x is the center of its sub-root cube, e the offset and ρ the half-width, all binary64. The first cell has $x = c_*$, $e = 0$ and ρ a little above h_*, and a cell that is not a leaf is split into 16 children of half-width a little above $\rho/2$; that the first box contains the cube and that the children cover their parent's box are tested in exact rational arithmetic. A leaf either has its box outside the ball $\|\eta\| \leq r_H$ of Section 6.4 (an exact test), so that it holds no member, or is excluded, decided, routed (Proposition 7.1) or decided through the chart (Proposition 8.8); the leaves of the last four kinds together hold the cube in the sense of Section 6.5. A cube near F_6 may instead be accepted as a whole, with no cells, by Lemma 8.7 with $\rho = 1999/10000$ in the lemma. The counts in Sections 6, 7 and 8 are those of the accepted lines.

A.4. **Software and deposit.** The runs used Python 3.12, NumPy 2.5.2, python-flint 0.8.0 [53] with FLINT 3.3.1 [51], PARI/GP 2.17.3 [52], SageMath 10.9 [54] with Singular 4.4.1 [9], and giac 1.9 [44]. The deposit [21] holds the programs of the runs and their logs, inputs and outputs. Its README gives each run's program and arguments and cites the files taken unchanged from [20]; for the files that runs read but the deposit does not hold (chiefly the sub-root files of the Root checker and the stored chart files of the interface runs), it says how these files are made again. It also gives the counts, run tables and case listings that this paper summarizes.

A.5. **Particular runs.** *Proposition 8.2.* Four steps use routines of the C library or of NumPy: a bound of $\|A\|_2$ formed with the C library's power function, valid whatever that function returns (if it is not finite, another bound is used); the constant 2^{-51} of the dyadic sub-boxes, formed with that function; the bounds of matrix norms from NumPy's singular value decomposition, certified in ball arithmetic up to their sign; and NumPy's test for a zero matrix in the bisections of the injectivity and value tests. Each computation on a chart cell was repeated by processes that printed this constant, exactly 2^{-51}, and checked at every call that each of these norm bounds is positive and that each test for a zero matrix is decided exactly. The repetitions reproduced the recorded outputs, which are what the proof uses, so those outputs rest only on the hypotheses of Appendix A.1.

Theorem 8.6. A repetition of the first program of the proof found each of the 3,456 bounds of spectral norms from NumPy's singular value decomposition positive and reproduced the recorded output.

Appendix B. The Fourier fiber

This appendix proves Theorem 2.8. Appendix B.1 turns the common unbiased lines of $\{I, F^{\mathsf{T}}(x,y)/\sqrt{6}\}$ into the real solutions of a polynomial system in three angles; Appendix B.2 shows that, at the points of part (a), every third basis is a Kramers basis or meets each orbit of Z in at most one line; Appendix B.3 reduces the parameters, by symmetries, to a fundamental domain; Appendix B.5 shows in ball arithmetic that the system has at most 16 real solutions there and that no third basis is of the second kind, using the fold lemma of Appendix B.4 near two vertices of the domain, where pairs of solutions merge; and Appendix B.6 assembles the proof, with an exact computation at the twelve points of part (b).

Throughout, $x = \mathrm{e}^{i\alpha}$ and $y = \mathrm{e}^{i\beta}$ with $\alpha, \beta \in \mathbb{R}$, $d = (1, x, y)$, $E = \operatorname{diag}(d)$ as in (2.2), and

$$e = (e_0, e_1, e_2) = (d_0\overline{d_1},\ d_1\overline{d_2},\ d_2\overline{d_0}) = (\bar{x},\ x\bar{y},\ y),$$
$$\eta = (\eta_0, \eta_1, \eta_2) = (\cos(2\beta - \alpha),\ \cos(\alpha + \beta),\ \cos(2\alpha - \beta)),$$

so that $\eta_j = \operatorname{Re}(e_{j+2}\overline{e_{j+1}})$, indices taken modulo 3. Let $V(x,y)$ be the set of common unbiased lines of $\{I, F^{\mathsf{T}}(x,y)/\sqrt{6}\}$. A line unbiased to the standard basis has a representative $v = (a; b)/\sqrt{6}$ with $a, b \in \mathbb{T}^3$, and its *ratio vector* $\nu = (b_j/a_j)_j \in \mathbb{T}^3$ depends only on the line. Powers of Z are written Z^q, $q \in \mathbb{Z}/3$.

The *F_6-type points*, those excluded in Theorem 2.8 (a), are the (x,y) with $(x^2, y^2) \in \{(1,1), (\omega, \omega^2), (\omega^2, \omega)\}$. They are the points at which e_0, e_1, e_2 are equal up to sign, since $e_0^2 = e_1^2 = e_2^2$ means $\bar{x}^2 = x^2\bar{y}^2 = y^2$, that is, $y^2 = \bar{x}^2$ and $x^6 = 1$. In the coordinates (α, β) they form the lattice

$$\Lambda = \mathbb{Z}(\pi, 0) + \mathbb{Z}(-\tfrac{\pi}{3}, \tfrac{\pi}{3}),$$

which contains $\pi\mathbb{Z}^2$ with index 3.

B.1. **The reduction.** For $u \in \mathbb{C}^3$ let g_k be column k of $\overline{F_3}$. Then

$$|\langle g_k, u\rangle|^2 = \sum_{j,l} \omega^{(j-l)k} u_j \overline{u_l} = \|u\|^2 + 2\operatorname{Re}(\omega^k R_u), \qquad R_u = \sum_j \overline{u_j}\, u_{j+1}, \tag{B.1}$$

so $|\langle g_k, u\rangle|^2 = 6$ for $k = 0, 1, 2$ if and only if $\|u\|^2 = 6$ and $R_u = 0$: the sum over k gives $\|u\|^2 = 6$, and then $\operatorname{Re} R_u = \operatorname{Re}(\omega R_u) = 0$ forces $R_u = 0$. This is [18, Lemma 2.1].

Lemma B.1. *Let $v = (a; b)/\sqrt{6}$ with $a, b \in \mathbb{T}^3$, $a_0 = 1$, ratio vector ν, and put $t_j = \overline{a_j} a_{j+1}$, so that $t \in \mathbb{T}^3$, $t_0 t_1 t_2 = 1$ and $a = (1, t_0, t_0 t_1)$; conversely every pair $(\nu, t) \in \mathbb{T}^3 \times \mathbb{T}^3$ with $t_0 t_1 t_2 = 1$ comes from exactly one such v. Put*

$$p_j = (1 + \overline{\nu_j})(1 + \nu_{j+1}), \qquad q_j = (1 - \overline{\nu_j})(1 - \nu_{j+1}),$$
$$m = (p_1 e_2 q_2 - p_2 e_1 q_1,\ p_2 e_0 q_0 - p_0 e_2 q_2,\ p_0 e_1 q_1 - p_1 e_0 q_0).$$

(a) *The line of v lies in $V(x,y)$ if and only if* (R0) $\sum_j \operatorname{Re}\nu_j = 0$, (R1) $\sum_j t_j p_j = 0$ *and* (R2) $\sum_j t_j e_j q_j = 0$.

