DENSITY OF BLOW-UP DATA AND REGULARITY AT THE BLOW-UP TIME FOR THE THREE-DIMENSIONAL EULER EQUATIONS

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Abstract. Alpöge, Buckmaster and Coiculescu have constructed smooth odd initial data on $\mathbb{T}^3$ whose solutions of the incompressible Euler equations blow up in finite time. Running their construction without oddness, with the singularity carried by a fluid particle, we show that data whose solutions blow up in arbitrarily short time are dense among smooth divergence-free fields in L^∞, H^1 and C^α, $0 < \alpha < 5/6$. The same holds inside classes of fields invariant under a finite group of isometries of $\mathbb{T}^3$, such as the Taylor–Green and Kida classes, with blow-up at the same time along a whole orbit. The Euler equations are not well posed in these topologies, and the result says nothing about instability in $C^{1,\alpha}$ or in H^s, $s > 5/2$. For both our solutions and those of Alpöge, Buckmaster and Coiculescu, the velocity stays continuous up to the blow-up time T in H^1 and in C^α for every $0 < \alpha < 5/6$, so the enstrophy converges, while $u(T) \notin C^{5/6}$, $\operatorname{curl} u(T)$ lies in no L^p with $p > 2$, and the vorticity on the particle grows faster than $(T-t)^{-2}$.

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1. Introduction

We consider the incompressible Euler equations on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$,

$$\partial_t u + (u \cdot \nabla)u + \nabla p = 0, \qquad \operatorname{div} u = 0. \tag{1.1}$$

Smooth divergence-free data have a unique maximal smooth solution [10]. Alpöge, Buckmaster and Coiculescu have recently shown that it can become singular in finite time. Smooth compactly supported data on $\mathbb{R}^3$ whose solutions blow up were constructed earlier in [17].

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Theorem 1.1 (Alpöge, Buckmaster and Coiculescu [1, Theorem 1.1])**.** *There is an odd divergence-free $u_0 \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ whose maximal classical solution u of (1.1) is smooth and odd on $\mathbb{T}^3 \times [0,T_*)$ for a finite $T_* > 0$, and satisfies*

$$\int_0^{T_*} \|\nabla u(t)\|_{L^\infty}\, dt = \infty, \qquad \int_0^{T_*} \|\operatorname{curl} u(t)\|_{L^\infty}\, dt = \infty. \tag{1.2}$$

Moreover,

$$\sup_{0\le t<T_*} \|u(t)\|_{L^\infty} < \infty, \qquad \lim_{t\uparrow T_*} |\operatorname{curl} u(t,0)| = \infty. \tag{1.3}$$

For every compact $K \subset \mathbb{T}^3 \setminus \{0\}$ and every integer $n \ge 0$, both $u(t)$ and $\partial_t u(t)$ converge in $C^n(K)$ as $t \uparrow T_*$.

The data are odd, $u_0(-x) = -u_0(x)$; the solution stays odd, by uniqueness, so its flow fixes the origin, and the vorticity becomes unbounded there. Without symmetry the particle need not be fixed, and the singularity is carried by a fluid particle.

Definition 1.2. Let $z \in \mathbb{T}^3$. A divergence-free $u_0 \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ *blows up along the particle from z* at time T_* if its maximal smooth solution u has lifespan $T_* < \infty$, satisfies (1.2) and $\sup_{0\le t<T_*} \|u(t)\|_{L^\infty} < \infty$, and the particle path $y'(t) = u(t,y(t))$, $y(0) = z$, satisfies:

(i) $y(t)$ has a limit y_* as $t \uparrow T_*$;
(ii) $|\operatorname{curl} u(t,y(t))| \to \infty$ as $t \uparrow T_*$;
(iii) for every compact $K \subset \mathbb{T}^3 \setminus \{y_*\}$ and every integer $n \ge 0$, both $u(t)$ and $\partial_t u(t)$ converge in $C^n(K)$ as $t \uparrow T_*$.

For finitely many initial points, u_0 blows up along the particles from them if (i) and (ii) hold for each particle and (iii) holds for every compact K that avoids all their endpoints.

Definition 1.2 asks that the curl diverge along the moving particle, not at the fixed endpoint of its path. If u_0 is odd and $z = 0$, then $y \equiv 0$, Definition 1.2 is the conclusion of Theorem 1.1, and we say that u_0 *blows up as in Theorem 1.1*.

Theorem 1.3. *Let $v \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ be divergence free, $z \in \mathbb{T}^3$, $0 < \alpha < 5/6$ and $\varepsilon > 0$. There is a divergence-free $u_0 \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ that blows up along the particle from z at a time $T_* < \varepsilon$, with endpoint y_* at distance less than ε from z, and*

$$\|u_0 - v\|_{L^\infty} + \|u_0 - v\|_{H^1} + \|u_0 - v\|_{C^\alpha} < \varepsilon.$$

Here $\|w\|_{C^\alpha} = \|w\|_{L^\infty} + [w]_\alpha$, where $[w]_\alpha$ is the supremum of $|w(x) - w(y)|/|x-y|^\alpha$ in the periodic distance. As the periodic distance is less than 1, $[w]_\alpha$ increases with α, and a diagonal choice of $\alpha \uparrow 5/6$ and $\varepsilon \downarrow 0$ in Theorem 1.3 gives a single sequence of such data converging to v in L^∞, in H^1 and in every C^α, $0 < \alpha < 5/6$, with blow-up times tending to zero and endpoints tending to z.

Theorem 1.4. *Let $v \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ be divergence free, $0 < \alpha < 5/6$ and $\varepsilon > 0$.*

(a) *If v is odd, there is an odd divergence-free $u_0 \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ that blows up as in Theorem 1.1 at a time $T_* < \varepsilon$, with $\|u_0 - v\|_{L^\infty} + \|u_0 - v\|_{H^1} + \|u_0 - v\|_{C^\alpha} < \varepsilon$.*
(b) *Let G be a finite group of isometries $gx = Q_g x + d_g$ of $\mathbb{T}^3$, acting on fields by $(g_* w)(x) = Q_g w(g^{-1}x)$, and suppose that $g_* v = v$ for $g \in G$. Let $z \in \mathbb{T}^3$ satisfy $\nabla v(z) \ne 0$ and $gz \ne z$ for $g \ne \mathrm{id}$. Then there is a G-invariant divergence-free $u_0 \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ that blows up along the particles from all the points gz, $g \in G$, at one time $T_* < \varepsilon$, with $\|u_0 - v\|_{L^\infty} + \|u_0 - v\|_{H^1} + \|u_0 - v\|_{C^\alpha} < \varepsilon$. The particle from gz ends at gy_*, where y_* is at distance less than ε from z, and these $|G|$ endpoints are distinct.*

If v is G-invariant and not constant, a point z as in (b) exists. An isometry other than the identity fixes no open set, so the points with trivial stabilizer form a dense open set, which meets the nonempty open set where $\nabla v \neq 0$. If v is constant, take a point z with trivial stabilizer, a ball B about it whose images gB are disjoint, and a small smooth divergence-free w supported in B with $\nabla w(z) \neq 0$; then $v + \sum_{g\in G} g_* w$ is G-invariant and divergence free, with gradient $\nabla w(z)$ at z. Blow-up data are therefore dense in every class of G-invariant fields. The Taylor–Green class [18] and Kida's class [15] consist of the fields invariant under groups of 64 and 192 isometries that preserve the two flows (Section 5).

Theorem 1.5. *Let u be the solution constructed in* [1, Sections 6–8]*, with $z = 0$, or the solution from the data constructed in the proof of Theorem 1.3 or of Theorem 1.4, with $z = 0$ in Theorem 1.4*(a)*. In each case, let N, the large parameter of the construction (Section 2), be sufficiently large. Let T be the blow-up time of u, y the particle from z, and $u(T)$ the uniform limit of $u(t)$ as $t \uparrow T$, which exists by* [1, Section 8] *and Section 4. Then:*

(a) *$u \in C([0,T]; C^\alpha)$ for every $0 < \alpha < 5/6$ and $u \in C([0,T]; H^1)$; in particular $\operatorname{curl} u(T) \in L^2$ and the enstrophy converges, $\|\operatorname{curl} u(t)\|_{L^2} \to \|\operatorname{curl} u(T)\|_{L^2}$ as $t \uparrow T$;*

(b) *$u(T) \notin C^{5/6}$, and $\operatorname{curl} u(T) \notin L^p$ for every $p > 2$;*

(c) *$(T-t)^2 |\operatorname{curl} u(t, y(t))| \to \infty$ as $t \uparrow T$.*

In particular, the data in Theorems 1.3 and 1.4 can be chosen so that their solutions have these properties.

Theorems 1.3–1.5 rest on the proof of [1, Theorem 1.1], which runs the induction of its Theorem 7.1, built from the step of its Sections 3–5, and passes to the limit in its Section 8. For Theorem 1.4(a) we change only the field over which the induction starts (Proposition 2.1 and Section 3). Sections 3 and 5 of [1] are stated for an odd parent field, and Theorem 1.3 and Theorem 1.4(b) use them for a parent that is not odd. Their proofs justify this, since oddness there only keeps the origin fixed and propagates parity (Lemma 4.1). Without a fixed point, each wave is centered at the label z of its own parent's flow, and two successive waves see the velocity gradient along two different particle paths. Lemma 4.3 compares them, using two bounds from the proofs of [1, Lemmas 5.9 and 5.10], which we state as Lemma 4.2. For a group, the waves run at all points of an orbit at once, with one phase-independent correction for all of them (Section 5).

In [1] the induction starts from the stationary shear $\sin(2\pi x_2)(e_1 + \varepsilon_1 e_3)$. Its first wave, of frequency $\lambda_1 = N^{640b}$ with N large and b a fixed integer, is set up in the frame (e_1, e_2, e_3), in which the two couplings of the shear's gradient are 2π and $2\pi\varepsilon_1$ [1, (6.1)–(6.4)]. Their ratio ε_1 is what the second wave needs. It tends to zero as $N \to \infty$, and by [1, (7.4)] and the counts of Lemma 3.2 the data of [1] then converge to $\sin(2\pi x_2)e_1$ in the norms of Theorem 1.3, with blow-up times tending to zero; so for this shear at $z = 0$, and hence, by scaling, translation and a Galilean change of frame, for constant fields, Theorem 1.3 already follows from their construction. Here the base is the Euler solution from θv, for a small $\theta > 0$, cut off in frequency at N^5 at each time. Its residual is smaller than any power of N, as the induction requires, whereas the residual of the stationary cutoff of θv does not tend to zero unless v is a steady state. The frame of the first wave is built from the gradient of the base at z, which is trace free and, once a small shear is added to v if necessary, nonzero. At time zero the ratio of the couplings is then again exactly ε_1, while the first coupling, no longer 2π, stays within a factor two of its initial value.

At time zero the data differ from the base by the first leading wave $V(x) = \chi(|x-z|/\ell_*)F_h(\lambda_1 E_1 \cdot (x-z))\mathbf{a}_*$, $h = \lambda_1^{-1/2}$, up to terms that are small in C^1; for a

group, by one such wave at each point of the orbit. Here F_h is the profile of [1, Lemma 3.1], E_1 is a unit vector, $\mathbf{a}_* \perp E_1$ has length between $\lambda_1^{-3/4}/2$ and $\lambda_1^{-3/4}$, and the envelope radius ℓ_* tends to zero. The size $\lambda_1^{-5/4}$ of V and the size $\lambda_1^{1/4}$ of its gradient give the bound in C^α for $0 < \alpha < 5/6$ (Lemma 3.2 and Remark 3.3). In H^1, the one-sided bound $F_h' \geq -Ch$ makes the mean of $|F_h'|^2$ at most of order h, which offsets the slope $\lambda_1|\mathbf{a}_*| \leq \lambda_1^{1/4}$ and gives $\|\nabla V\|_{L^2} = O(\ell_*^{3/2})$ (Lemma 3.2).

The same two counts, applied to each wave uniformly for $0 \leq t \leq T$, give Theorem 1.5(a); the statements in (b) use the strength and wave vector of the waves at their crossings [1, Proposition 2.8]. For the solution of [1], the approximate solutions converge uniformly on $[0,T] \times \mathbb{T}^3$ [1, Section 8], so that $u \in C([0,T]; C^0)$, and their auxiliary note shows that, for a suitable choice of the function $\mathcal{B}$, their velocity extends to a smooth field on a space-time neighborhood of $\{T\} \times (\mathbb{T}^3 \setminus \{0\})$ [2, Theorem 1.1], so that the singular set is $\{(T,0)\}$. For the other solutions the uniform convergence is proved in the same way. The lower bound on the particle uses $D^2 p \leq \mathrm{Id}$, the limit of the pressure bound (T3) of [1, Theorem 7.1] and of its version (T3′) in Section 2, which gives $|\omega|'' \geq -|\omega|$ for the vorticity ω on the particle (Section 6), and the strength of the waves at their crossings, as in the proof of [1, (8.2)].

