AN EFFECTIVE ANDRÉ–OORT THEOREM FOR SHIMURA VARIETIES

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Abstract. We give effective bounds for the special subvarieties V that are maximal in a subvariety Z of a Shimura variety, where Z is geometrically irreducible and defined over a number field and the level is neat and of product form: for the degree of V, with no hypothesis; and for the discriminant of a number field L_V attached to V and the Klingler–Yafaev invariant α_V of the level, if for some $\delta > 0$ the Dedekind zeta function of L_V has no zero with real part greater than $1-\delta$. The bound for the degree depends only on the degree of Z, and the other two on the degree of Z, the degree of its field of definition and δ; all three are effectively computable, given representatives of the connected components of the adjoint Shimura variety. Cureton and Thorner prove that such a half-plane exists with $\delta = 1/32$ for every number field; with their theorem, all these bounds hold with no further hypothesis. For abelian type, an algorithm lists these subvarieties from the bounds, given a finite étale cover by a Shimura variety of Hodge type inside one of Mumford's moduli spaces and equations for the preimage of Z.

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1. Introduction

Let S be a Shimura variety and $Z \subset S$ a closed subvariety. A special subvariety $V \subset Z$ is *maximal in* Z if no special subvariety V' with $V \subsetneq V' \subset Z$ exists. The André–Oort conjecture [3, 63] says that Z contains only finitely many maximal special subvarieties. By the o-minimal strategy of Pila and Zannier [67], Tsimerman proved it for $\mathcal{A}_g$ [83], and Pila,

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Shankar and Tsimerman for every Shimura variety, by bounding the heights of special points [65, Thm. 1.2], to which Binyamini, Schmidt and Yafaev had reduced it [14, Thms. 1 and 2]. Earlier, Klingler and Yafaev [50] and Ullmo and Yafaev [85] proved it under the generalized Riemann hypothesis (GRH) for CM fields, along a route through Hecke correspondences and Galois orbits that goes back to Edixhoven [27], Edixhoven and Yafaev [28] and Yafaev [88].

Neither proof bounds the maximal special subvarieties of Z effectively in terms of Z. Given the theorem, Scanlon's automatic uniformity shows that their number and their degrees are bounded in terms of the degree of Z [75, Thms. 2.4 and 4.2], without a way to compute the bounds. In the o-minimal proofs the counting theorem of Pila and Wilkie [66] for sets definable in $\mathbb{R}_{\mathrm{an,exp}}$ is not effective in general [9, §1.4], and the lower bounds for Galois orbits of special points rest on the Brauer–Siegel theorem [14, Prop. 6], [83, Cor. 3.2], whose constants are ineffective [76] unless Siegel zeros are excluded [78]. In the proof under GRH the hypothesis is used to find small primes that split completely in the splitting field of a torus; the existence of such a prime is proved for the members of a Zariski-dense set of special subvarieties of Z, after the set is replaced by a subset with the same Zariski closure [50, Def. 3.0.4, Prop. 10.2.3], not for each special subvariety. The other half of the proof is equidistribution, through Ratner's theorem [17], [85, Thm. 3.8], which gives no bound. We give such bounds along the route through Hecke correspondences: for the degrees with no hypothesis, and for the discriminants and level invariants that Klingler and Yafaev attach to a special subvariety, with GRH replaced by a zero-free half-plane for one Dedekind zeta function.

1.1. **Results.** Let (G, X) be a Shimura datum and $K = \prod_p K_p \subset G(\mathbb{A}_f)$ a neat compact open subgroup, and fix a faithful representation $\rho\colon G \to \mathrm{GL}_{n_\rho}$ with $\rho(K) \subset \mathrm{GL}_{n_\rho}(\widehat{\mathbb{Z}})$. Degrees on $\mathrm{Sh}_K(G, X)$ are taken for the Baily–Borel bundle $\mathcal{L}_K$, the extension of the bundle of top-degree holomorphic forms to the Baily–Borel compactification (Section 2). A special subvariety V has a generic Mumford–Tate group H_V; let T_V be the connected center of its image in the adjoint group G^{ad}, let L_V be the splitting field of T_V and d_{L_V} its absolute discriminant, and let $\alpha_V \geq 1$ be the invariant of Klingler and Yafaev that measures how far the level is from maximal on T_V (Definition 2.2); in Theorems 1.1 and 1.3 it is taken on the adjoint Shimura variety. For $\delta \in (0, 1/2]$ and a number field L we write

$$\mathrm{QRH}(\delta; L)\colon \quad \zeta_L(s) \neq 0 \quad \text{for } \operatorname{Re} s > 1 - \delta,\ s \neq 1,$$

a quasi-Riemann hypothesis for ζ_L.