(b) *If the line of v lies in $V(x,y)$, no entry of ν is 1 or -1.*

(c) *Let (x,y) not be an F_6-type point, and let $\nu \in \mathbb{T}^3$ satisfy* (R0). *Then $m \neq 0$, and the lines of $V(x,y)$ with ratio vector ν are three lines forming one orbit of Z if $|m_0| = |m_1| = |m_2|$, and there are none otherwise.*

(d) *With* $\nu_j = C_j + iS_j$, $|m_j|^2 = 32N_j$, *where*

$$N_j = S_{j+2}^2(1 - C_jC_{j+1} - \eta_j S_j S_{j+1}).$$

Hence, if (x, y) *is not an* F_6*-type point,* $|V(x, y)|$ *is three times the number of real solutions* $(C, S) \in \mathbb{R}^6$ *of the system*

$$\Sigma(\eta): \qquad C_j^2 + S_j^2 = 1 \ (j \in \mathbb{Z}/3), \qquad C_0 + C_1 + C_2 = 0, \qquad N_0 = N_1 = N_2.$$

Proof. (a) With the columns $c_k = (g_k; g_k)$ and $c'_k = (Eg_k; -Eg_k)$ of $F^{\mathsf{T}}(x, y)$, $\langle c_k, (a; b)\rangle = \langle g_k, u^+\rangle$ and $\langle c'_k, (a; b)\rangle = \langle g_k, u^-\rangle$ for $u^+ = a + b$ and $u^- = \overline{E}(a - b)$ (as in [18, Section 2.1] and [35, Section 2.2]), and the line is unbiased to the columns of $F^{\mathsf{T}}(x, y)/\sqrt{6}$ if and only if all these have modulus $\sqrt{6}$. Now $u_j^+ = a_j(1 + \nu_j)$ and $u_j^- = \overline{d_j}a_j(1 - \nu_j)$, so $\|u^\pm\|^2 = 6 \pm 2\sum_j \operatorname{Re}\nu_j$, $R_{u^+} = \sum_j t_j p_j$ and $R_{u^-} = \sum_j d_j\overline{d_{j+1}}t_j q_j = \sum_j t_j e_j q_j$, and (B.1) gives the claim. This is [18, Lemma 2.2]: in the order of coordinates of the proof of the counts after Proposition 2.10, its vector $\mathbf{u} = (1, \overline{c_1}, \dots, \overline{c_5})/\sqrt{6}$, with $c_j \in \mathbb{T}$ (not the columns above), is v with $a = (1, \overline{c_4}, \overline{c_2})$ and $b = (\overline{c_3}, \overline{c_1}, \overline{c_5})$, and (R0)–(R2) are its (15), (12) and (14).

(b) If $\nu_k = -1$, then $p_k = p_{k-1} = 0$, and (R1) leaves $t_{k+1}p_{k+1} = 0$, so $\nu_{k+1} = -1$ or $\nu_{k+2} = -1$; two entries -1 give $\sum_j \operatorname{Re}\nu_j \le -1$, against (R0). If $\nu_k = 1$, the same argument with q and (R2) gives $\sum_j \operatorname{Re}\nu_j \ge 1$.

(c) First, p and $e \circ q = (e_j q_j)_j$ are linearly independent. By (R0) at most one entry of ν is -1 and at most one is 1. If no entry is ± 1, write $\nu_j = \mathrm{e}^{i\psi_j}$ and $\tau_j = \tan(\psi_j/2) \in \mathbb{R} \setminus \{0\}$; then $(1-\nu_j)/(1+\nu_j) = -i\tau_j$, so p has no zero entry and $q_j/p_j = \tau_j\tau_{j+1}$ is real, and $e \circ q = \mu p$ would make every $e_j = \mu p_j/q_j$ a real multiple of μ, which is excluded. If $\nu_k = -1$, then $p_k = p_{k-1} = 0 \ne p_{k+1}$, and $e \circ q \in \mathbb{C}p$ would need $q_k = q_{k-1} = 0$, that is $\nu_{k+1} = \nu_{k-1} = 1$, against (R0); the case $\nu_k = 1$ is the same with p and q exchanged. So the solutions $t \in \mathbb{C}^3$ of (R1) and (R2) form the line $\mathbb{C}m$, and $m \ne 0$. A solution with $t \in \mathbb{T}^3$ and $t_0t_1t_2 = 1$ is $t = \lambda m$ with $|\lambda||m_j| = 1$ for each j and $\lambda^3 m_0m_1m_2 = 1$: there is none unless $|m_0| = |m_1| = |m_2|$, and then there are exactly the three cube roots λ of $1/(m_0m_1m_2)$. Replacing λ by $\omega\lambda$ multiplies t by ω, hence a and $b = \nu \circ a$ by $(1, \omega, \omega^2)$, which is the action of Z.

(d) Since $|1 \pm \nu_j|^2 = 2 \pm 2C_j$, $|p_1q_2|^2 + |p_2q_1|^2 = 16S_2^2((1+C_1)(1-C_0) + (1+C_0)(1-C_1)) = 32S_2^2(1 - C_0C_1)$, and $p_1q_2\overline{p_2q_1} = [(1 + \overline{\nu_1})(1 - \nu_1)]\,[(1 - \nu_0)(1 + \overline{\nu_0})]\,[(1 + \nu_2)(1 - \overline{\nu_2})]^2 = (-2iS_1)(-2iS_0)(2iS_2)^2 = 16S_0S_1S_2^2$. So $|m_0|^2 = |p_1q_2|^2 + |p_2q_1|^2 - 2\operatorname{Re}(e_2\overline{e_1})\,16S_0S_1S_2^2 = 32N_0$, and the other two follow by shifting the indices. The real points of the circle equations correspond bijectively to $\nu \in \mathbb{T}^3$, (R0) is $\sum_j C_j = 0$, and by (c) each solution of $\Sigma(\eta)$ carries exactly three lines and each line of $V(x, y)$ comes from one. □

B.2. **Kramers bases and the other third bases.** Let $X(a; b) = (b; a)$, which is the map V^3 of [18, (39)] in their order of coordinates. The maps Z, J and X map the lines of the standard basis to lines of the standard basis, and the column lines of $F^{\mathsf{T}}(x, y)$ to column lines: for Z and J this is the proof of Proposition 2.6, and $Xc_k = c_k$, $Xc'_k = -c'_k$. Modulo scalars they generate a group of twelve maps of lines, $Z^qJ^sX^r$, with $Z^3 = 1$, $J^2 = -1$, $X^2 = 1$, $JZ = Z^{-1}J$, $ZX = XZ$ and $XJ = -JX$; the twelve are distinct, since Z^qX^r is a scalar only for $q = r = 0$ and J is antiunitary.

Lemma B.2. *Let $v = (a;b)/\sqrt{6}$ represent a line of $V(x,y)$ with ratio vector ν. Then Zv, Jv and Xv represent lines of $V(x,y)$ with ratio vectors ν, $-\nu$ and $\overline{\nu}$, and $Xv \perp v$. The Kramers basis $K(v) = \{Z^q v, Z^q Jv : q \in \mathbb{Z}/3\}$ satisfies $K(u) = K(v)$ for every $u \in K(v)$, so the Kramers bases partition $V(x,y)$.*

Proof. The first claim is the remark above. $Jv = (E\bar{b}; -E\bar{a})/\sqrt{6}$ has ratio vector $(-\overline{a_j}/\overline{b_j})_j = -\nu$, and Xv has ratio vector $\overline{\nu}$. $\langle v, Xv\rangle = \frac{1}{3}\sum_j \operatorname{Re}(\overline{a_j} b_j) = \frac{1}{3}\sum_j \operatorname{Re}\nu_j = 0$ by Lemma B.1 (a); this is part of [18, Corollary 4.7]. For $u = Z^q v$ we have $Ju = Z^{-q} Jv$, and for $u = Z^q Jv$ we have $Ju = -Z^{-q} v$; so $K(u) = K(v)$. □

For the next lemma write the entries of a line of $V(x,y)$ in half-angle form, $a_j = \zeta_j \mathrm{e}^{-i\psi_j/2}$ and $b_j = \zeta_j \mathrm{e}^{i\psi_j/2}$ with $\zeta_j \in \mathbb{T}$ and $\nu_j = \mathrm{e}^{i\psi_j}$, and put $P_j = \cos\frac{\psi_j}{2}\cos\frac{\psi_{j+1}}{2}$, $Q_j = \sin\frac{\psi_j}{2}\sin\frac{\psi_{j+1}}{2}$ and $\tilde{\zeta}_j = \overline{\zeta_j}\zeta_{j+1}$. A direct computation gives $t_j p_j = 4P_j\tilde{\zeta}_j$ and $t_j q_j = 4Q_j\tilde{\zeta}_j$, so (R0)–(R2) read

$$\textstyle\sum_j \cos\psi_j = 0, \qquad \sum_j P_j\tilde{\zeta}_j = 0, \qquad \sum_j Q_j e_j \tilde{\zeta}_j = 0, \tag{B.2}$$

and by Lemma B.1 (b) no P_j or Q_j vanishes.