Theorems 1.3 and 1.4 are about topologies in which the Euler equations are not well posed. Bardos and Titi give shear flows that lie in $C^{0,\alpha}$ at $t = 0$ and in no $C^{0,\beta}$, $\beta > \alpha^2$, at any $t \neq 0$ [3, Theorem 2.2(iii)], and conclude that the equations are not well posed in C^β, $0 < \beta < 1$ [3, Section 5]; with shear flows of the same form, DiPerna and Lions showed that a $W^{1,p}$ norm can grow from one to any size in any given time, as recalled in [3, Theorem 2.1]; and bounded weak solutions are not unique [9, Theorem 1.1]. Smooth data, on the other hand, have unique local solutions in $C^{1,\alpha}$ [16, 14], and the equations are well posed in H^s for every integer $s > 5/2$ [10, Section 15.2]; Theorems 1.3 and 1.4 say nothing about instability in $C^{1,\alpha}$ or in H^s, $s > 5/2$.

Near the Biot–Savart velocity of every smooth compactly supported vorticity on $\mathbb{R}^3$, Bourgain and Li find smooth perturbations, arbitrarily small in $H^{5/2}$, whose solutions have unbounded $H^{5/2}$ norm on every interval $(0, t_0)$ [5, Theorem 1.8]. Near any smooth solution on $\mathbb{T}^3$, Enciso, Peñafiel-Tomás and Peralta-Salas construct weak solutions that are uniformly close to it at time zero and singular in a prescribed region [11, Theorem 1.5]. Theorem 1.3 is a statement of the same kind for blow-up in finite time from smooth data, in topologies weaker than C^1.

2. The construction over a general base

The construction of [1], which follows the multiscale approach of Córdoba and Martínez-Zoroa [6, 7], adds to a parent field, at frequencies $\lambda_1 < \lambda_2 < \cdots$, Craik–Criminale waves [8] localized at the origin. Each wave grows in the shear of its predecessor, and an endpoint problem for the phase-independent correction chooses the initial velocity of each step so that the limit solves the unforced equations [1, Sections 1 and 4]. We recall what the proof uses, in the notation of [1]: $(\nabla U)_{ij} = \partial_j U_i$, $a \otimes b = ab^{\mathsf{T}}$, $\mathbb{P}$ is the Leray projection and pressures have zero mean [1, Section 1, Conventions]. Matrix norms are operator norms, and matrices are compared as quadratic forms. As in [1], this section and the next work at the origin with odd fields; Section 4 moves the center to a label z.

Other inputs. Besides Sections 3 and 5 of [1], Theorem 1.3 and Theorem 1.4(b) use from [1], as stated, [1, (1.14), (1.15), Section 2, Theorem 4.2, Lemmas 1.3, 3.1, B.1 and D.1–D.3, and Appendices A and C] and the continuation criterion cited in [1, Corollary 8.3].

Their proofs run the arguments of [1, Sections 6–8] again with the moving center and do not use [1, Theorems 7.1 and 1.1].

Parameters [1, Section 1.5]. An integer $b \geq 1$, a constant $\Gamma_P \geq 1$ [1, Section 5.6] and a nondecreasing function $\mathcal{B}$ bounding the fixed-order constants are fixed once, before N, large enough for [1], for Lemmas 4.2 and 4.3 and, for a group G, for the constants of Lemma 5.1, which depend on G; [1, (1.14), (5.43), (7.8)] hold for every such choice. Then $K(s) = 320b(s+1)$ and $N \geq \mathcal{B}(1)$. With orders q_j as in [1, (1.14)], $q_1 = 1$, put $\lambda_0 = N$, $\lambda_j = \lambda_{j-1}^{K(q_j)}$, $\Lambda_j = \lambda_j^5$, $L_j = 4\log\lambda_{j+1}$ and $\varepsilon_{j-1} = L_j^{-2}$ for $j \geq 2$. Thus $\lambda_1 = N^{640b}$, and $L_{j+1} \leq (\log\lambda_{j-1})^2$ by [1, (1.15)]. The times of [1, Section 7] are $t_1 = t_2 = 0$, $\tau_j = \lambda_{j-1}^{-1/8} L_{j+1}^3$, $S_0 = 1$ and $S_j = t_j + 2\tau_j$.

The profile [1, Lemma 3.1]. For $0 < h \leq 1$, F_h is odd, smooth and 1-periodic, with zero-mean periodic primitive P_h, and

$$F_h'(0) = 1, \qquad -Ch \leq F_h' \leq 1, \qquad \|F_h\|_\infty \leq Ch, \qquad \|F_h''\|_\infty \leq C_2 h^{-1}. \tag{2.1}$$

The step [1, Sections 3.1 and 5.1]. The parent U is a smooth odd divergence-free field on $[0,T] \times \mathbb{T}^3$, $T \leq 1$, with flow X, $X(0,a) = a$, and $M = (\nabla U) \circ X$. A number $\mathsf{D} \geq 2$ bounds it: U is supported on frequencies at most $2\mathsf{D}$, $\|\nabla U\|_\infty \leq \mathsf{D}$ and $\sup|(D_a X)^{\pm 1}| \leq \mathsf{D}$ [1, (5.1)]. With p_U the zero-mean solution of $\Delta p_U = -\operatorname{tr}((\nabla U)^2)$ and $R_U = U_t + \mathbb{P}\operatorname{div}(U \otimes U)$, the step assumes $D^2 p_U - \nabla R_U \leq \mathrm{Id}$ and $\|R_U\|_{C_t L^\infty} \leq \lambda^{-5/2}$ [1, (5.2), (5.6)]. The wave has a constant vector k_*, the transported wave vector $\kappa = (D_a X)^{-\mathsf{T}} k_*$, an activation time t_0, frequency $\lambda = |\kappa(t_0, 0)|$ and $h = \lambda^{-1/2}$. Its amplitude, the seed, solves

$$\mathbf{a}_t = -(\mathrm{Id} - 2P_\kappa) M \mathbf{a}, \qquad P_\kappa = \frac{\kappa \otimes \kappa}{|\kappa|^2},$$

by [1, (3.2)]. At the origin it is perpendicular to κ and normalized by $\max_{[t_0,T]} |\mathbf{a}(t,0)| = \lambda^{-3/4}$; at the other labels it starts from the projection of $\mathbf{a}(t_0,0)$ onto $\kappa(t_0,a)^\perp$ [1, (5.4)]. With $\gamma(t) = \lambda^{3/4}|\mathbf{a}(t,0)|$ and $\nu(t,0) = (\kappa/|\kappa|) \cdot M(\mathbf{a}/|\mathbf{a}|)$ at the origin, the two central conditions [1, (5.5)] ask that, for $t_0 \leq t' \leq t \leq T$,

$$\frac{|b(t)|}{\gamma(t)} \leq \mathsf{D}\,\frac{|b(t')|}{\gamma(t')}, \qquad \nu(t,0) \geq \mathsf{D}^{-1}, \tag{2.2}$$

the first for every solution $b \perp \kappa$ of the seed equation at the origin. The envelope is $g(a) = \chi(|a|/\ell_*)$, with χ smooth, $0 \leq \chi \leq 1$, $\chi = 1$ on $[0,1]$ and $\chi = 0$ on $[2,\infty)$, and $\ell_* \leq \mathsf{D}^{-3}$ is a fixed power of D [1, Section 5.1 and Appendix D]. The leading oscillation is $A_1 = g\mathbf{a}F_h$, with potential and divergence correction

$$Q[A_1] = gP_h\,\frac{\mathbf{a} \times \kappa}{|\kappa|^2}, \qquad C[A_1] = \mathbf{d} \times Q[A_1], \qquad \mathbf{d} = (D_a X)^{-\mathsf{T}} \nabla_a, \tag{2.3}$$

by [1, (3.3), (3.5)]. A profile $f(t,a,s)$ with phase s gives the Eulerian field $f^\natural(t, X(t,a)) = f(t,a,k_* \cdot a)$, and $(A_1 + C[A_1])^\natural = \operatorname{curl}(Q[A_1]^\natural)$ by [1, (3.1), (3.6)].

At the first step $\mathsf{D} = N^9$ [1, Section 7.1], so $\ell_* \leq N^{-27}$. At every step, [1, (5.44)] for phase-independent fields at order one and $t = 0$, where $X = \mathrm{id}$, applied to g, with $\|g\|_{[1]} \leq C$ (proof of [1, Lemma 5.3(iii)]), gives $\ell_*^{-1} \leq \|\chi'\|_\infty \ell_*^{-1} = \|\nabla g\|_\infty \leq C\mathsf{D}^b$, as $\|\chi'\|_\infty \geq 1$ and b is at least the exponent of [1, Lemma 5.7]; at the first step $C\mathsf{D}^b \leq \lambda_1^{1/32}$ by [1, (7.8)]. Thus, at the first step,

$$\lambda_1^{-1/32} \leq \ell_* \leq N^{-27}. \tag{2.4}$$

Section 4 of [1] is written for a parent "not assumed to solve any equation", and the step of Section 5 for any parent and wave as above, given $\Gamma_P \mathsf{D}^b \lambda^{-1/12} \leq 1$ [1, (5.43)]. A

particular field, the shear of Section 1, is chosen in [1, Section 6.1] and starts the induction of [1, Section 7.2]; its first wave has $k_* = \lambda_1 e_1$ and seed direction e_2 [1, (6.1), (6.3)]. The following proposition replaces it by a general base.

Proposition 2.1. *Let $0 < m \le 1$. There is N_0, depending only on m and the fixed choices of* [1, Section 1.5]*, such that the following holds for every $N \ge N_0$. Let $z \in \mathbb{T}^3$, and let U_0 be a smooth field on $[0,1] \times \mathbb{T}^3$ such that*

(a) *U_0 is divergence free and supported on frequencies at most $2N^5$;*
(b) *$|U_0|$, $|\nabla U_0|$ and $|\partial_t \nabla U_0|$ are at most m^{-1} on $[0,1] \times \mathbb{T}^3$;*
(c) *$\|R_0\|_{C([0,1];L^\infty)} \le \lambda_1^{-3}$, where $R_0 = \partial_t U_0 + \mathbb{P}\operatorname{div}(U_0 \otimes U_0)$;*
(d) *$D^2\pi_0 \le \frac{3}{4}\operatorname{Id}$ on $[0,1] \times \mathbb{T}^3$, where π_0 has zero mean and $\Delta\pi_0 = -\operatorname{tr}((\nabla U_0)^2)$;*
(e) *$|(\operatorname{Id} - e \otimes e)\nabla U_0(0,z)e| \ge m$ for a unit vector e.*

Then there are a unit vector $E_1 \perp e$, a vector $\mathbf{a}_*$ in the direction of e with $\lambda_1^{-3/4}/2 \le |\mathbf{a}_*| \le \lambda_1^{-3/4}$, and a divergence-free $u_0 \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ that blows up along the particle from z at a time

$$T_* \le \tfrac{4}{3}\lambda_1^{-1/8}(\log\lambda_1)^6, \tag{2.5}$$

with $\sup_{t<T_*}\|u(t)\|_{L^\infty} \le m^{-1} + 1$ and

$$u_0 - U_0(0) = \operatorname{curl}\Big(\lambda_1^{-1} g(x) P_h(\lambda_1 E_1 \cdot (x - z))\,(\mathbf{a}_* \times E_1)\Big) + H, \qquad \|H\|_{C^{1,1/2}} \le 2\lambda_1^{-1}. \tag{2.6}$$

Here $h = \lambda_1^{-1/2}$, $g(x) = \chi(|x-z|/\ell_*)$ on the cell $z + [-1/2,1/2]^3$ with ℓ_* as in (2.4), and the first term is extended periodically. If U_0 is odd and $z = 0$, then u_0 is odd and blows up as in Theorem 1.1.

Proof when U_0 is odd and $z = 0$. Section 4 gives the changes for the general case. We follow the proof of [1, Theorem 7.1], with U_0 as the field of index zero. Constants written $c(m)$, $C(m)$ are positive and depend only on m, and "for large N" means for $N \ge N_0(m)$. Three statements of [1, Theorem 7.1] change. Its pressure bound (T3) becomes (T3′), namely $D^2\pi_j \le (\frac{3}{4} + \sum_{i=1}^{j}\lambda_{i-1}^{-1})\operatorname{Id}$ on $[0, S_j] \times \mathbb{T}^3$; the range $\pi \le \nu_j \le 4\pi$ in [1, (7.3)] becomes $\nu_*/2 \le \nu_j \le 2\nu_*$, with ν_* from (2.7) below; and the first wave is the one chosen below.

Index zero. (T1) is hypothesis (a) and the oddness of U_0, with $\Lambda_0 = N^5$ and $S_0 = 1$. For (T2), $\|U_0\|_\infty \le m^{-1} \le N$; Gronwall's inequality for $\partial_t D_a X_0 = (\nabla U_0 \circ X_0) D_a X_0$ and for the inverse gives $|(D_a X_0)^{\pm 1}| \le e^{1/m} \le N$; and the residual bound is hypothesis (c). (T3′) is hypothesis (d), and hypothesis (b) gives [1, (7.6)] at index zero.

Item (1), the parent, for $j \ge 1$. As in [1, Section 7.2], (T1), (T2) and [1, (7.6)] at $j-1$ give [1, (5.1)] with $\mathsf{D} = \mu^9$, $\mu = \lambda_{j-1}$, and [1, (5.6)]. Bernstein's inequality [1, (B.1)] gives $\|\nabla R_{j-1}\|_\infty \le C\mu^5\lambda_j^{-3}$, and $\sum_{i\ge1}\lambda_{i-1}^{-1} \le 2/N$, so (T3′) gives $D^2\pi_{j-1} - \nabla R_{j-1} \le (\frac{3}{4} + 2/N + C\mu^5\lambda_j^{-3})\operatorname{Id} \le \operatorname{Id}$ for large N. This is [1, (5.2)].

The first wave. Let $B(t) = \nabla U_0(t,0)$. Since U_0 is odd, $U_0(t,0) = 0$, the flow fixes the origin and $M(t,0) = B(t)$, a smooth trace-free matrix with $|B|, |B'| \le m^{-1}$. Put $E_2 = e$, $w = (\operatorname{Id} - e \otimes e)B(0)e$, $n_1 = w/|w|$, $n_3 = n_1 \times E_2$, $\beta = \arctan\varepsilon_1$ and

$$E_1 = n_1\cos\beta - n_3\sin\beta, \qquad E_3 = E_1 \times E_2 = n_3\cos\beta + n_1\sin\beta.$$

Since $w \perp E_2$ and $m \le |w| \le m^{-1}$, $\varepsilon_1 \le 1$,

$$\nu_* := E_1 \cdot B(0)E_2 = |w|\cos\beta \in [m/\sqrt{2},\, m^{-1}], \qquad E_3 \cdot B(0)E_2 = |w|\sin\beta = \varepsilon_1\nu_*. \tag{2.7}$$

The first wave has $k_* = \lambda_1 E_1$, activation time zero and time cutoff $\widetilde{\chi}_1 \equiv 1$; at the origin its seed is the solution of the seed equation with initial direction E_2, scaled by a positive constant so that $\max_{[0,S_1]} |\mathbf{a}(t,0)| = \lambda_1^{-3/4}$. Put $\mathbf{a}_* = \mathbf{a}(0,0)$, a positive multiple of e. At time zero $D_a X = \mathrm{Id}$, so $\kappa(0,\cdot) = \lambda_1 E_1 \perp \mathbf{a}_*$, and [1, (5.4)] gives $\mathbf{a}(0,a) = \mathbf{a}_*$ at every label. The pair $(\kappa, \mathbf{a})(t,0)$ solves [1, (2.1)] over B, and [1, Lemma 2.2], which uses only $\operatorname{tr} B = 0$, applies to its frame $E_i(t)$, with $E_i(0)$ as above. By (2.7), its couplings at time zero are $\nu_1 = \nu_*$ and $\eta_1 = \varepsilon_1 \nu_*$.

We claim that if $2S_1/m \le \log 2$ and $(2m^{-2} + m^{-1})S_1 \le m/(2\sqrt{2})$, then (2.2) holds on $[0, S_1]$ for every $\mathsf{D} \ge 2\sqrt{2}/m$, and $\lambda_1^{1/4}/2 \le \lambda_1 |\mathbf{a}_*| \le \lambda_1^{1/4}$. We follow the proof of [1, Lemma 6.1]. At the origin the seed equation reads $b_t = -(\mathrm{Id} - 2P_\kappa)Bb$, and $\mathrm{Id} - 2P_\kappa$ is a reflection, so $|\partial_t \log |b|| \le |B| \le m^{-1}$ for every nonzero solution, the seed included. Hence, for $0 \le t' \le t \le S_1$,

$$\frac{|b(t)|}{\gamma(t)} \le e^{2(t-t')/m} \frac{|b(t')|}{\gamma(t')} \le 2 \frac{|b(t')|}{\gamma(t')}, \qquad \lambda_1^{-3/4} = \max_{[0,S_1]} |\mathbf{a}(\cdot,0)| \le e^{S_1/m} |\mathbf{a}_*| \le 2|\mathbf{a}_*|,$$

and the strength at time zero is $|\mathbf{a}_*||\kappa(0,0)| = \lambda_1 |\mathbf{a}_*|$. By [1, (2.2)], $|E_1'| = (\nu^2 + B_{13}^2)^{1/2} \le |B|$ and $|E_2'| = (\nu^2 + \eta^2)^{1/2} \le |B|$, so $\nu(t,0) = E_1(t) \cdot B(t) E_2(t)$ satisfies $|\partial_t \nu| \le 2|B|^2 + |B'| \le 2m^{-2} + m^{-1}$ and

$$\nu(t,0) \ge \nu_* - (2m^{-2} + m^{-1})S_1 \ge \nu_*/2 \ge m/(2\sqrt{2}) \ge \mathsf{D}^{-1}.$$

This proves the claim. By [1, (1.15)], $S_1 = 2\tau_1 \le 2N^{-1/8}(\log N)^6$, so the claim applies with $\mathsf{D} = N^9$ for large N. It replaces item (2) of [1, Section 7.2] for $j = 1$ and gives [1, (7.3)] at $t_2 = t_1 = 0$ with $\eta_1/\nu_1 = \varepsilon_1$, $\nu_1 = \nu_*$ and strength in $[\lambda_1^{1/4}/2, \lambda_1^{1/4}]$.

Item (2), choosing wave $j \ge 2$. By (T4) at $j-1$, the gradient at the origin splits as in [1, (6.5)] on $[t_j, S_j]$, with $B = \nabla U_{j-2}(t,0)$ and $\|E_c\|_\infty \le \mu^{-1/4}$. At t_j the frozen couplings and strength satisfy $\varepsilon = \varepsilon_{j-1}$, $\nu_*/2 \le \nu \le 2\nu_*$ and $c\mu^{1/4} \le \sigma \le C\mu^{1/4} L_j^2$; for $j = 2$ these are the values of the first wave at time zero. For $j = 2$, $B(t) = \nabla U_0(t,0)$ satisfies $\|B\|_\infty, \|B_t\|_\infty \le m^{-1} \le \mu^{1/32}$, and for $j \ge 3$ the bounds $\|B\|_\infty \le \mu^{1/32}$ and $\|B_t\|_\infty \le \mu^{1/16}$ hold as in [1]. Hence $r = (2\sigma\nu)^{1/2}$ satisfies $c(m)\mu^{1/8} \le r \le C(m)\mu^{1/8} L_j$, the normalized length $r(S_j - t_j) = 2r\mu^{-1/8} L_{j+1}^3$ lies between $2c(m)L_{j+1}^3 \ge 2$ and $C(m)L_{j+1}^4$, and the three terms of [1, (6.6)], with $\eta = \nu\varepsilon \ge \nu_*\varepsilon/2$, give

$$\delta \le C(m)(\varepsilon^{-1} L_{j+1}^3 \mu^{-1/16} + \mu^{-3/32} + \mu^{-1/4}).$$

As in [1], $C_0(2r\tau_j)^{C_0}\delta/\varepsilon$ is at most $C(m)\mu^{-1/16}$ times a fixed power of $\log \mu$, so both conditions of [1, (2.7)] hold for large N. In [1, (6.7)], $CS = Cr/(2\nu) \le C(m)\mu^{1/8} L_j$ and $2/(\nu\varepsilon) \le 4\nu_*^{-1} L_j^2$ are at most μ for large N, which gives (2.2) with $\mathsf{D} = \mu^9$.

Item (3), the next activation time. The upper end of the bracket of [1, Proposition 2.8] lies at most $2(\varepsilon\varepsilon_j)^{-1/2}/r \le C(m)\mu^{-1/8} L_{j+1}^2 \le \tau_j$ after t_j, for large N. The strength bounds in [1, (7.3)] follow from [1, (2.23)] with absolute constants, as in [1]. For the coupling, [1, (2.19)] gives $|\log(\nu_j(t_{j+1})/\nu_{j-1}(t_j))| \le 2\varepsilon_{j-1}$; summing from $\nu_1(t_2) = \nu_*$ and using $\varepsilon_i \le 32^{1-i}\varepsilon_1$ from [1, (7.9)],

$$\left|\log \frac{\nu_j(t_{j+1})}{\nu_*}\right| \le 4\varepsilon_1 \le 4(\log N)^{-2} < \log 2,$$

so $\nu_*/2 \le \nu_j(t_{j+1}) \le 2\nu_*$. Finally [1, (7.10)] holds, since $Ce^3 r \le C(m)\mu^{1/8} L_j \le \mu$ for large N and $\widetilde{\chi}_1' \equiv 0$.

The other items. Items (4) and (5) of [1, Section 7.2], with [1, (7.11), (7.12)], see the base only through the hypotheses of [1, Section 5.1], now supplied by items (1)–(3), and through (T1), (T2) and [1, (B.1)] in the bound for $\|U\|_{C_t C^{P,1/2}}$ preceding [1, (7.12)]; they are unchanged. Item (6) adds the same increment $\mu^{-1}\,\mathrm{Id}$ to (T3′) for the parent as to (T3), and so proves (T3′) at step j; its bound for the central gradient is unchanged. The bounds [1, (7.7), (7.8)] concern the parameters only, and the paragraph after item (6) gives the limit T_∞ of the activation times with $0 < T_\infty - t_j \le \frac{4}{3}\tau_j$. Section 8 of [1] uses Theorem 7.1 through (T1)–(T4), (7.4)–(7.7) and the bound on the normalized length in item (2), now $C(m)L_{j+1}^4$; so the constant c in [1, (8.1), (8.2)] depends on m, and for $j = 2$ the older gradient in [1, (8.2)] is at most m^{-1}. Thus the proofs of [1, Propositions 8.1, 8.2, 8.4 and Corollary 8.3] apply. The data $U_j(0)$ converge in every $C^{n,1/2}$ to an odd divergence-free $u_0 \in C^\infty$ that blows up as in Theorem 1.1 at $T_* = T_\infty$, and $\|u\|_\infty \le \|U_0\|_\infty + 2\sum_j \lambda_j^{-1} \le m^{-1} + 1$ by [1, (7.4)]. Since $t_2 = 0$ and $L_3 \le (\log\lambda_1)^2$, this gives (2.5).

The data. By the induction above, the oscillation O_j of step j vanishes on $[0, t_j]$ for $j \ge 2$, as in [1, Theorem 7.1], and $t_j \ge 0$. Hence

$$u_0 - U_0(0) = O_1(0) + H, \qquad H = \sum_{j\ge 1}(U_j - U_{j-1} - O_j)(0),$$

and [1, (7.4)], with $q_j \ge q_1 = 1$, gives $\|H\|_{C^{1,1/2}} \le \sum_{j\ge1}\lambda_j^{-1} \le 2\lambda_1^{-1}$. For the first wave, $O_1 = W_{\mathrm{osc}}^\natural = \sum_{p\le P}(V_p + C_{p+1})^\natural$ with $V_1 = A_1$ and $C_2 = C[A_1]$ [1, (3.7) and Proposition 5.4]. The transverse corrections are solved from zero data at time zero [1, Section 5.4], so $V_p(0) = 0$ and $C_{p+1}(0) = C[V_p](0) = 0$ for $p \ge 2$. Thus $O_1(0) = (A_1 + C[A_1])^\natural(0) = \mathrm{curl}(Q[A_1]^\natural(0))$. At time zero $X = \mathrm{id}$, $\kappa = \lambda_1 E_1$ and $\mathbf{a} = \mathbf{a}_*$, so (2.3) gives

$$Q[A_1]^\natural(0,x) = \lambda_1^{-1} g(x) P_h(\lambda_1 E_1 \cdot x)\,(\mathbf{a}_* \times E_1), \tag{2.8}$$

which proves (2.6). □

3. Odd data: proof of Theorem 1.4(a)

The data of Proposition 2.1 differ from the base by the first leading wave and a remainder small in C^1. The gradient of the wave in L^2 rests on the following count.