Lemma B.3. *Let (x,y) not be an F_6-type point, and let v, w represent lines of $V(x,y)$ with $w \perp Z^{q_1} v$ and $w \perp Z^{q_2} v$ for some $q_1 \neq q_2$. Then the line of w lies in $K(v)$.*

Proof. Expand $w = \sum_q \beta_q Z^q v + \sum_q \beta'_q Z^q Jv$ in the orthonormal basis $K(v)$ (Proposition 2.6), put $\hat{\beta}(j) = \sum_q \beta_q \omega^{qj}$ and define $\hat{\beta}'$ likewise. In the slice of the coordinates $(0,j),(1,j)$ the vectors $Z^q v$ and $Z^q Jv$ are ω^{qj} times $v_j = (a_j, b_j)/\sqrt{6}$ and $(Jv)_j = d_j(\overline{b_j}, -\overline{a_j})/\sqrt{6}$, which are orthogonal of squared norm $\frac{1}{3}$. So $w_j = \hat{\beta}(j)v_j + \hat{\beta}'(j)(Jv)_j$, and since every coordinate of w has modulus $1/\sqrt{6}$,

$$|\hat{\beta}(j)|^2 + |\hat{\beta}'(j)|^2 = 1 \qquad \text{and} \qquad \operatorname{Re}(\hat{\beta}(j)\,\overline{\hat{\beta}'(j)}\,\overline{d_j}\,\zeta_j^2) = 0,$$

the second because $6(|w_{(0,j)}|^2 - |w_{(1,j)}|^2) = 4\operatorname{Re}(\hat{\beta}(j)\overline{\hat{\beta}'(j)}\,\overline{d_j} a_j b_j)$ and $a_j b_j = \zeta_j^2$. The hypothesis gives $\beta_{q_1} = \beta_{q_2} = 0$, so $\hat{\beta}(j) = \beta\omega^{qj}$ with q the third index and $\beta = \beta_q$, and $|\hat{\beta}'(j)| = \rho' = (1-|\beta|^2)^{1/2}$ for every j.

If $\beta = 0$, then w_j is a multiple of $(Jv)_j$ for each j, so w has the ratio vector of Jv, and by Lemma B.1 (c) its line is in the Z-orbit of the line of Jv. If $|\beta| = 1$, then $\hat{\beta}' = 0$, so $\beta' = 0$ and $w = \beta Z^q v$. Otherwise $0 < |\beta| < 1$, and the second identity gives $\hat{\beta}'(j) = -i\varepsilon_j \rho'(\beta/|\beta|)\omega^{qj}\overline{d_j}\zeta_j^2$ with $\varepsilon_j \in \{\pm 1\}$. Substituting, with $\tau = \rho'/|\beta|$ and $\phi = \arctan\tau \in (0, \pi/2)$,

$$w_{(0,j)} = c\,\omega^{qj}\zeta_j \mathrm{e}^{-i(\psi_j/2 + \varepsilon_j\phi)}, \qquad w_{(1,j)} = c\,\omega^{qj}\zeta_j \mathrm{e}^{i(\psi_j/2+\varepsilon_j\phi)}, \qquad |c| = 1/\sqrt{6}.$$

So w has half-angles $\psi'_j/2 = \psi_j/2 + \varepsilon_j\phi$ and $\tilde{\zeta}' = \omega^q\tilde{\zeta}$, and (B.2) holds for v and for w. If $\operatorname{Re}\tilde{\zeta}$ and $\operatorname{Im}\tilde{\zeta}$ are linearly independent over $\mathbb{R}$, the real solutions z of $\sum_j z_j\tilde{\zeta}_j = 0$ form a line, so $P' = \kappa P$; as no entry vanishes, $\cos^2(\psi'_j/2) = \kappa\cos^2(\psi_j/2)$ for each j (for instance $\cos^2\frac{\psi'_0}{2} = P'_0P'_2/P'_1$), and $\sum_j \cos^2(\psi_j/2) = \frac{3}{2}$ for both lines by (R0), so $\kappa = 1$. If they are dependent, then $\tilde{\zeta} \in \lambda\mathbb{R}^3$ for some λ; if also $e \circ \tilde{\zeta} \in \lambda'\mathbb{R}^3$, every e_j is a real multiple of λ'/λ, which is excluded, so the real solutions of $\sum_j z_j e_j \tilde{\zeta}_j = 0$ form a line, $Q' = \kappa' Q$, and as before $\sin^2(\psi'_j/2) = \sin^2(\psi_j/2)$. In both cases $\psi_j/2 + \varepsilon_j\phi \equiv \pm\psi_j/2 \pmod{\pi}$. The sign $+$ is impossible since $0 < \phi < \pi/2$, so $\psi_j \equiv -\varepsilon_j\phi \pmod{\pi}$ and $\cos\psi_j = \pm\cos\phi$ for each j. Then $\sum_j \cos\psi_j$ is $\cos\phi$ times an odd integer, which is not 0, against (R0). □

Proposition B.4. *Let (x, y) not be an F_6-type point. Every third basis of $\{I, F^{\mathsf{T}}(x, y)/\sqrt{6}\}$ is a Kramers basis or meets every orbit of Z on lines in at most one line. The Kramers bases are exactly the third bases invariant under Z.*

Proof. If a third basis K contains $Z^{q_1}v$ and $Z^{q_2}v$ with $q_1 \neq q_2$, every other line of K is orthogonal to both, so lies in $K(v)$ by Lemma B.3; hence $K \subset K(v)$, and both have six lines. A Z-invariant third basis contains the orbit of each of its lines, which consists of three lines by Proposition 2.6, and Kramers bases are Z-invariant. □

B.3. **Symmetries.** Let $\mathcal{W}$ be the group of affine maps of $\mathbb{R}^2$ generated by the translations by $(\pi, 0)$, $(0, \pi)$ and $(\frac{2\pi}{3}, -\frac{2\pi}{3})$ and the linear maps $(\alpha, \beta) \mapsto (\beta, \alpha)$, $(\alpha, \beta) \mapsto (-\alpha, \beta - \alpha)$ and $(\alpha, \beta) \mapsto (-\alpha, -\beta)$. The translations generate Λ, and the linear maps generate a group of order 12 that preserves Λ and the quadratic form $\alpha^2 - \alpha\beta + \beta^2$.

Lemma B.5. *For every $\gamma \in \mathcal{W}$ and every (α, β) there is a unitary or antiunitary map W that carries the lines of the standard basis onto themselves and the column lines of $F^{\mathsf{T}}(x, y)$ onto those of $F^{\mathsf{T}}(x', y')$, where (x', y') corresponds to $\gamma(\alpha, \beta)$, and with WZW^{-1} a scalar multiple of Z or of Z^{-1}. So W carries $V(x, y)$ onto $V(x', y')$, third bases to third bases, orbits of Z to orbits of Z and Z-invariant third bases to Z-invariant ones. Every point of $\mathbb{R}^2$ is mapped by some $\gamma \in \mathcal{W}$ into the triangle*

$$\Omega = \{(\alpha, \beta) : \alpha \leq 0,\ \alpha + \beta \geq 0,\ \beta - \alpha \leq \tfrac{\pi}{3}\}$$

with vertices $(0, 0)$, $(0, \frac{\pi}{3})$ and $(-\frac{\pi}{6}, \frac{\pi}{6})$, and $\Omega \cap \Lambda = \{(0, 0)\}$.