Lemma 3.1. *Let $0 < h \le 1$ with $Ch \le 1$, where C is the constant of (2.1). For $\lambda \ge 1$, a unit vector E and $R > 0$,*

$$\int_{|x|\le R} |F_h'(\lambda E\cdot x)|^2\,dx \le 4\pi Ch\,R^2(R + \lambda^{-1}).$$

Proof. By (2.1) and $Ch \le 1$, $|F_h'| \le 1$. Being the derivative of a periodic function, F_h' has mean zero, so $\int_0^1 |F_h'| = 2\int_0^1 (F_h')_- \le 2Ch$ and $\int_0^1 |F_h'|^2 \le \sup|F_h'| \int_0^1 |F_h'| \le 2Ch$. Write $x = sE + y$ with $y \perp E$, $|y| \le R$. The slice $\{s : |sE + y| \le R\}$ is an interval I_y of length at most $2R$, and λI_y meets at most $\lambda|I_y| + 2$ intervals $[k, k+1]$, $k \in \mathbb{Z}$. By periodicity,

$$\int_{I_y} |F_h'(\lambda s)|^2\,ds = \lambda^{-1}\int_{\lambda I_y} |F_h'|^2 \le (2R + 2\lambda^{-1})\,2Ch.$$

Integrating over the disc $|y| \le R$, of area πR^2, gives the bound. □

Lemma 3.2. *In the setting of Proposition 2.1, write $\lambda = \lambda_1$, $\ell = \ell_*$, $E = E_1$, and put*

$$V(x) = g(x)F_h(\lambda E\cdot(x - z))\,\mathbf{a}_*, \qquad V^{\mathrm{c}}(x) = \lambda^{-1}P_h(\lambda E\cdot(x-z))\,\nabla g(x) \times (\mathbf{a}_* \times E).$$

For large N, $u_0 - U_0(0) = V + V^{\mathrm{c}} + H$ with $\|V^{\mathrm{c}}\|_{C^1} \le 2\lambda^{1/16-3/4}$, and, with C depending only on χ and the profiles,

$$\|V\|_{L^\infty} \le C\lambda^{-5/4}, \qquad \tfrac{1}{2}\lambda^{1/4} \le \|\nabla V\|_{L^\infty} \le C\lambda^{1/4}, \qquad [V]_\alpha \le C\lambda^{3\alpha/2-5/4} \quad (0<\alpha\le 1), \tag{3.1}$$

$$\|\nabla V\|_{L^2}^2 \le C(\lambda^{-5/2}\ell + \ell^3 + \lambda^{-1}\ell^2). \tag{3.2}$$

Proof. Since $E\times(\mathbf{a}_*\times E) = \mathbf{a}_*$, the curl in (2.6) is $\lambda^{-1}\nabla(gP_h(\lambda E\cdot(x-z)))\times(\mathbf{a}_*\times E) = V^{\mathrm{c}}+V$. By (2.3) and (2.8), $V^{\mathrm{c}} = C[A_1]^\natural(0)$ for the first wave. The bounds [1, (5.36)] at $p=1$ and [1, (5.44)] at orders 0 and 1 and $t=0$, where $\gamma\le 1$ (for a parent that is not odd, by Lemma 4.1), bound $\|V^{\mathrm{c}}\|_\infty$ and $\|\nabla V^{\mathrm{c}}\|_\infty$ by $\lambda^{-9/4}$ and $\lambda^{3/2-9/4}$ times products of two fixed-order constants and powers D^C with $C\le b$. By [1, (7.8)] at $j=1$, $\mathcal{B}(q_2)\mathsf{D}^b \le \lambda^{1/32}$, which gives the bound for V^{c}.

For large N, $Ch\le 1$, so $|F_h'|\le 1$. The bound for V is $|\mathbf{a}_*|\|F_h\|_\infty \le C\lambda^{-3/4}h$. Next,

$$\nabla V = \mathbf{a}_*\otimes\big(F_h(\lambda E\cdot(x-z))\nabla g + \lambda g F_h'(\lambda E\cdot(x-z))E\big).$$

The second term is at most $\lambda|\mathbf{a}_*| \le \lambda^{1/4}$, and at z, where $\nabla g = 0$, it is $\lambda\mathbf{a}_*\otimes E$, of norm at least $\lambda^{1/4}/2$. On the support of ∇g, $|x-z|\le 2\ell$, so $F_h(0)=0$ gives $|F_h(\lambda E\cdot(x-z))|\le 2\lambda\ell$, while $|\nabla g| \le \|\chi'\|_\infty \ell^{-1}$; the first term is therefore at most $2\|\chi'\|_\infty\lambda|\mathbf{a}_*|$. The Hölder bound follows from $|V(x)-V(y)| \le (2\|V\|_\infty)^{1-\alpha}(\|\nabla V\|_\infty|x-y|)^\alpha$.

For (3.2), V is supported in $|x-z|\le 2\ell$, and $\|\nabla g\|_{L^2}^2 \le C\ell$, so

$$\begin{aligned}\|\nabla V\|_{L^2}^2 &\le 2|\mathbf{a}_*|^2\Big(\|F_h\|_\infty^2\|\nabla g\|_{L^2}^2 + \lambda^2\int_{|x|\le 2\ell}|F_h'(\lambda E\cdot x)|^2\,dx\Big)\\ &\le C\lambda^{-3/2}\big(h^2\ell + \lambda^2 h(\ell^3+\lambda^{-1}\ell^2)\big)\end{aligned}$$

by Lemma 3.1, and $\lambda^{1/2}h = 1$. □

The one-sided bound in (2.1) is what makes (3.2) small. A sine profile with the same slope at zero has $\int_0^1 |F'|^2 = 1/2$, and for it the count in the proof of Lemma 3.2 gives $\|\nabla V\|_{L^2}$ of order $\lambda_1^{1/4}\ell_*^{3/2}$, which grows, since $\ell_* \ge \lambda_1^{-1/32}$ by (2.4).

Remark 3.3. With $\delta = 1/(2C_2)$ and $y = z + \delta\lambda^{-3/2}E$, (2.4) and (2.1) give $g(y) = 1$ and $|V(y)| \ge \frac{\delta}{4}\lambda^{-5/4} = \frac{1}{4}\delta^{1/6}|y-z|^{5/6}$, so $[V]_{5/6} \ge \frac{1}{4}\delta^{1/6}$. Near z the other terms of $u_0 - v$ in the proofs below have gradient bounded uniformly in N, and each changes the $C^{5/6}$ quotient at (y,z) by $O(|y-z|^{1/6})$; so for these data the $C^{5/6}$ distance to v stays bounded below.

Proof of Theorem 1.4(a). Scaling. For $c>0$, $u_c(t,x) = cu(ct,x)$ is the maximal solution from cu_0; it blows up along the particle path $y(ct)$, with the same endpoint, at time T_*/c.

A nonzero gradient. If $\nabla v(0) = 0$, then $v + s\sin(2\pi x_2)e_1$ is smooth, odd and divergence free, has gradient $2\pi s\, e_1\otimes e_2 \ne 0$ at the origin for $s\ne 0$, and tends to v in C^∞ as $s\to 0$. So we may assume $\nabla v(0)\ne 0$.

The base. Let u_v be the smooth solution from v on $[0,T_0]$, $T_0>0$, with pressure p_v; it is odd, since $-u_v(t,-x)$ solves (1.1) with the same data. Let $\theta = \min\{1, T_0, (2\sup|D^2p_v|)^{-1/2}\}$, omitting the last term if $D^2p_v \equiv 0$. Then $f(t,x) = \theta u_v(\theta t, x)$ solves (1.1) on $[0,1]$ with pressure $p_f(t,x) = \theta^2 p_v(\theta t,x)$, $D^2p_f \le \frac{1}{2}\,\mathrm{Id}$ and $f(0) = \theta v$. The matrix $\nabla f(0,0) = \theta\nabla v(0)$ is nonzero and trace free, and a trace-free matrix for which every vector is an eigenvector vanishes. We fix once and for all a unit vector e with $d = |(\mathrm{Id} - e\otimes e)\nabla f(0,0)e| > 0$. With S_Λ the frequency cutoff of [1, Appendix B], put $U_0 = S_{N^5}f$ and $\delta_N = U_0 - f$.

Let $0 < m \le \min\{1, d/2\}$ be so small that $m^{-1} \ge C \sup_{[0,1]\times\mathbb{T}^3}(|f| + |\nabla f| + |\partial_t \nabla f|)$, with C the norm of S_Λ on L^∞. We check the hypotheses of Proposition 2.1 for $z = 0$ and this e and m; its proof then builds the frame of the first wave from $\nabla U_0(0,0)$. By [1, Lemma B.1], U_0 is odd, divergence free and supported on frequencies at most $2N^5$, and hypothesis (b) holds, since S_Λ commutes with ∇ and ∂_t. As $\partial_t f = -\mathbb{P}\operatorname{div}(f \otimes f)$ and S_Λ commutes with $\mathbb{P}\operatorname{div}$,

$$R_0 = \mathbb{P}\operatorname{div}(U_0 \otimes U_0 - S_{N^5}(f \otimes f)) = \mathbb{P}\operatorname{div}(\delta_N \otimes U_0 + f \otimes \delta_N + (\mathrm{Id} - S_{N^5})(f \otimes f)).$$

Fix J with $5(J-1) > 1920b$. The tail estimate [1, (B.2)] at $s = 1$, the product estimate [1, (5.59)] and the boundedness of $\mathbb{P}$ on $C^{0,1/2}$ give $\|R_0\|_{C_t L^\infty} \le C_{f,J} N^{5(1-J)} \le N^{-1920b} = \lambda_1^{-3}$ for large N, which is hypothesis (c). Next, $\Delta(\pi_0 - p_f) = -\operatorname{tr}(\nabla\delta_N \nabla U_0 + \nabla f \nabla \delta_N)$ and $D^2\Delta^{-1}$ is bounded on $C^{0,1/2}$, so $\|D^2(\pi_0 - p_f)\|_\infty \le C_{f,J} N^{5(1-J)} \le \frac{1}{4}$ for large N, which gives hypothesis (d). Finally $\nabla U_0(0,0) \to \nabla f(0,0)$ by [1, (B.2)], so hypothesis (e) holds for large N, as $m \le d/2$.

Conclusion. For large N, Proposition 2.1 and Lemma 3.2 give an odd divergence-free $u_0^N \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ that blows up as in Theorem 1.1 at a time at most $\frac{4}{3}\lambda_1^{-1/8}(\log\lambda_1)^6$, with $u_0^N - \theta v = V + (V^{\mathrm{c}} + H + S_{N^5}(\theta v) - \theta v)$. By Lemma 3.2, (2.6) and [1, (B.2)], the bracket tends to zero in C^1 as $N \to \infty$, and on $\mathbb{T}^3$ the norms of H^1 and C^α are bounded by that of C^1. By (3.1), (3.2) and (2.4), V tends to zero in L^∞, in H^1 and in C^α for $\alpha < 5/6$. By the scaling step with $c = \theta^{-1}$, $u_0 = \theta^{-1} u_0^N$ blows up as in Theorem 1.1 at a time at most $\frac{4}{3}\theta\lambda_1^{-1/8}(\log\lambda_1)^6$, and $u_0 - v = \theta^{-1}(u_0^N - \theta v)$. Taking N large proves Theorem 1.4(a). □

4. Without symmetry

This section proves Proposition 2.1 without oddness, and with it Theorem 1.3. Section 4.1 adapts the step of [1, Section 5] to a wave centered at a moving particle, and Section 4.2 runs the induction along it.

4.1. **One step over a parent that is not odd.**

We use the notation of [1, Sections 5.5–5.6]: $W = W_{\mathrm{osc}} + \bar{W}$ is the profile of the velocity increment of a step, with oscillatory part W_{osc} and phase-independent part $\bar{W}$, $V_1 = \widetilde{\chi} A_1$ is its leading term (at the first step $\widetilde{\chi} \equiv 1$, as in [1, (6.3)], so this is $V_1 = A_1$ of Section 2), and $U + W^{\natural}$ is the approximate solution of [1, Lemma 5.8].

Lemma 4.1. *Let U, X, D and λ be as in the step of Section 2, except that U need not be odd, and let $z \in \mathbb{T}^3$. Center the wave at the label z: its phase is $k_* \cdot (a - z)$, and its oscillatory profiles are defined on the lift of the cell $z + [-1/2, 1/2]^3$, where they vanish near the faces, and extended periodically; its envelope is $g(a) = \chi(|a - z|/\ell_*)$; $\lambda = |\kappa(t_0, z)|$; the seed is normalized by $\max_{[t_0,T]} |\mathbf{a}(t,z)| = \lambda^{-3/4}$ and extended to the other labels by* [1, (5.4)] *with z in place of 0; and γ, ν and* (2.2) *are taken at z. Then the statements of* [1, Sections 3 and 5] *hold for this wave, with the label z in place of 0, the point $X(t,z)$ in place of $x = 0$ in* [1, (5.50)]*, and without the assertions of parity.*

Proof. Oddness enters [1, Sections 3 and 5] in three ways, none of them an estimate. It makes the origin a fixed particle and $D_a X$, M, κ, the seed and the envelope even in a [1, end of Section 3.1, and Section 5.1]. It makes the construction symmetric under $(a,s) \mapsto (-a,-s)$, by the paragraph before [1, Proposition 5.4], which is used only for the parity assertions of [1, Lemma 5.8 and Proposition 5.11]. And in the proof of [1, (5.50)] the flow fixes the origin, so the gradient is evaluated at $a = 0$, $s = 0$. Here we evaluate at $a = z$,

$s = 0$, $x = X(t,z)$. As $g(z) = F_h'(0) = 1$ and $F_h(0) = 0$, the chain rule [1, (3.1)] gives $D_x V_1^\natural(t, X(t,z)) = \widetilde{\chi}(t)\mathbf{a}(t,z) \otimes \kappa(t,z)$, and the bound for E_c is proved as in [1].

The rest does not use parity. [1, Theorem 4.2] is stated for every smooth divergence-free parent with $D^2 p_U - \nabla R_U \le \mathrm{Id}$; [1, Propositions 5.1 and 5.2] bound coefficients and the mean inverse at all labels; the proof of [1, Lemma 5.3] compares the equation at a label a with the one at the center, now z, with $|a - z|$ in place of $|a|$; [1, Proposition 5.4, Lemma 5.5 and Theorem 5.6] use cancellations at fixed labels and the global estimate of [1, Proposition 5.2] for the mean; [1, Lemma 5.7] is the chain rule, with profiles extended periodically as in [1, Section 3.1]; and the proof of [1, Lemma 5.10] uses the volume preservation of X and a phase primitive. Finally, odd fields have zero mean and U need not. The estimates of [1, Sections 3–7] do not use the mean, since $\mathbb{P}$ is the identity on constants, the inverse of [1, Theorem 4.2] includes the zero Fourier mode (proof of [1, Lemma A.2]), and S_Λ preserves it. □

Below, H denotes a quantity $C_{2,P}\mathsf{D}^C$ with C absolute, which may change from line to line; b and Γ_P were taken at least as large as these exponents and constants (Section 2), so [1, (5.43)] gives $\mathsf{H} \le \lambda^{1/12}$.

Lemma 4.2 (from the proofs of [1, Lemmas 5.9 and 5.10])**.** *In the setting of Lemma 4.1, let v_Λ be a smooth divergence-free field on $[0,T]\times\mathbb{T}^3$ with $\|v_\Lambda\|_{C_t C^2} \le \lambda^{-5/2}$, and $U^+ = U + W^\natural + v_\Lambda$. Write*

$$U^+ - U = (\widetilde{\chi} g \mathbf{a} F_h)^\natural + r, \qquad c = (D_a X)^{-1}(W + v_\Lambda \circ X) = c_{\mathrm{osc}} + \bar{c},$$

with $\bar{c}$ the phase-independent part of c. Then, on $[0,T]\times\mathbb{T}^3$,

$$|r| \le \mathsf{H}\lambda^{-23/12}, \qquad |\nabla r| \le \mathsf{H}\lambda^{-5/12}, \tag{4.1}$$

$$\|c_{\mathrm{osc}}\|_{[2]} \le \mathsf{H}\lambda^{-5/4}, \qquad \|\bar{c}\|_{[2]} \le \mathsf{H}\lambda^{-5/2}, \qquad \|k_* \cdot c\|_{[1]} \le \mathsf{H}\lambda^{-5/4}. \tag{4.2}$$

Proof. Here $r = (W_{\mathrm{osc}} - V_1)^\natural + \bar{W}^\natural + v_\Lambda$. The proof of [1, (5.50)] bounds $\|W_{\mathrm{osc}} - V_1\|_{[1],\gamma}$ by $\mathsf{H}\lambda^{-23/12}$, using [1, (5.34), (5.36)], and $\|\bar{W}\|_{[1]}$ by $\mathsf{H}\lambda^{-5/2}$, using [1, (5.37)]. The bound [1, (5.44)] is a supremum over all x; at orders 0 and 1, with $\gamma \le 1$, it gives $|(W_{\mathrm{osc}} - V_1)^\natural| \le \mathsf{H}\lambda^{-23/12}$ and $|\nabla(W_{\mathrm{osc}} - V_1)^\natural| \le \mathsf{H}\lambda^{3/2-23/12}$, and its version for phase-independent fields gives $\mathsf{H}\lambda^{-5/2}$ for $\bar{W}^\natural$ and its gradient. With $\|v_\Lambda\|_{C^1} \le \lambda^{-5/2}$ this proves (4.1); before t_0 the oscillations vanish. The bounds (4.2) are displayed in the proof of [1, Lemma 5.10], from $\widetilde{\mathbf{d}} \cdot (W + v_\Lambda \circ X) = 0$, the Piola identity and the zero-mean phase primitive; they do not involve the center. □

Lemma 4.3. *In the setting of Lemma 4.2, let Y be the flow of U^+, and $y(t) = X(t,z)$, $y^+(t) = Y(t,z)$. Then $|y^+ - y| \le \lambda^{-1}$ on $[0,T]$, and on $[t_0,T]$*

$$\nabla U^+(t, y^+(t)) = \nabla U(t, y(t)) + \widetilde{\chi}(t)\,\mathbf{a}(t,z) \otimes \kappa(t,z) + E(t), \qquad |E(t)| \le \lambda^{-1/3}.$$

Proof. Let $A(t)$ be the label with $X(t, A(t)) = y^+(t)$, lifted continuously with $A(0) = z$, and $s(t) = k_* \cdot (A(t) - z)$, its phase. Differentiating $X(t,A) = Y(t,z)$ and subtracting $U(t, y^+)$ gives $(D_a X)A' = (U^+ - U)(t, X(t,A))$, that is, the system [1, (5.54)] along one trajectory, $A' = c(t,A,s)$, $s' = k_* \cdot c(t,A,s)$. By (4.2) and $T \le 1$, $|A - z|$ and $|s|$ are at most $\mathsf{H}\lambda^{-5/4}$; in particular $|A - z| \le \ell_*$, since $\ell_*^{-1} \le C\mathsf{D}^b \le \lambda^{1/12}$ by Section 2 and [1, (5.43)]. At the label A the leading term of $U^+ - U$ is $\widetilde{\chi} g(A)\mathbf{a}(t,A)F_h(s)$, with $|F_h(s)| \le |s|$, as $F_h(0) = 0$ and $|F_h'| \le 1$, and $|\mathbf{a}(t,A)| \le 2\lambda^{-3/4}$ by [1, (5.19)]; so it is at most $\mathsf{H}\lambda^{-2}$, and $|r| \le \mathsf{H}\lambda^{-23/12}$ by (4.1). Hence $|A'| \le \mathsf{D}\mathsf{H}\lambda^{-23/12}$, and

$$|A - z| \le \mathsf{H}\lambda^{-23/12}, \qquad |y^+ - y| = |X(t,A) - X(t,z)| \le \mathsf{D}|A - z| \le \mathsf{H}\lambda^{-23/12} \le \lambda^{-1}.$$

For $t \geq t_0$, put $p = g\mathbf{a}$. By [1, (3.1)],

$$\nabla U^+(t, y^+) = \nabla U(t, y^+) + \widetilde{\chi}[F_h(s)\,\mathbf{d}p + F_h'(s)\,p \otimes \kappa](t, A) + \nabla r(t, y^+).$$

On the envelope, [1, (5.20)] for the seed, as in the proof of [1, (5.22)], gives $|\nabla_a p| \leq \mathsf{H}\lambda^{-3/4}$, so $|\mathbf{d}p| \leq \mathsf{H}\lambda^{-3/4}$; [1, Proposition 5.1 and (5.3)] give $|\nabla_a \kappa| \leq \mathsf{H}\lambda$ and $|\kappa| \leq \mathsf{D}^2\lambda$; and $|p| \leq 2\lambda^{-3/4}$. So $p \otimes \kappa$ has label derivative at most $\mathsf{H}\lambda^{1/4}$, and $p(z) = \mathbf{a}(t, z)$. By (2.1), $|F_h'(s) - 1| \leq C_2|s|/h$, and $\|\nabla^2 U\|_\infty \leq C\mathsf{D}^2$ by [1, (B.1)]. Hence

$$|E| \leq \mathsf{H}(\lambda^{-3/4}|s| + \lambda^{1/4}|A - z| + \lambda^{1/4}|s|/h + |y^+ - y| + \lambda^{-5/12}),$$

and the bounds above give the exponents -2, $-5/3$, $-1/2$, $-23/12$ and $-5/12$. Thus $|E| \leq \mathsf{H}\lambda^{-5/12} \leq \lambda^{-1/3}$. □

4.2. **The induction along the particle.** Let U_0, z and m be as in Proposition 2.1, without oddness. We run the proof of Section 2, with wave j centered as in Lemma 4.1 at the label z of the flow X_{j-1} of U_{j-1}. Write $y_i(t) = X_i(t, z)$ and $M_i(t) = \nabla U_i(t, y_i(t))$, and let $(\mathbf{a}_j, \kappa_j)$ be the seed and wave vector of wave j at the label z, with couplings [1, (7.1)] taken for M_{j-1}. For $j \geq 1$ and $\mu = \lambda_{j-1}$, the iteration gives fields U_j on $[0, S_j] \times \mathbb{T}^3$ with the following properties, proved in Section 2 for odd U_0 and $z = 0$, where $y_j \equiv 0$ and (T4) gives (T4′).

(I1) (T1), (T2) and (T3′), without the oddness in (T1) (Section 2, index zero and the other items);

(I2) [1, (7.3)] at t_{j+1}, with $\nu_*/2 \leq \nu_j \leq 2\nu_*$ (Section 2, items (2) and (3));

(I3) [1, (7.4), (7.6)–(7.8)], and $0 < T_\infty - t_j \leq \frac{4}{3}\tau_j$, so $T_\infty < S_j$ (Section 2, the other items; [1, Section 7.1] for (7.6));

(I4) (T4), with its first sentence replaced by (T4′) (Lemma 4.3, applied below), namely $M_j = M_{j-1} + \widetilde{\chi}_j \mathbf{a}_j \otimes \kappa_j + E_j$ with $|E_j| \leq \lambda_j^{-1/4}$ on $[t_j, S_j]$, and $|y_j - y_{j-1}| \leq \lambda_j^{-1}$ on $[0, S_j]$;

(I5) O_j vanishes on $[0, t_j]$ for $j \geq 2$ and lies in the ball of radius μ^{-1} about $y_{j-1}(t)$ ([1, Section 7.2, item (5)]; the ball below); for a group, this holds with O_j the packet at z;

(I6) the U_j converge uniformly on $[0, T_\infty] \times \mathbb{T}^3$ (proof of Proposition 2.1 below; for odd U_0, by summing the bounds [1, (7.4)] of (I3)).

The proof in Section 2 applies, with Section 5 of [1] replaced by Lemma 4.1 and the following changes.

The first wave. Now $B(t) = M_0(t)$, and B' is the derivative of ∇U_0 along the particle y_0. As $R_0 = \partial_t U_0 + (U_0 \cdot \nabla)U_0 + \nabla \pi_0$, $B' = (-(\nabla U_0)^2 - D^2\pi_0 + \nabla R_0)(t, y_0(t))$. By hypothesis (d), the eigenvalues of $D^2\pi_0$ are at most $\frac{3}{4}$, and their sum is $-\operatorname{tr}((\nabla U_0)^2) \geq -3m^{-2}$, so each is at least $-3m^{-2} - \frac{3}{2}$; with $\|\nabla R_0\|_\infty \leq CN^5\lambda_1^{-3} \leq 1$ this gives $|B'| \leq 4m^{-2} + \frac{5}{2}$. The claim in Section 2 then holds with $(6m^{-2} + \frac{5}{2})S_1 \leq m/(2\sqrt{2})$ in place of $(2m^{-2} + m^{-1})S_1 \leq m/(2\sqrt{2})$, since its proof uses only $|B| \leq m^{-1}$, the bound on $|B'|$, $\operatorname{tr} B = 0$, and that $(\kappa, \mathbf{a})(t, z)$ solves [1, (2.1)] over B.