Compare [2, Section 2], where equivalences of complex Hadamard matrices divide the parameter square of the Fourier family into 144 triangles of equal area.

Proof. It suffices to treat the generators. Write $F^{\mathsf{T}}(d)$ for (2.2) with $E = \operatorname{diag}(d)$ for any $d \in \mathbb{T}^3$. Replacing x by $-x$ exchanges the rows $(0, 1)$ and $(1, 1)$, and similarly for y; this permutation commutes with Z. For a permutation σ of $\mathbb{Z}/3$, which is affine, $j \mapsto uj + c$, the permutation matrix P_σ satisfies $P_\sigma \overline{F_3} = \overline{F_3} M_\sigma$ with M_σ monomial, so $\operatorname{diag}(P_\sigma, P_\sigma) F^{\mathsf{T}}(d) = F^{\mathsf{T}}(d^\sigma) \operatorname{diag}(M_\sigma, M_\sigma)$ with d^σ the permuted vector, and $P_\sigma \operatorname{diag}(1, \omega, \omega^2) P_\sigma^{-1} = \omega^{c'} \operatorname{diag}(1, \omega, \omega^2)^{\pm 1}$; with $F^{\mathsf{T}}(\lambda d) = F^{\mathsf{T}}(d) \operatorname{diag}(I, \lambda I)$ this gives the first two linear maps, which permute $(1 : x : y)$. Since $\operatorname{diag}(1, \omega, \omega^2)\overline{F_3} = \overline{F_3} S$ with S as in (2.3), $F^{\mathsf{T}}(\omega x, \omega^2 y) = F^{\mathsf{T}}(x, y) \operatorname{diag}(I, S)$, which gives the translation by $(\frac{2\pi}{3}, -\frac{2\pi}{3})$ with $W = I$. Coordinatewise conjugation carries $F^{\mathsf{T}}(x, y)$ to $F^{\mathsf{T}}(\bar{x}, \bar{y})$ with the columns k and $-k$ of each block exchanged, and Z to Z^{-1}.

In the coordinates $(\alpha - \beta/2, \sqrt{3}\beta/2)$ the form becomes Euclidean, and Λ becomes a hexagonal lattice whose six shortest vectors are, in the coordinates (α, β), $\pm(-\frac{\pi}{3}, \frac{\pi}{3})$, $\pm(\frac{\pi}{3}, \frac{2\pi}{3})$, $\pm(\frac{2\pi}{3}, \frac{\pi}{3})$; the twelve linear maps form the symmetry group of the Voronoi cell of 0, a regular hexagon. Every point is a translate by Λ of a point of this hexagon, and the twelve maps permute transitively the twelve triangles formed by the center, a vertex and the midpoint of an adjacent edge. Ω is one of them: $(0, \frac{\pi}{3})$ is equidistant from 0, $(-\frac{\pi}{3}, \frac{\pi}{3})$ and $(\frac{\pi}{3}, \frac{2\pi}{3})$, and $(-\frac{\pi}{6}, \frac{\pi}{6})$ is the midpoint of the segment from 0 to $(-\frac{\pi}{3}, \frac{\pi}{3})$. The closed Voronoi cell of 0 contains no other point of Λ. □

B.4. **The fold lemma.** Near each of the vertices $(0, 0)$ and $(0, \frac{\pi}{3})$ of Ω two pairs of real solutions of $\Sigma(\eta)$ merge, one pair the image of the other under $S \mapsto -S$ (at $(0, \frac{\pi}{3})$ this is Remark B.8). The following lemma counts the solutions of each pair.

Lemma B.6. *Let $K = [-R, R]^2$ and let $G\colon K \to \mathbb{R}^2$ be a C^2 map with $G(-\chi) = G(\chi)$. Let H_1, H_2 be real symmetric 2×2 matrices, $\kappa = \frac{1}{2}\min_{|w|=1} \sigma_{\min}([H_1 w, H_2 w])$, and $L \geq \sup_{\xi \in K} \|D^2 G(\xi) - (H_1, H_2)\|_F$, where $\|\cdot\|_F$ is the Frobenius norm of the eight entries of a pair, which bounds the norm of the bilinear map $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ it defines. If $L < \kappa$, any two zeros χ_1, χ_2 of G in K satisfy $\chi_2 = \pm\chi_1$; so G has at most two zeros in K.*

Proof. Let $Q(\chi) = \frac{1}{2}(\chi^{\mathsf{T}} H_1 \chi, \chi^{\mathsf{T}} H_2 \chi)$ and $\epsilon = G - G(0) - Q$. Then ϵ is even, so $D\epsilon(0) = 0$, and $D^2\epsilon = D^2 G - (H_1, H_2)$; hence $|D\epsilon(\xi)v| = |\int_0^1 D^2\epsilon(\tau\xi)[\xi, v]\, d\tau| \leq L|\xi||v|$ for $\xi \in K$. Let $G(\chi_1) = G(\chi_2) = 0$, $u = \chi_1 - \chi_2$, $w = \chi_1 + \chi_2$ and $m = \max(|\chi_1|, |\chi_2|)$. The segments $[\chi_2, \chi_1]$ and $[-\chi_2, \chi_1]$ lie in K and in the ball of radius m, and $\epsilon(\chi_2) = \epsilon(-\chi_2)$, so $|\epsilon(\chi_1) - \epsilon(\chi_2)| \leq Lm \min(|u|, |w|)$. On the other hand $Q(\chi_1) - Q(\chi_2) = \frac{1}{2}(u^{\mathsf{T}} H_1 w, u^{\mathsf{T}} H_2 w)$, whose norm is at least $\kappa|u||w|$, and $Q(\chi_1) - Q(\chi_2) = \epsilon(\chi_2) - \epsilon(\chi_1)$. If $u \neq 0$ and $w \neq 0$, this gives $\kappa \max(|u|, |w|) \leq Lm$, while $|u| + |w| \geq \max(|u + w|, |w - u|) = 2m$ gives $\max(|u|, |w|) \geq m$; so $\kappa m \leq Lm$, and $m = 0$, a contradiction. □

B.5. **The computation.** For $\psi \in \mathbb{R}^3$ put $C = \cos\psi$, $S = \sin\psi$ entrywise and $\mathcal{F}(\psi; \alpha, \beta) = (C_0 + C_1 + C_2,\ N_0 - N_1,\ N_1 - N_2)$, with N_j as in Lemma B.1 (d). The real solutions of $\Sigma(\eta)$ are the points $(\cos\psi, \sin\psi)$ for the zeros $\psi \in (\mathbb{R}/2\pi\mathbb{Z})^3$ of $\mathcal{F}(\cdot; \alpha, \beta)$, and ψ is the vector of ratio angles of the lines it carries. By Lemma B.1 (b) and (d) no $\sin\psi_j$ of a zero vanishes, so each zero has a *sign vector* $(\operatorname{sign}\sin\psi_j)_j \in \{\pm 1\}^3$; the map $\psi \mapsto -\psi$ maps zeros to zeros and the sign vector to its negative.