Choosing wave $j \geq 2$. By (T4′) at $j - 1$, $M_{j-1} = B + \mathbf{a}_{j-1} \otimes \kappa_{j-1} + E_{j-1}$ on $[t_j, S_j]$, with $B = M_{j-2}$; this is the splitting [1, (6.5)] along the particle. The preceding pair solves [1, (2.1)] over B, because it is built at the label z of X_{j-2}, and the new pair solves it over M_{j-1}, because it is built at the label z of X_{j-1}; [1, Section 2] uses nothing else about these matrices. The new wave has $k_* = \lambda_j (D_a X_{j-1}(t_j, z))^{\mathsf{T}} E_3(t_j)$, as in [1, Section 6.2]. For $j \geq 3$, $B' = (\partial_t \nabla U_{j-2} + (U_{j-2} \cdot \nabla)\nabla U_{j-2})(t, y_{j-2})$, and [1, (7.6), (T2), (B.1)] give

$\|B'\|_\infty \le \lambda_{j-2}^{18} + C\lambda_{j-2}^{12} \le \mu^{1/16}$; for $j = 2$, $\|B'\|_\infty \le 4m^{-2} + \frac{5}{2}$. Items (2) and (3) then hold as in Section 2.

Supports and the central gradient. In item (5) of [1, Section 7.2], O_j lies in $X_{j-1}(t, \operatorname{supp} g)$, and [1, (5.1)] with $\ell_* \le \mathsf{D}^{-3}$ places it in the ball of radius $2\mathsf{D}\ell_* \le \mu^{-1}$ about $y_{j-1}(t)$. For the central gradient, item (5) shows that $U_j = U_{j-1} + W^\natural + v_\Lambda$ with v_Λ as in Lemma 4.2, and Lemma 4.3 gives (T4′), with $|E_j| \le \lambda_j^{-1/3}$. This closes the induction, and $T_\infty \le \frac{4}{3}\lambda_1^{-1/8}(\log\lambda_1)^6$ as before.

Proof of Proposition 2.1. Lemma 4.1 and [1, Proposition 8.1], whose proof does not use parity, give $u_0 = \lim U_j(0)$, smooth and divergence free, with maximal solution u, $T_* \ge T_\infty$, and $U_j \to u$ in $C([0,T']; C^{n,1/2})$ for $T' < T_\infty$. By [1, (7.4)], the velocities converge uniformly on $[0,T_\infty]\times\mathbb{T}^3$ to a continuous w with $\|w\|_\infty \le m^{-1} + 2\sum_j \lambda_j^{-1} \le m^{-1} + 1$.

The particle. By (T4′), $\sup_t |y_i - y_j| \le \sum_{k>j}\lambda_k^{-1} \le 2\lambda_{j+1}^{-1}$ for $i > j$, so the y_j converge uniformly on $[0,T_\infty]$ to a path y. Passing to the limit in $y_j(t) = z + \int_0^t U_j(s, y_j(s))\,ds$, using the uniform convergence of U_j and the uniform continuity of w, gives $y(t) = z + \int_0^t w(s, y(s))\,ds$. Before T_∞, y is the particle of u from z, and $y_* = y(T_\infty)$ is its limit, with $|y_* - z| \le (m^{-1}+1)T_\infty$.

The vorticity on the particle. On $[t_j, t_{j+1}]$, (T4′) at steps $j-1$ and j gives $M_j = M_{j-2} + \mathbf{a}_{j-1}\otimes\kappa_{j-1} + \widetilde{\chi}_j\mathbf{a}_j\otimes\kappa_j + E_{j-1} + E_j$. This is the splitting at the origin in the proof of [1, Proposition 8.2], along the particle. The preceding pair solves [1, (2.1)] over M_{j-2}, so [1, (2.2)] gives $|\partial_t \log\sigma_{j-1}| \le |M_{j-2}|$, and that proof gives $|\operatorname{curl} U_j(t, y_j(t))| \ge c(m)\lambda_{j-1}^{1/8} - C\lambda_{j-2}^9 - C$ on $[t_j, t_{j+1}]$, with $C\lambda_{j-2}^9$ replaced by Cm^{-1} for $j = 2$. By [1, (B.1)], $\|\nabla^2 U_j\|_\infty \le C\Lambda_j^2\lambda_j = C\lambda_j^{11}$, so moving the evaluation from $y_j(t)$ to $y(t)$ costs at most $C\lambda_j^{11}\lambda_{j+1}^{-1}$. For $l > j$ the oscillation O_l vanishes on $[t_j, t_{j+1}]$, so the remaining increments change the curl by at most $C\sum_{l>j}\lambda_l^{-1}$, as in the proof of [1, Proposition 8.2]. Hence $|\operatorname{curl} u(t, y(t))| \to \infty$ as $t \uparrow T_\infty$. As in [1, Corollary 8.3], this gives $T_* = T_\infty$, and (1.2) by the periodic Beale–Kato–Majda criterion [4] in the formulation of [12, Theorem 1] used there; for data with nonzero mean the criterion is applied to $u(t, x + \bar{u}t) - \bar{u}$, with $\bar{u}$ the mean of u_0.

Away from the endpoint. Let $K \subset \mathbb{T}^3 \setminus \{y_*\}$ be compact, at distance $\delta > 0$ from y_*. When O_j is active, $t \ge t_j$, and its support lies within $\lambda_{j-1}^{-1} + 2\lambda_j^{-1} + (m^{-1}+1)(T_\infty - t_j)$ of y_*, which is less than $\delta/2$ for large j. So for large j, O_j vanishes on the $\delta/2$-neighborhood of K throughout $[0,T_\infty]$, and the proof of [1, Proposition 8.4], with a cutoff equal to one near K and zero near y_*, gives the convergence of $u(t)$ and $\partial_t u(t)$ in $C^n(K)$.

The data. The paragraph on the data in Section 2 applies, with the phase $\lambda_1 E_1 \cdot (x - z)$ in (2.8), and gives (2.6). □

Proof of Theorem 1.3. We follow the proof of Theorem 1.4(a) without oddness, at the point z. If $\nabla v(z) = 0$, replace v by $v + s\sin(2\pi(x_2 - z_2))e_1$, whose gradient at z is $2\pi s\, e_1\otimes e_2$. The base $U_0 = S_{N^5} f$ and the bounds of hypotheses (a)–(d) are as there, the vector e is chosen from $\nabla f(0,z) = \theta\nabla v(z)$, and (e) holds at $(0,z)$ for large N. Proposition 2.1 and Lemma 3.2 give u_0^N blowing up along the particle from z at a time $T_*^N \le \frac{4}{3}\lambda_1^{-1/8}(\log\lambda_1)^6$ with endpoint at distance at most $(m^{-1}+1)T_*^N$ from z, and the bounds on $u_0^N - \theta v$ are those of the proof of Theorem 1.4(a). The scaling step gives $u_0 = \theta^{-1}u_0^N$, with the same endpoint and blow-up time θT_*^N. Taking N large, and s small first if $\nabla v(z) = 0$, proves Theorem 1.3. □

5. Symmetry classes

Let G be a finite group of isometries $gx = Q_g x + d_g$ of $\mathbb{T}^3$. Each Q_g is orthogonal and preserves $\mathbb{Z}^3$, so it is a signed permutation matrix; put $e_g = \det Q_g$. If u solves (1.1), then so does $g_* u$, with the pressure composed with g^{-1}, so by uniqueness the solution from G-invariant data is G-invariant. For a G-invariant field U with flow X, uniqueness of trajectories gives $X(t, ga) = gX(t, a)$, so that

$$D_a X(t, ga) = Q_g D_a X(t, a) Q_g^{\mathsf{T}}, \qquad M(t, ga) = Q_g M(t, a) Q_g^{\mathsf{T}},$$

and $\operatorname{curl} U(t, gx) = e_g Q_g \operatorname{curl} U(t, x)$. The cutoff S_Λ commutes with g_*, since Q_g preserves $|\xi|$ and d_g multiplies each Fourier mode by a phase.

Let z have trivial stabilizer, and fix $0 < r_0 < 1/16$ such that the balls $B(gz, 8r_0)$, $g \in G$, are disjoint.

Lemma 5.1. *Let U, D, λ be as in Lemma 4.1, with U G-invariant and $2\ell_* < r_0/2$, and take the wave of Lemma 4.1 at z. At gz take the wave with $k_{*,g} = e_g Q_g k_*$, phase $k_{*,g} \cdot (a - gz)$, envelope $\chi(|a - gz|/\ell_*)$, seed $e_g Q_g \mathbf{a}(t, a)$ at the label ga, and the same activation time and cutoff. Run the construction of* [1, Section 5] *for the sum of these $|G|$ waves, with one phase-independent correction at each level, and put $\mathcal{N}_g = B(gz, 3\ell_*)$. Then the resulting field, its oscillation and each phase-independent correction are G-invariant, so that the packet at gz is g_* of the packet at z, and the oscillation is supported in the union of the closed balls $\overline{B}(gz, 2\ell_*)$ in the labels. The statements of Lemma 4.1 on the phase, the support and the gradient at the center hold for each wave in its own chart $\mathcal{N}_g$, and the identities for the divergence and the residual and the bounds on the velocity, pressure, expansion residual* [1, (5.48)]*, truncation and deformation hold for the whole field, with constants multiplied by a factor depending only on $|G|$, the order, the level and the depth of the expansion, with $\mathcal{B}$, and so N_0, chosen for these constants; they depend on G. Provided the Fourier tail v_Λ* [1, (5.55)] *satisfies $\|v_\Lambda\|_{C_t C^2} \le \lambda^{-5/2}$, Lemmas 4.2 and 4.3 hold at the label z for the field with the oscillation at z only, the common means and v_Λ; this field coincides with the child field in a neighborhood of the path $Y(t, z)$.*

Proof. The waves. With $\kappa = (D_a X)^{-\mathsf{T}} k_*$ and $\kappa_g = (D_a X)^{-\mathsf{T}} k_{*,g}$, the identities above give $\kappa_g(t, ga) = e_g Q_g \kappa(t, a)$, and $e_g Q_g \mathbf{a}(t, a)$ solves the seed equation at ga, since $P_{\kappa_g}(t, ga) = Q_g P_\kappa(t, a) Q_g^{\mathsf{T}}$. The phases satisfy $k_{*,g} \cdot (ga - gz) = e_g k_* \cdot (a - z)$, and F_h is odd, so the leading wave at gz is g_* of the one at z. The frame vectors at gz are $e_g Q_g$ times those at z, and $e_g Q_g$ is a rotation, so the couplings, strength and γ are the same at every point of the orbit.

Charts and levels. The charts $\mathcal{N}_g$ lie in the disjoint balls $B(gz, 8r_0)$, since $3\ell_* < r_0$, and in $\mathcal{N}_g$ a profile $f(t, a, s)$ is evaluated at the linear phase $s = k_{*,g} \cdot (a - gz)$, so the calculus of [1, Sections 3 and 5] applies there as in Lemma 4.1, with $k_{*,g}$, κ_g and $\widetilde{\mathbf{d}}_g = \mathbf{d} + \kappa_g \partial_s$. Each field of the construction is one phase-independent field on $\mathbb{T}^3$ plus, for each g, a field with zero phase mean supported in $\overline{B}(gz, 2\ell_*) \subset \mathcal{N}_g$ [1, Proposition 5.4]; oscillatory fields in different balls have zero product, also after derivatives, and phase averages $\langle \cdot \rangle$ are taken ball by ball. With $Y_{p,g} = V_{p,g} + C_{p,g}$ the oscillatory terms of level p at gz and B_p the common mean [1, Section 5.5], the source H_p of [1, (5.25), (5.26)] is formed in each ball from the terms there and the means, and outside the balls it is the sum of the products of means. One endpoint problem [1, (5.11)] gives B_p, with

$$-\bar{f}_p = \mathbf{1}_{p=2} R_U \circ X + \sum_{i+j=p} (B_i \cdot \mathbf{d}) B_j + \sum_{g \in G} \sum_{i+j=p} \langle (Y_{i,g} \cdot \mathbf{d}) Y_{j,g} + (\mathbf{d} \cdot V_{i,g}) Y_{j,g} \rangle,$$

since products with one oscillatory factor have zero average and $\langle (C_{i+1,g}\cdot\kappa_g)\partial_s Y_{j,g}\rangle = \langle (\mathbf{d}\cdot V_{i,g})Y_{j,g}\rangle$ [1, (5.42)]; each product of means and the parent's residual enter once. Then $V_{p,g}$ solves [1, (5.15)] with κ_g from zero data, forced by $-(H_p-\langle H_p\rangle)$ in the ball at gz [1, (5.27)] and by $-\sum_{i=2}^{p}(B_i\cdot\kappa_g)\partial_s Y_{p+1-i,g}$ [1, (5.28)], and $C_{p+1,g}=C[V_{p,g}]$; the cancellations of [1, Proposition 5.4] hold in each ball and, for the means, outside them.