Proposition B.7 (computer-assisted)**.** *Let $(\alpha, \beta) \in \Omega \setminus \{(0,0)\}$. Then:*

(a) *$\Sigma(\eta)$ has at most 16 real solutions;*

(b) *$\{I, F^{\mathsf{T}}(x, y)/\sqrt{6}\}$ has no third basis that meets every orbit of Z on lines in at most one line.*

Both parts cover $\Omega \setminus \{(0,0)\}$ by small parameter squares and certify, on each square and for all its parameters at once, the zeros of $\mathcal{F}$ for (a), and the common unbiased lines and the pairs of them that may be orthogonal for (b). Near the vertices $(0,0)$ and $(0, \frac{\pi}{3})$, where pairs of zeros merge, other arguments take over: the fold lemma for (a), and for (b) a derivative bound near $(0,0)$ and a separation of the merging lines from the others near $(0, \frac{\pi}{3})$.

Enclosures are formed in Arb ball arithmetic at 64 bits. Box and square endpoints are binary64 numbers, bisection tiles a box exactly, and a box $[l, h]$ is enclosed by a ball formed in binary64 with the enlargement $10^{-15}\max(1, |l|, |h|)$, which contains it enlarged at each end by $6.6 \cdot 10^{-16}\max(1, |l|, |h|)$, more than the rounding gaps between adjacent squares and at region edges (at most $2.3 \cdot 10^{-16}$). The tests against Ω and the excluded discs are exact, and the fold argument's containment tests are in ball arithmetic. The comparisons made in binary64 (that a piece lies in a cube of Lemma 6.3, that two such cubes are disjoint, that a box touches a face $\psi_j = \pm\pi$), and the radii and centers that the fold argument forms in binary64, carry margins larger than their rounding, which the deposit's README gives with their checks [21]; Arb bounds compared in binary64 are rounded in the safe direction (L up, κ down). The group table of Appendix B.2 and the sign vector s, found numerically, equal the exact ones. Every other floating-point number is used only as a proposal.

Proof of (a). Three computations cover $\Omega \setminus \{(0,0)\}$.

(K1) Ω minus the closed discs of radius 0.03 about $(0,0)$ and 0.003 about $(0, \frac{\pi}{3})$ is covered by parameter squares of side 0.005, quartered where needed. On each square P, boxes of $[-\pi, \pi]^3$ are bisected, and a box is discarded when some component of $\mathcal{F}$ excludes

0 on the box times P; no remaining box touches a face $\psi_j = \pm\pi$, so every zero has exactly one representative, interior to $[-\pi,\pi]^3$. The remaining boxes form clusters. For each, Lemma 6.3 with the parameter in P gives exactly one zero of $\mathcal{F}(\cdot;\alpha,\beta)$ in a cube about a numerical zero for every $(\alpha,\beta)\in P$, a finer exclusion puts every zero of the cluster in that cube, and the cubes are pairwise disjoint. Every square has exactly 16 clusters, hence at most 16 zeros for each parameter in it.

(K2) On the square of half-side $3\cdot10^{-3}$ about $(0,\frac{\pi}{3})$ the same procedure gives 12 clusters with exactly one zero each, and every other remaining box lies in one of the two cubes $\pm\psi_D+[-0.12,0.12]^3$, $\psi_D=(-\frac{\pi}{2},-\frac{\pi}{2},\frac{\pi}{2})$, each with at most two zeros by the fold argument below, with $\kappa\geq 0.8980$ and $L\leq 0.7607$.

(K3) On the square of half-side 0.03 about $(0,0)$: 12 clusters with one zero each, and the cubes $\pm\psi_0+[-0.1,0.1]^3$, $\psi_0=(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$, with at most two zeros each, with $\kappa\geq 0.5987$ and $L\leq 0.4434$.

A zero counted twice in (K2) or (K3) only raises the bound, so there are at most 16 zeros.

The fold argument. Let $\psi^*\in\{\pm\frac{\pi}{2}\}^3$, $s=(\operatorname{sign}\sin\psi^*_j)_j$ and $\rho<\frac{\pi}{2}$. For $\psi=\psi^*+\delta$ in the cube $\psi^*+[-\rho,\rho]^3$, $C_j=-s_j\sin\delta_j$ and $S_j=s_j\cos\delta_j=s_j(1-C_j^2)^{1/2}$, and the first component of $\mathcal{F}$ vanishes if and only if $\sin\delta_2=-s_2(s_0\sin\delta_0+s_1\sin\delta_1)$. Since $|\delta_j|\leq\rho<\frac{\pi}{2}$ determines δ_j from $\sin\delta_j$, the map $\psi\mapsto(\sin\delta_0,\sin\delta_1)$ is injective on the zeros in the cube, and maps them to zeros in $[-\sin\rho,\sin\rho]^2$ of the map $G(c_0,c_1)=(N_0-N_1,N_1-N_2)$ evaluated at $C=(-s_0c_0,-s_1c_1,s_0c_0+s_1c_1)$, $S_j^2=1-C_j^2$ and $S_jS_{j+1}=s_js_{j+1}((1-C_j^2)(1-C_{j+1}^2))^{1/2}$. This G is even, since $C\mapsto -C$ changes no term. Lemma B.6 is applied with (H_1,H_2) the Hessian of G at 0 for the center of the square:

$$H_1=\begin{pmatrix}0&-\frac{9}{2}\\-\frac{9}{2}&-\frac{9}{2}\end{pmatrix},\quad H_2=\begin{pmatrix}-\frac{9}{2}&0\\0&\frac{9}{2}\end{pmatrix}\quad\text{at }(0,\tfrac{\pi}{3}),$$

$$H_1=\begin{pmatrix}0&-3\\-3&-3\end{pmatrix},\quad H_2=\begin{pmatrix}-3&0\\0&3\end{pmatrix}\quad\text{at }(0,0),$$

enclosed in balls of radius at most $3\cdot10^{-12}$; κ is bounded below on arcs covering the half-circle through $\sigma_{\min}\geq|\det|/\|\cdot\|_F$, and L bounds $\|D^2G-(H_1,H_2)\|_F$, by second-order automatic differentiation, over sub-boxes of $[-\sin\rho,\sin\rho]^2$ times the square, so for every parameter in it. The cube about $-\psi^*$ is the image of the cube about ψ^* under $\psi\mapsto-\psi$. □

Remark B.8. Let $h=(h_1,h_2)=(\alpha,\beta-\frac{\pi}{3})$, so that D is the parameter point $h=0$, and let $K_0=K(v_0)/\sqrt{6}$, $v_0=(1,1,-\mathrm{e}^{i\pi/6};i,i,i\mathrm{e}^{i\pi/6})$, the Kramers basis of D whose lines have ratio vectors $\pm i(1,1,-1)$. By (K2) the 12 zeros of $\mathcal{F}$ outside the two cubes $\pm\psi_D+[-0.12,0.12]^3$, whose sign vectors are not $\pm(-1,-1,1)$, continue uniquely near D; they carry 36 lines in six Kramers bases, which tend to the six Kramers bases of D other than K_0. Near the zero $C=0$, $S=\pm(1,1,-1)$, in the chart $c=(\sin\delta_0,\sin\delta_1)$ of the fold argument with $\psi^*=\pm\psi_D$, the even map G satisfies

$$G(c;h)=\tfrac{3\sqrt{3}}{2}(h_2,-h_1)+\tfrac{1}{2}(c^{\mathsf{T}}H_1c,\,c^{\mathsf{T}}H_2c)+O(|h|^2+|h||c|^2+|c|^4)$$

with H_1,H_2 as above at $(0,\frac{\pi}{3})$. The quadratic part maps the unit circle onto an ellipse containing 0 in its interior, around which it turns once while the angle runs over a half-circle; so $\frac{1}{2}(c^{\mathsf{T}}H_1c,c^{\mathsf{T}}H_2c)=y$ has exactly two real solutions $\pm c$, both nondegenerate, for every $y\neq0$. Rescaling $c=|h|^{1/2}\hat{c}$ and the implicit function theorem give, for $h\neq0$ small, two zeros $\pm c_+(h)$ near 0, with $|c_+|$ of order $|h|^{1/2}$, and by Lemma B.6, with the constants of (K2), the cube contains no others. Write $\nu(c)=C+iS$ for the ratio vector of the zero with

chart coordinate c. The lines of the zeros $\pm c_+$, with those of their images under $\psi \mapsto -\psi$, form two Kramers bases K' and $K'' = XK'$, with ratio vectors $\pm\nu(c_+)$ and $\pm\overline{\nu(c_+)}$; each has one line near each line of K_0, and the two lines near a line of K_0 are at distance of order $|h|^{1/2}$. So $|V(x,y)| = 48$ for $h \neq 0$ small, and as $h \to 0$ along any path the lines with a common limit are these six pairs; two lines of one Kramers basis, being orthogonal, never have a common limit. Going once around D exchanges K' and K''.