Bounds. With the maximum over the balls as the norm of oscillatory terms, the induction in the proof of [1, Theorem 5.6] runs as there, with the same powers of D and λ, since each forcing is a sum of at most $|G|$ packet terms of one-packet size and of the products of means. As the oscillatory supports are disjoint, [1, (5.44)], which holds in each ball with $k_{*,g}$, $|k_{*,g}|=|k_*|$, holds for the sum, and [1, (5.45)] with a factor $|G|$. In [1, Lemma 5.9] the leading pressures have disjoint supports, and the mean pressure is estimated once.

Deformation and the particle. By these bounds, (4.2) for c_{osc} and $\bar c$ holds for the sum, so the child flow moves labels by at most $\mathsf{H}\lambda^{-5/4}<r_0/4$, and trajectories from a ball of labels of radius $r_0/4$ stay in a tube meeting at most one $\mathcal{N}_g$. There the child velocity is that of the field with only the oscillation at gz, the means and the Fourier tail, so the two flows agree on that ball. This field is divergence free, since each B_p is by [1, (5.11)] and each $V_{p,g}+C_{p+1,g}$ is by [1, (3.6)], and the proof of [1, Lemma 5.10] for it, with the phase bound of (4.2) for $k_{*,g}$, gives [1, (5.53)].

Invariance. Under $a\mapsto ga$, $s\mapsto e_g s$, vector profiles transform by Q_g and scalars by composition; so does $\kappa_g\partial_s$, since both factors acquire e_g, and the potential $-\partial_s^{-1}(\kappa\times V)/|\kappa|^2$ transforms by e_gQ_g, so that its curl transforms by Q_g. The endpoint problem and the transverse equations have unique solutions, so by induction on p each B_p is G-invariant and the corrections at gz are the transforms of those at z. As S_Λ commutes with g_*, the truncated field is G-invariant. □

Proof of Theorem 1.4(b). The base $U_0=S_{N^5}f$ of the proof of Theorem 1.3 is G-invariant, and no preliminary shear is needed. With $\mathcal{B}$, and so N_0, as fixed in Section 2 for this G, take N so large that $2\ell_*<r_0/2$ and $(m^{-1}+1)S_1<r_0/2$, and run the induction of Section 4 with Lemma 5.1 in place of Lemma 4.1; the iteration supplies the hypothesis of Lemma 5.1 on the Fourier tail by [1, Section 7.2, item (5)]. Each U_j is G-invariant, (I1)–(I6) hold at z, with O_j in (I5) the packet at z, and the oscillation of step j at gz is supported within λ_{j-1}^{-1} of $gy_{j-1}(t)$. The proof of Proposition 2.1 in Section 4 gives the particle y from z, with $|\operatorname{curl}u(t,y(t))|\to\infty$; the particle from gz is gy, and $|\operatorname{curl}u(t,gy(t))|=|\operatorname{curl}u(t,y(t))|$. The endpoint gy_* lies within $(m^{-1}+1)T_\infty<r_0/2$ of gz, so the endpoints are distinct. For a compact K avoiding all of them, the argument away from the endpoint applies to each of the $|G|$ oscillations of step j. At time zero $u_0-U_0(0)$ is the sum of g_* applied to $V+V^{\mathrm c}$ of Lemma 3.2, plus H, so the bounds of Lemma 3.2 hold with a factor $|G|$. The scaling step preserves invariance, and the proof of Theorem 1.3 gives the bounds. □

We turn to the two classes. Write $s_i=\sin 2\pi x_i$, $c_i=\cos 2\pi x_i$ and $c_i'=\cos 6\pi x_i$. The Taylor–Green velocity [18] and a field with the symmetries of Kida's class [15] are

$$v_{\mathrm{TG}}=(s_1c_2c_3,\,-c_1s_2c_3,\,0),\qquad (v_{\mathrm{K}})_i=s_i(c_j'c_k-c_jc_k'),$$

for $(i,j,k)=(1,2,3),(2,3,1),(3,1,2)$. Both are divergence free. Let R_i change the sign of x_i, and let P_{12} and P_c be the transposition $(x_1,x_2,x_3)\mapsto(x_2,x_1,x_3)$ and the cyclic permutation $(x_1,x_2,x_3)\mapsto(x_2,x_3,x_1)$.

Definition 5.2. G_{TG} is the group of isometries of $\mathbb{T}^3$ generated by R_1, R_2, R_3, the translations by $(\frac12,\frac12,0)$ and $(0,\frac12,\frac12)$, and $x\mapsto P_{12}x+(\frac12,0,0)$; G_{K} is generated by these and

P_c. The Taylor–Green class and Kida's class are the smooth divergence-free fields invariant under G_{TG} and under G_{K}.

The generators of G_{TG} preserve v_{TG}, and those of G_{K} preserve v_{K}. The R_i preserve both fields, P_{12} changes the sign of both, P_c preserves v_{K}, and, as every frequency of every component is odd in each coordinate, a translation by $d \in \{0, \frac{1}{2}\}^3$ multiplies both by $(-1)^{2(d_1+d_2+d_3)}$. The maps $x \mapsto Qx + d$, $d \in \{0, \frac{1}{2}\}^3$, with Q a signed permutation matrix whose permutation has sign $(-1)^{2(d_1+d_2+d_3)}$, form a group containing the generators, which give all sign changes, the four translations with $2(d_1 + d_2 + d_3)$ even, and an element over each permutation fixing x_3, or, with P_c, over each permutation; so $|G_{\mathrm{TG}}| = 8 \cdot 2 \cdot 4 = 64$ and $|G_{\mathrm{K}}| = 8 \cdot 6 \cdot 4 = 192$.

6. The solutions at the blow-up time

Let u be as in Theorem 1.5. Its three statements are unchanged by the scaling $u_c(t, x) = cu(ct, x)$, which takes T to T/c and the particle $y(t)$ to $y(ct)$ (proof of Theorem 1.4(a)). We may therefore take for u the solution of [1], or the solution of Proposition 2.1 or of the proof of Theorem 1.4(b), built from fields U_j for a sufficiently large N. For the solution of [1], $z = 0$ and $y \equiv y_j \equiv 0$, and (T3′) and (T4′) below stand for (T3) and (T4) of [1, Theorem 7.1], which imply them, and (I5), (I6) are in [1, Theorem 7.1 and Section 8]; for the others, (I1)–(I6) of Section 4.2 hold; in the group case (I5) holds for each packet about its own point $gy_{j-1}(t)$. Positive constants c, C may depend on m and G, not on j.

By (I6), and by [1, Proposition 8.1] or, for the other solutions, its proof in Section 4, $U_j \to u$ uniformly on $[0, T] \times \mathbb{T}^3$ and in $C([0, T']; C^{n,1/2})$ for $T' < T = T_\infty$. Fix a large j and write $\lambda = \lambda_j$, $\mu = \lambda_{j-1}$, $\mathsf{D} = \mu^9$, $h = \lambda^{-1/2}$ and $\ell = \ell_*$; let X be the flow of U_{j-1}, so that $y_{j-1}(t) = X(t, z)$, and let $\mathbf{a}(t, a)$ and $\kappa(t, a) = (D_a X)^{-\mathsf{T}} k_*$ be the seed and wave vector of wave j at the label a. Its phase is $k_* \cdot (a - z)$. In the group case, O_j is the sum of the $|G|$ packets of Lemma 5.1, with disjoint supports, and V_1 denotes the sum of the leading terms of all of them. We use five facts.

(F1) By [1, (7.8)], $C_{n,P}\mathsf{D}^b \le \mu^{10b} = \lambda^{1/(32(q_j+1))}$ for $n \le 2$, and $q_j \to \infty$ [1, Lemma 1.3]; also $\mathsf{D} = \lambda^{9/K(q_j)}$. So every bound $C_{n,P}\mathsf{D}^C\lambda^\beta$ with $C \le b$, and every $\mathsf{D}^{-C}\lambda^\beta$, has the form $\lambda^{\beta+o(1)}$ as $j \to \infty$.

(F2) On $[0, S_j] \supset [0, T]$, $\|O_j\|_\infty \le \lambda^{-5/4+o(1)}$ and $\|\nabla O_j\|_\infty \le \lambda^{1/4+o(1)}$, by [1, (5.47)] and by [1, (5.44)] at order one with [1, (5.37)] and $\gamma \le 1$, for each packet. By (I5), O_j vanishes before t_j and lies in the ball of radius μ^{-1} about $y_{j-1}(t)$, or about the points $gy_{j-1}(t)$, and $\|U_j - U_{j-1} - O_j\|_{C^{1,1/2}} \le \lambda^{-1}$ [1, (7.4)]. Also $O_j = V_1^\natural + (W_{\mathrm{osc}} - V_1)^\natural$ with $|\nabla(W_{\mathrm{osc}} - V_1)^\natural| \le \lambda^{-1/3}$, by the proof of (4.1), where for the solution of [1] the factor H is $\lambda^{o(1)}$ by (F1); and $\widetilde{\chi}_j = 1$ on $[t_{j+1}, S_j] \ni T$. Summing (T4′), $|y_i - y_{j-1}| \le 2\lambda_j^{-1}$ on $[0, T]$ for $i \ge j$, and $|y - y_{j-1}| \le 2\lambda_j^{-1}$.

(F3) The proof of [1, Lemma 5.3] gives $|\mathbf{a}(t, a) - \mathbf{a}(t, z)| \le C_1|a - z||\mathbf{a}(t, z)|$ and $|\kappa(t, a) - \kappa(t, z)| \le C_1|a - z||\kappa(t, z)|$ for $|a - z| \le 2\ell$, with $C_1 = C\mathsf{D}^C$. Put $\rho = \min\{\ell, 1/(4C_1)\} = \lambda^{-o(1)}$. For $|a - z| \le \rho$, $|\mathbf{a}(t, a)| \ge \frac{3}{4}|\mathbf{a}(t, z)|$ and $|\kappa(t, a)| \ge \frac{3}{4}|\kappa(t, z)|$; and $\mathbf{a} \perp \kappa$ at every label.

(F4) By (2.1) and the proof of Lemma 3.1, $\int_0^1 |F_h'|^2 \le 2Ch$, and $F_h' \ge \frac{1}{2}$ on $\{s : \mathrm{dist}(s, \mathbb{Z}) \le c_1 h\}$, $c_1 = 1/(2C_2)$.

(F5) Let $k = |\kappa(T, z)|$ and $\sigma_T = |\mathbf{a}(T, z)|k$. Then $\sigma_T \ge c\lambda^{1/4} L_j L_{j+1}$ and $k \le C\lambda L_{j+1}^2$. Indeed, at t_{j+1}, [1, (2.23)] with $\max|\mathbf{a}(\cdot, z)| = \lambda^{-3/4}$ and $(\varepsilon_{j-1}\varepsilon_j)^{-1/2} = L_j L_{j+1}$ gives the

first bound, and [1, (2.10)] gives $|\kappa| \le \sqrt{3/2}\,\lambda/\cos\varphi$, where $\tan\varphi = \varepsilon_{j-1}\tau^2/2$ in normalized time τ [1, (2.4)], with $\cos\varphi \ge \varepsilon_j/3$ at a crossing by the proof of [1, Proposition 2.8]. From t_{j+1} to T, $\kappa_t = -M^{\mathsf T}\kappa$ and the seed equation at z give $|\partial_t \log|\kappa||, |\partial_t \log|\mathbf{a}|| \le |M_{j-1}| \le \mu^9$, while $T - t_{j+1} \le \frac{4}{3}\tau_{j+1} \le \frac{4}{3}\lambda^{-1/16} \le \frac{4}{3}\mu^{-40}$ by [1, (7.7)]; so both logarithms change by $o(1)$.