Proof of (b). By (a) and Lemma B.1 (d), $|V(x,y)| \leq 48$ on $\Omega \setminus \{(0,0)\}$. Four computations cover this set:

(T1) Ω minus the closed discs of radius 0.02 about $(0,0)$ and $(0, \frac{\pi}{3})$, by squares of side 0.002, quartered where needed, each passing the box test below;

(T2) $\Omega \cap ([-0.025, 0] \times [0, 0.025])$ minus the closed disc of radius 10^{-5} about $(0,0)$, by squares passing the box test;

(T3) the square $[-10^{-5}, 10^{-5}]^2$ minus $(0,0)$;

(T4) $\Omega \cap ([-0.025, 0] \times [1.0222, 1.0472])$, by the squares of side 0.002 that meet Ω.

The parts of Ω in the two discs of (T1) lie in the windows of (T2) and (T4), since $\alpha \leq 0 \leq -\alpha \leq \beta \leq \frac{\pi}{3} + \alpha \leq \frac{\pi}{3} < 1.0472$ on Ω and $\frac{\pi}{3} - 0.02 > 1.0222$; so the four sets cover $\Omega \setminus \{(0,0)\}$.

The box test on a square P uses the lines $w(\theta)$ and the functions $R(\theta; H)$ of Section 6.3 for the matrix H obtained from $F^{\mathsf{T}}(x,y)$ by the column permutation $k \mapsto -k$ in each half (same column lines; the six residuals sum to 0, so any five have the same zeros). (1) For four numerically found lines, Lemma 6.3 with the parameter in P gives for every $(\alpha, \beta) \in P$ exactly one zero θ_s in a box X_s, $s = 1, \ldots, 4$. (2) The 48 lines $g\, w(\theta_s)$, for g in the group of Appendix B.2, are pairwise distinct for every parameter in P (so they are all of $V(x,y)$): a pair $\{g\, w(\theta_s), g' w(\theta_s)\}$ with $g^{-1}g' \in \{Z^{\pm 1}, Z^q J, X\}$ is orthogonal by Proposition 2.6 and Lemma B.2, and for every other pair the group law reduces the overlap to some $|\langle w(\theta_s), g\, w(\theta_{s'})\rangle|$ whose enclosure, divided by 6, has upper bound below 1. (3) The graph on the 48 labels, with an edge where the pair is $\{u, gu\}$ with $g \in \{Z^{\pm 1}, Z^q J, X\}$ or where the enclosure of the overlap contains 0, contains the orthogonality graph of $V(x,y)$ for every parameter in P. (4) An exhaustive search finds no six-clique of this graph whose six labels lie in six distinct orbits of Z.

The square (T3). The 48 lines at $(0,0)$ pass Lemma 6.3 on the whole square, each on its own box, and they are pairwise distinct for every parameter in the square: every pair of them has an overlap enclosure, divided by 6, with upper bound below 1. So at every $(\alpha, \beta) \in \Omega \setminus \{(0,0)\}$ in the square they are all of $V(x,y)$, and the graph of step (3) of the box test on them contains the orthogonality graph of $V(x,y)$; this graph has exactly six six-cliques with labels in six distinct orbits of Z. Each of them is, at $(0,0)$, the third basis $U\, K(v)$ for a Kramers basis $K(v)$, where $U = \overline{F^{\mathsf{T}}(1,1)}/\sqrt{6}$ is a unitary map exchanging the standard basis and the columns of $F^{\mathsf{T}}(1,1)/\sqrt{6}$ ($F^{\mathsf{T}}(1,1) = F_2 \otimes \overline{F_3}$ is symmetric), so maps third bases to third bases. This identification, and the labels, are certified: every image of a line enclosure under the maps used lies in the uniqueness box, of radius $2 \cdot 10^{-3}$, of the line it is identified with. For each of the six cliques, the vector $f \in \mathbb{C}^{15}$ of the inner products, each divided by 6, of the 15 pairs of its six lines normalized to first coordinate 1, vanishes at $(0,0)$, and its real 30×2 derivative in (α, β), enclosed over the square through $\partial\theta = -(\partial_\theta R)^{-1} \partial_{(\alpha,\beta)} R$, lies within Frobenius distance 0.097 of a binary64 matrix whose smallest singular value is at least 0.4939, both bounds formed in ball arithmetic; so every matrix in the enclosure is injective, and $f(\alpha, \beta) = \bar{A}\, (\alpha, \beta)^{\mathsf{T}}$ with $\bar{A} = \int_0^1 Df(\tau\alpha, \tau\beta)\, d\tau$ in

the convex enclosure gives $f(\alpha,\beta) \neq 0$ for $(\alpha,\beta) \neq (0,0)$. So none of the six cliques is a basis at another point of the square, and as the graph contains the orthogonality graph, no third basis there meets six distinct orbits of Z.

The squares (T4), near the degenerate vertex $(0, \frac{\pi}{3})$, where the box test needs boxes much smaller than their distance to the vertex. Call a zero of $\mathcal{F}$ and its lines *special* if its sign vector is $\pm(-1,-1,1)$, and *regular* otherwise. On each square P: (1) the exclusion of (K1) leaves 12 clusters of regular zeros and some clusters of special ones; (2) for each regular cluster Lemma 6.3 gives one zero in a cube and a finer exclusion puts every zero of the cluster in it, the cubes being disjoint: so there are exactly 12 regular zeros, hence by Lemma B.1 (c) exactly 36 regular lines, for every parameter in P, and the 36 lines $g\,w$, for g in the group of Appendix B.2, with w one of three lines certified by Lemma 6.3, have certified regular sign vectors and are pairwise distinct for every parameter in P, as in step (2) of the box test; so they are these; (3) every remaining box of the special clusters is turned into enclosures of the lines whose ratio angles lie in it, by Lemma B.1 (c) with $a = (1, \lambda m_0, \lambda^2 m_0 m_1)$; Test A: each such enclosure has an overlap enclosure excluding 0 with each of the 36 regular lines (lower bound 0.19); Test B: no six-clique of the graph of step (3) of the box test on the regular lines has its labels in six distinct orbits of Z. Since there are at most 48 lines, at most 12 are special, in at most four orbits of Z. A third basis containing a special line consists, by Test A, of special lines, and cannot meet six distinct orbits. One without special lines is excluded by Test B.