Proof of Theorem 1.5. Hölder continuity. By (F2) and interpolation, on $[0,T]$, $[O_j(t)]_\alpha \le 2\|O_j\|_\infty^{1-\alpha}\|\nabla O_j\|_\infty^\alpha \le \lambda_j^{3\alpha/2-5/4+o(1)}$, which is summable in j for $0 < \alpha < 5/6$, and the other increments are summable in $C^{1,1/2}$. Each U_j is smooth on $[0,T]$, so $U_j \to u$ in $C([0,T];C^\alpha)$.

H^1. By [1, (3.1)], $\nabla V_1^\natural = \widetilde{\chi}[F_h\,\mathbf{d}(g\mathbf{a}) + gF_h'\,\mathbf{a}\otimes\kappa]^\natural$ for each packet. The first term is at most $\lambda^{-5/4+o(1)}$, by $|F_h| \le Ch$ and [1, (5.22)]. For the second, change to labels, which preserves volume, and use $|\mathbf{a}| \le 2\lambda^{-3/4}$, $|\kappa| \le \mathsf{D}^2\lambda$ and Lemma 3.1 with $|k_*| \ge \mathsf{D}^{-1}\lambda$ in place of λ and $R = 2\ell$, where $|k_*|^{-1} \le \ell$:

$$\|gF_h'\,\mathbf{a}\otimes\kappa\|_{L^2}^2 \le 4\mathsf{D}^4\lambda^{1/2}\int_{|a-z|\le 2\ell} |F_h'(k_*\cdot(a-z))|^2\,da \le C\mathsf{D}^4\lambda^{1/2}h\,\ell^3 \le C\mathsf{D}^{-5},$$

since $\ell \le \mathsf{D}^{-3}$; in the group case the packets add a factor $|G|$. With (F2), $U_j \to u$ in $C([0,T];H^1)$, since $\sup_{t\le T}\|U_j - U_{j-1}\|_{H^1} \le C\lambda_{j-1}^{-45/2} + 3\lambda_j^{-1/3}$ is summable; hence $\operatorname{curl} u(t) \to \operatorname{curl} u(T)$ in L^2.

The exponent 5/6. Put $x^0 = y_{j-1}(T)$ and $x^1 = x^0 + c\,h\,\kappa(T,z)/k^2$ with $c = c_1/2$, so $|x^1 - x^0| = ch/k$. The phase $\vartheta(x) = k_*\cdot(X(T,\cdot)^{-1}(x) - z)$ has $\vartheta(x^0) = 0$, $\nabla\vartheta(x^0) = \kappa(T,z)$ and, by the proof of [1, Proposition 5.1], $|D^2\vartheta| \le \lambda\mathsf{D}^C$; with $k \ge \mathsf{D}^{-2}\lambda$ this gives $\vartheta(x^1) = ch(1 + O(\mathsf{D}^C\lambda^{-3/2}))$, so $F_h(\vartheta(x^1)) \ge ch/4$ by (F4). The label a^1 of x^1 has $|a^1 - z| \le \mathsf{D}|x^1 - x^0| < \rho$, so $g(a^1) = 1$ and $|\mathbf{a}(T,a^1)| \ge \frac{3}{4}|\mathbf{a}(T,z)|$ by (F3). As $V_1^\natural(T,x^0) = 0$, and in the group case the other packets vanish near x^0 and x^1, $|V_1^\natural(T,x^1) - V_1^\natural(T,x^0)| \ge \frac{3}{16}c\,h\,|\mathbf{a}(T,z)|$. The other parts of $u(T)$ change between x^0 and x^1 by at most $(\mu^9+2)|x^1 - x^0|$, from U_{j-1}, $(W_{\mathrm{osc}} - V_1)^\natural$ and $U_j - U_{j-1} - O_j$, plus $8\lambda_{j+1}^{-1}$ from the later increments. Relative to $h|\mathbf{a}(T,z)|$ these are $C(\mu^9+2)/\sigma_T$ and $C\lambda_{j+1}^{-1}\lambda^{5/4}L_{j+1}$, by (F5), and both tend to zero. Hence, for large j,

$$\frac{|u(T,x^1) - u(T,x^0)|}{|x^1 - x^0|^{5/6}} \ge c'h^{1/6}k^{5/6}|\mathbf{a}(T,z)| = c'\sigma_T h^{1/6}k^{-1/6} \ge c''L_jL_{j+1}^{2/3},$$

by (F5), and x^0, x^1 tend to y_*. So $u(T) \notin C^{5/6}$.

L^p, $p > 2$. Let Q be a cube of side $\rho/8$, with a face perpendicular to k_*, inside the shell $\rho/4 \le |a-z| \le \rho/2$, and $\mathcal{S} = \{a \in Q : \operatorname{dist}(k_*\cdot(a-z),\mathbb{Z}) \le c_1h\}$. As $|k_*|\rho \ge \mathsf{D}^{-1}\lambda\rho \to \infty$, $|\mathcal{S}| \ge c\rho^3h$. Its image $\mathcal{E} = X(T,\mathcal{S})$ has $|\mathcal{E}| = |\mathcal{S}|$ and lies at distance at least $\rho/(4\mathsf{D}) > 4\lambda^{-1}$ from x^0 and within $\mathsf{D}\rho$ of it. At time T, for each $i > j$, the packet of O_i at z is supported within λ_{i-1}^{-1} of $y_{i-1}(T)$, hence within $3\lambda^{-1}$ of x^0 by (F2); in the group case the other packets lie near the points $gy_{i-1}(T)$, far from $\mathcal{E}$. So the later oscillations vanish near $\mathcal{E}$, and $u(T)$ is C^1 there, with later increments bounded by $\sum_{i>j}\lambda_i^{-1}$ in C^1. On $\mathcal{E}$ the curl of $V_1^\natural$ is $F_h'\,\kappa\times\mathbf{a} + F_h\,\mathbf{d}\times(g\mathbf{a})$, with $g = 1$, $F_h' \ge \frac{1}{2}$ and $|\kappa\times\mathbf{a}| \ge \frac{9}{16}\sigma_T$ by (F3) and (F4). Everything else contributes at most $2\mu^9 + 3$, so $|\operatorname{curl} u(T)| \ge \sigma_T/4 \ge c\lambda^{1/4}$ on $\mathcal{E}$, and

$$\int_{\mathbb{T}^3} |\operatorname{curl} u(T)|^p \ge c^p\lambda^{p/4}|\mathcal{E}| \ge c_p\lambda^{p/4-1/2}\rho^3 = c_p\lambda^{p/4-1/2-o(1)} \to \infty.$$

The lower bound on the particle. Let $\omega(t) = \operatorname{curl} u(t,y(t))$ and $\Omega(t) = |\omega(t)|$ for $t < T$. By (T3′), $D^2\pi_j \le (\frac{3}{4} + 2/N)\,\mathrm{Id} \le \mathrm{Id}$ on $[0,S_j]\times\mathbb{T}^3$; for $t < T$, $\operatorname{tr}((\nabla U_j)^2) \to \operatorname{tr}((\nabla u)^2)$

in $C^{0,1/2}$ and $D^2\Delta^{-1}$ is bounded on $C^{0,1/2}$, so the zero-mean pressure π of u satisfies $D^2\pi \le \mathrm{Id}$. With $\mathsf{A} = \nabla u(t, y(t))$ and $\mathsf{K} = D^2\pi(t, y(t))$, differentiating (1.1) along the particle gives $\mathsf{A}' = -\mathsf{A}^2 - \mathsf{K}$ and $\omega' = \mathsf{A}\omega$, hence Ohkitani's relation $\omega'' = -\mathsf{K}\omega$ [13, (2.8)]. By the lower bound at t_j below, $\omega \neq 0$, and $\omega' = \mathsf{A}\omega$ is linear, so ω never vanishes. Then $\Omega'' = (|\omega'|^2 - \Omega'^2 - \omega \cdot \mathsf{K}\omega)/\Omega \ge -\Omega$. As $T < S_1 < 1$, $\cos t \ge \cos 1 > 0$ on $[0, T]$, and $\Phi = \Omega' \cos t + \Omega \sin t$ satisfies $\Phi' = (\Omega'' + \Omega)\cos t \ge 0$ and $(\Omega/\cos t)' = \Phi/\cos^2 t$. If $\Phi \le 0$ throughout, Ω is bounded, which the lower bound at t_j excludes; so $\Phi > 0$ on some $[t^*, T)$, and $\Omega(t) \ge \cos(1)\,\Omega(s)$ for $t^* \le s \le t < T$.

Let $r_j = (2\sigma_{j-1}\nu_{j-1})^{1/2}(t_j)$ be the rate of wave j [1, Section 6.2], so that $r_j^2 \le C\sigma_{j-1}(t_j)$ by [1, (7.3)], and $d_j = t_{j+1} - t_j$. Moving the evaluation from $y_j(t)$ to $y(t)$ changes the curl by $o(1)$, as in the proof of Proposition 2.1 in Section 4. At t_j, $\widetilde{\chi}_j = 0$, so (T4′) at steps $j-1$ and j gives $\omega(t_j) = \kappa_{j-1} \times \mathbf{a}_{j-1} + O(\lambda_{j-2}^9 + 1)$; with $\sigma_{j-1}(t_j) \ge c\mu^{1/4}$, $\Omega(t_j) \ge \sigma_{j-1}(t_j)/2 \ge c\, r_j^2$, as in the proof of [1, (8.2)]. At the crossing t_{j+1}, the proof of [1, Proposition 2.8] gives $\tan\varphi \ge 1/(4\varepsilon_j)$, and $\tan\varphi = \varepsilon_{j-1}\tau^2/2$ with $\tau = r_j d_j$, so $r_j d_j \ge L_j L_{j+1}/\sqrt{2}$.

At the midpoint $s_j = t_j + d_j/2$ the normalized time is $\tau_c/2 > 1$, $\tau_c = r_j d_j$, so $\widetilde{\chi}_j = 1$. By [1, (2.10), (2.18)], σ_j lies between absolute multiples of $b_j \lambda a_{\mathrm{ref}}/\cos\varphi$, where $b_j > 0$ is the scale of the seed [1, Section 6.2], and by [1, (2.14)], with a_{ref} the reference amplitude there,

$$\frac{(a_{\mathrm{ref}}/\cos\varphi)(\tau_c/2)}{(a_{\mathrm{ref}}/\cos\varphi)(\tau_c)} = \Big(\frac{\cos\varphi(\tau_c)}{\cos\varphi(\tau_c/2)}\Big)^{1/2} \exp\Big(-\int_{\tau_c/2}^{\tau_c} (1+e)\cos\varphi\, d\tau\Big).$$

Since $\tan\varphi(\tau_c/2) = \tan\varphi(\tau_c)/4$, the first factor is at least $1/2$, and, as $0 \le e \le \varepsilon_{j-1}/8$ [1, (2.13)] and $\cos\varphi \le 1/\tan\varphi$, the integral is at most $\frac{9}{8}\int_{\tau_c/2}^{\tau_c} 2/(\varepsilon_{j-1}\tau^2)\, d\tau = \frac{9}{4}/(\varepsilon_{j-1}\tau_c) \le 1$, since $\varepsilon_{j-1}\tau_c \ge L_{j+1}/(\sqrt{2}L_j) \ge 4$ by [1, (7.9)]. So $\sigma_j(s_j) \ge c\,\sigma_j(t_{j+1})$. The preceding wave contributes at most $2\sigma_{j-1}(t_j) \le C\mu^{1/4}L_j^2$, as in the proof of [1, Proposition 8.2], and $\sigma_j(t_{j+1}) \ge c\lambda_j^{1/4}$, so $\Omega(s_j) \ge c\,\sigma_j(t_{j+1}) \ge c\, r_{j+1}^2$.

For large j, $t_j \ge t^*$. On $[t_j, s_j]$, $T - t \ge d_j/2$ and $\Omega \ge \cos(1)\Omega(t_j)$, so $(T-t)^2\Omega(t) \ge c\, d_j^2 r_j^2 \ge c(L_j L_{j+1})^2$. On $[s_j, t_{j+1}]$, $T - t \ge t_{j+2} - t_{j+1} = d_{j+1}$ and $\Omega \ge \cos(1)\Omega(s_j)$, so $(T-t)^2\Omega(t) \ge c\, d_{j+1}^2 r_{j+1}^2 \ge c(L_{j+1}L_{j+2})^2$. Both tend to infinity. □

Declarations

Use of AI. This paper and its results were produced by an AI research system that the author designed, built and directs, running on Anthropic's Claude Opus 5.5 in Claude Code and OpenAI's GPT-6.2 Astra in Codex. The author chooses the problems and prioritizes the work. Separate AI agents checked each result. The system also searched the literature and checked the citations against reliable sources. The author read the paper and takes full responsibility for all of the paper's contents.

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