So no square of (T1)–(T4) has a third basis meeting six distinct orbits of Z, and a third basis meeting every orbit at most once meets six distinct orbits. □

B.6. **Proof of Theorem 2.8.**

Proof. (a) Let (x,y) not be an F_6-type point, so that $(\alpha,\beta) \notin \Lambda$, and choose $\gamma \in \mathcal{W}$ with $\gamma(\alpha,\beta) \in \Omega$ by Lemma B.5. As $\mathcal{W}$ preserves Λ, $\gamma(\alpha,\beta) \in \Omega \setminus \{(0,0)\}$. The map W of Lemma B.5 carries $V(x,y)$ onto the set of common unbiased lines at $\gamma(\alpha,\beta)$, which has at most 48 lines by Lemma B.1 (d) and Proposition B.7 (a). It carries a third basis K to a third basis, and if K met every orbit of Z at most once, so would its image, against Proposition B.7 (b). So K contains two lines of one orbit of Z and is a Kramers basis by Proposition B.4; by Proposition 2.6, Z is a clock of the triplet.

(b) The twelve points are the points of Λ modulo $2\pi\mathbb{Z}^2$, all mapped to $(0,0)$ by translations in $\mathcal{W}$, so by Lemma B.5 each pair is carried to $\{I, F^{\mathsf{T}}(1,1)/\sqrt{6}\} = \{I, F_6/\sqrt{6}\}$. At $(1,1)$, $e = (1,1,1)$ and $\eta = (1,1,1)$. The proof of Lemma B.1 (c) used that (x,y) is not an F_6-type point only where ν has no entry ±1 and $e \circ q = \mu p$; at $(1,1)$ this means $\tau_0\tau_1 = \tau_1\tau_2 = \tau_2\tau_0$, so $\nu_0 = \nu_1 = \nu_2$, and by (R0) $\nu = \pm i(1,1,1)$. For these two ratio vectors $p = q = (2,2,2)$, and (R1), (R2) both say $t_0 + t_1 + t_2 = 0$: three numbers of modulus one with sum 0 and product 1, that is, the six permutations of $(1, \omega, \omega^2)$; these are 12 lines, and the corresponding solutions of $\Sigma(1,1,1)$ have $N = 0$. For every other ν satisfying (R0), Lemma B.1 (c) holds as stated, and $|m_j|^2 = 32N_j$ as in (d). The system $\Sigma(1,1,1)$ is solved exactly: its ideal in $\mathbb{Q}[C_0, C_1, C_2, S_0, S_1, S_2]$ is zero-dimensional of degree 80, with 38 distinct complex solutions, of which exactly 14 are real (Singular through Sage, over $\mathbb{Q}$; the real count is confirmed by a Sturm sequence, in Sage and in PARI/GP, of the characteristic polynomial of multiplication by $10C_0 - 18C_1 + 19S_0 + 18S_1 - 15S_2$ on the quotient by the radical, which is squarefree of degree 38). Besides $C = 0$, $S = \pm(1,1,1)$, they are the cyclic shifts of

$$C = \varepsilon(s,-s,0), \quad S = \varepsilon'(-t,-t,1), \qquad s = (2\sqrt{3}-3)^{1/2},\ t = \sqrt{3}-1,\ \varepsilon,\varepsilon' \in \{\pm1\},$$

at each of which $N_0 = N_1 = N_2 = 4\sqrt{3} - 6 > 0$. So $|V(1,1)| = 3 \cdot 12 + 12 = 48$. All 48 lines are explicit vectors over $\mathbb{Q}(\zeta_{12}, 12^{1/4})$, a field of degree 8, and in exact arithmetic over it each is unbiased to both bases, they are pairwise distinct, and an exhaustive search of the orthogonality graph finds exactly 16 six-cliques: ten are invariant under Z, eight of them Kramers bases, and the other six are invariant under T_3. Since $T_3 = I_2 \otimes S$, for the shift S, multiplies the columns c_k and c'_k of $F^{\mathsf{T}}(1,1)$ by ω^{-k}, it fixes every column line. At the other eleven points the same holds for the images of Z and T_3 under W.

(c) By Lemma 3.1 (a) it suffices to treat triplets containing $\{I, F^{\mathsf{T}}(x,y)/\sqrt{6}\}$ or $\{I, F(x,y)/\sqrt{6}\}$, and since $F(x,y) = F^{\mathsf{T}}(x,y)^{\mathsf{T}} \sim F^{\mathsf{T}}(x,y)$, Lemma 3.1 (b) and (a) reduce the second kind to the first. So let K be a third basis of $\{I, F^{\mathsf{T}}(x,y)/\sqrt{6}\}$. If (x,y) is not an F_6-type point, the triplet has the clock Z by (a). Otherwise the map W of (b) carries the triplet to $\{I, F^{\mathsf{T}}(1,1)/\sqrt{6}, K'\}$ with K' invariant under Z or under T_3. Both are non-scalar with cube I; Z fixes every line of the standard basis and permutes the column lines of $F^{\mathsf{T}}(1,1)$ by (2.3), and T_3, a permutation matrix, permutes the lines of the standard basis and fixes every column line of $F^{\mathsf{T}}(1,1)$. So whichever of them maps K' to itself is a clock, and Lemma 3.1 (a) carries it back. □

Appendix C. Statements from [19]

We restate, in our notation, the statements of [19] that our proofs use, each with its place in [19] and the places where we use it. A real 6×6 matrix is a vector of $\mathbb{R}^{36}$, in row-major order and with the Euclidean norm, and a matrix acting as a linear map, real or complex (on $\mathbb{R}^k$, $\mathbb{R}^{30}$, $\mathbb{R}^{36}$ or $\mathbb{C}^6$), has the operator norm. Horizontal matrices and the projection Π onto them are those of Section 3.3, and equivalence is that of Section 3. For a real 6×6 matrix P let $F(P) \in \mathbb{C}^{15} \cong \mathbb{R}^{30}$ be the vector with the entries

$$F_{jk}(P) = \frac{1}{2\pi i} \sum_{l=0}^{5} e(P_{jl} - P_{kl}) \qquad (0 \le k < j \le 5)$$

[19, (2.1)], so that $F(P) = 0$ exactly when the rows of $e(P)$ are orthogonal, that is, when $e(P) \in \mathcal{H}$.

(U1) *Root certificates and records* [19, Section 2.4 and Appendix A.4]. A Root certificate consists of $P \in \mathbb{R}^{36}$, a matrix $T \in \mathbb{R}^{36\times k}$ with $0 \le k \le 10$, and nonnegative numbers $n_0, \iota, C, \omega, \chi, \kappa, \varphi$; put $n_P(t) = n_0 + \iota t + Ct^2$. It is *certified* if $\|T^{\mathsf{T}}T - I\| \le \omega$, $\|(I - \Pi)T\| \le \chi$ and $\|F(P)\| \le \varphi$, which is [19, (A.1)], and if every horizontal δ with $F(P + \delta) = 0$ satisfies, with $\eta = T^{\mathsf{T}}\delta$,

$$\|(I - TT^{\mathsf{T}})\delta\| \le n_P(\|\delta\|), \qquad \|\delta\|^2 \le (1 + \omega)\|\eta\|^2 + \|(I - TT^{\mathsf{T}})\delta\|^2,$$

which are [19, (A.2) and (A.3)], and, for every horizontal v,

$$\|(I - TT^{\mathsf{T}})(\delta - v)\| \le (\iota + C(\|\delta\| + \|v\|))\|\delta - v\| + \kappa\|F(P + v)\|,$$

which is [19, (A.4)]. A *record* (P, r) is a certified Root certificate with a radius $r \ge 0$. Its solution set $S(P, r)$ is the set of the matrices $e(P + \delta)$ with δ horizontal, $\|\delta\| \le r$ and $F(P + \delta) = 0$, the members of the record. Used in Sections 4.1 and 6.4.

(U2) *Sufficient conditions for certification* [19, Lemma C.23]. Let $P, T, n_0, \iota, C, \omega, \chi, \kappa, \varphi$ be as in (U1) and satisfy [19, (A.1)]. Let L_P be the derivative of F at P, $(L_P x)_{jk} = \sum_l e(P_{jl} - P_{kl})(x_{jl} - x_{kl})$, and let $M \in \mathbb{R}^{36\times 30}$ satisfy $\|M\| \le \kappa$, $\kappa\varphi \le n_0$ and $\|\Pi - TT^{\mathsf{T}} - ML_P\Pi\| \le \iota$. Suppose that $C \ge \pi b$ for a number b with $\sum_{j>k} w_{jk}(y)\|x_j - x_k\|^2 \le b\|x\|^2$ for every horizontal x, with rows x_j, and every unit vector $y \in \mathbb{R}^{36}$,

where $w_{jk}(y)$ is the norm of the two coordinates of $M^{\mathsf{T}}y$ that belong to the pair (j,k) [19, Lemma C.21]. Then [19, (A.2)–(A.4)] hold, so the Root certificate is certified. Used in the proof of Proposition 4.1.

(U3) *Symmetries* [19, Lemma C.24]. Let $g \in G$, acting on $\mathbb{R}^{36}$ as in Section 3.3 and on each column of a matrix in $\mathbb{R}^{36\times k}$. Then g is an orthogonal map of $\mathbb{R}^{36}$ that commutes with Π and maps $\mathbb{Z}^{6\times 6}$ onto itself. For every real Y, $\|F(gY)\| = \|F(Y)\|$, and $e(gY)$ is obtained from $e(Y)$ by permutations of rows and columns, possibly transposition and possibly complex conjugation. If $e(Y)$ is a complex Hadamard matrix, so is $e(gY)$, and the two are equivalent. If $(P,T,n_0,\iota,C,\omega,\chi,\kappa,\varphi)$ is a certified Root certificate, so is $(gP,gT,n_0,\iota,C,\omega,\chi,\kappa,\varphi)$, and for $r \geq 0$, $e(gP+\delta') \in S(gP,r)$ if and only if $e(P+g^{-1}\delta') \in S(P,r)$, and these two matrices are equivalent. Used in the proofs of Proposition 4.1 and Theorem 6.12.

(U4) *The frame* [19, Lemma C.32]. For the frame T and the constant ω of a certified Root certificate (the setting of [19, Appendix C.9]), $\|T\| \leq (1+\omega)^{1/2}$, $\|I - TT^{\mathsf{T}}\| \leq 1+\omega$, and $\|x\|^2 \leq (1+\omega)\|T^{\mathsf{T}}x\|^2 + \|(I-TT^{\mathsf{T}})x\|^2$ for every $x \in \mathbb{R}^{36}$. Used in Lemma 6.7.

(U5) *Members of a record* [19, Lemma C.36 (i), (ii)]. Let (P,r) be a record whose Root certificate satisfies [19, (A.1) and (A.2)], and put $r_H = (1+\omega)^{1/2}r$ and $n = n_0+\iota r+Cr^2$, which Section 6.4 writes n_R when $k=4$. Every horizontal δ with $\|\delta\| \leq r$ and $F(P+\delta) = 0$ satisfies $\|\eta\| \leq r_H$ and $\|\nu\| \leq n$ for $\eta = T^{\mathsf{T}}\delta$ and $\nu = \delta - T\eta$, and $\|e(P)/\sqrt{6}\| \leq (1+2\pi\sqrt{2}\,\varphi/6)^{1/2}$. Used in Section 6.4 and Lemma 6.8.

(U6) *Refinement* [19, Lemma 3.9], in the terms of [19, Sections 2.6 and 3.6]. Consider a process of a refinement to radius τ' that starts with some stored children, each of radius at least τ', and some blockers, and suppose that the solution set of each of these blockers is contained, up to equivalence, in the union of the solution sets of these stored children. If the process completes each of its parents, then, up to equivalence, the solution set of each of its parents and of each of its blockers is contained in the union of the solution sets of the children stored at its end. [19] assumes here its Assumption (A) and Condition (B), but its proof, from [19, Lemmas A.2, C.24 and C.33–C.35], uses them and the properties [19, (P6)] of the refinement program only for what that program computed and did; we apply it to processes whose computations and Root certificates independent programs have confirmed. Used in (U7) and in the proof of Proposition 4.1.

(U7) *The cover* [19, Section 4.3], from the proof of [19, Theorem 1.1], as far as the mid records. We follow that proof, but use no property of the programs that wrote the files of [20]: three checks by independent programs confirm each step from these files, under the hypotheses of Appendix A.1.

First, the seed cover. A program visits all 160,389,488 lists of bins for x [19, Section 2.3] and, for the 37 shared bins of those that pass the filters on x, all lists for y. It rejects a bin pair only when integer arithmetic or Arb decides that a filter of [19, Appendix A.2] fails, and it found each of the 151,742,926 surviving pairs among the bin pairs of the subseeds listed in [20] or within the bound of [19, Lemma 3.7] of a center listed there with the same selected position, not listed as unresolved. By [19, Lemmas C.5–C.9 and 3.7], every complex Hadamard matrix of order six is therefore equivalent to one arranged with the selected position of such a center or subseed and whose seed coordinates lie in its seed or subseed ball.

Second, the balls. For each of these 1,079,091 seed and subseed balls, an independent program confirmed in Arb its frames and a complete tree of cells, each leaf excluded

by a test of [19, Appendix A.3] or covered, as in [19, Appendix A.5], by one of the 46,040 records of the atlas of [19], transported by a map of (U3), at a radius at most the record's; programs built on the code of Stage A only propose all this. The Root certificates of these records, and of all the records of the refinement below, passed the check of the proof of Proposition 4.1 [19, Section 5]. By [19, Lemmas C.11–C.20, C.24–C.30 and A.1], every complex Hadamard matrix of order six is therefore equivalent to a member of one of these records.

Third, the refinement. The atlas is refined, in two passes, to the 1,406 mid records [19, Section 4.2]. The processes of these passes completed all their parents other than the 3,685 atlas records of [19, (F4)], whose solution sets are contained in those of parents completed in the earlier pass, and each started with stored children and blockers as (U6) requires [19, (F3) and (F5)]. An independent program confirmed these facts from the stored records and the command lines, and the program of the proof of Proposition 4.1 confirmed every step of these processes from the logs of reruns with the same stores [19, Section 5]; the program reads the grid half-sizes from the logs, where they lie between 6 and 15, and the two conditions it needs (that they are nonnegative, and that the number of boxes fits in its 64-bit integers) hold for these values. So, by (U6), applied in their order, every complex Hadamard matrix of order six is equivalent to a member of one of the mid records. Used in the proof of Proposition 4.1.

(U8) *Stage C* [19, Section 2.7 and Appendix A.7]. The tests of Stage C on bins, vector balls and edges are (SC1)–(SC5) of Section 6.4. Lemma 6.8 proves what our proofs use of them, for the members of a record (case (i)) and for those of a cube with a sub-root (case (ii)). For a record, [19] proves the same in Appendix A.7 and Lemmas C.36 (iii), (iv) and C.37–C.44. Used in Sections 4.3 and 6.4.

Declarations

Use of AI. This paper and its results were produced by an AI research system that the author designed, built and directs, running on Anthropic's Claude Opus 5.5 in Claude Code and OpenAI's GPT-6 Astra in Codex. The author chooses the problems and prioritizes the work. Separate AI agents checked each result, and a second program confirmed the computations the proofs use, with any exceptions listed in the public deposit. The system also searched the literature and checked the citations against reliable sources. The author read the paper, reran the computations from the deposit and takes full responsibility for all of the paper's contents.

Data availability. The programs and data used in this paper, with instructions for rerunning them, are deposited at Zenodo [21].

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