Theorem 1.1. *Let (G, X), K and ρ be as above, and let $\delta \in (0, 1/2]$. For all integers $D, d \geq 1$ there are numbers $c_{\deg}(D)$ and $c(\delta, D, d)$, effectively computable from (G, X, K, ρ), D, d, δ and fixed representatives of the connected components of the adjoint Shimura variety, with the following property. Let $F \supset E(G, X)$ be a number field with $[F : \mathbb{Q}] = d$, let $Z \subset \mathrm{Sh}_K(G, X)$ be a closed subvariety that is geometrically irreducible, defined over F and of degree at most D, and let V be a special subvariety maximal in Z. Then*

(i) $\deg V \leq c_{\deg}(D)$;
(ii) if $\mathrm{QRH}(\delta; L_V)$ *holds, then* $d_{L_V} \leq c(\delta, D, d)$ *and* $\alpha_V \leq c(\delta, D, d)$.

Remark. In Theorems 1.1 and 1.3, α_V is formed for the adjoint representation, with the constant B of Table 1 for it, also when G is adjoint (Theorem 9.12). The numbers $c_{\deg}(D)$ and $c(\delta, D, d)$ are defined, and Theorem 1.1 is proved, in Section 9, through Theorem 9.12 at $z = \dim \mathrm{Sh}_K(G, X)$.

By the André–Oort theorem [65] the set of special subvarieties maximal in Z is finite. Finiteness alone gives no procedure that finds its members; with bounds as in Theorem 1.1, Theorem 1.4 gives an algorithm that lists them in Mumford's moduli spaces, and Theorem 10.21(c) for abelian type, given a finite étale cover by a Shimura variety of Hodge type.

The only analytic hypothesis is the zero-free half-plane for the one Dedekind zeta function ζ_{L_V} of the subvariety in question. It is used only to produce primes that split completely in L_V, so no lower bound for class numbers is needed. Cureton and Thorner prove $\mathrm{QRH}(1/32; L)$ for every number field L [19, Theorem 1.1], in a preprint whose proof has been formalized in Lean [6].

Corollary 1.2 (with [19, Theorem 1.1])**.** *In the setting of Theorem 1.1,* $\deg V \le c_{\deg}(D)$, $d_{L_V} \le c(1/32, D, d)$ *and* $\alpha_V \le c(1/32, D, d)$ *for every special subvariety* V *maximal in* Z.

Hu's zero-free half-plane $\operatorname{Re} s > 7/8$ for Hecke L-functions [46, Theorem 1.2] gives $\mathrm{QRH}(1/8; L)$ for every L, and with it Corollary 1.2 holds with $c(1/8, D, d)$.

For $V = \{x\}$ a special point and G adjoint, T_V is the Mumford–Tate group T of x, so d_{L_V} is the d_L in the discriminant $\operatorname{disc}(x) = [K_T^m : K_T]\, d_L$ of Binyamini, Schmidt and Yafaev [14, Def. 1], the quantity in [65, Thm. 1.2]; for x maximal in Z and in the image of $X^+ \times \{1\}$, under $\mathrm{QRH}(\delta; L_V)$, Theorem 1.1(ii) bounds $\operatorname{disc}(x)$ by a constant times a power of $c(\delta, D, d)$, with the constant and the exponent depending only on (G, X, K, ρ) (the remark after Proposition 6.6).

For the Siegel modular varieties the constants simplify.

Theorem 1.3. *Let* $g \ge 1$, $N \ge 3$, *and let* $\mathcal{A}_{g,K(N)}$ *be the Shimura variety of* $(\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$ *at the principal congruence level* $K(N)$.

(i) Theorem 1.1 and Corollary 1.2 hold for $\mathcal{A}_{g,K(N)}$, *with* L_V *the splitting field of the connected center of* H_V *itself,* α_V *computed at the image of* $K(N)$ *in* $\mathrm{PGSp}_{2g}(\mathbb{A}_f)$, *and with constants given in Theorem 9.13(a). There the factor* h_K *of Theorem 9.12, the largest degree of the map from a connected component onto its image in the adjoint Shimura variety, is at most* $\varphi(N)/2$.

(ii) The same holds for $\mathcal{A}_{g,1} = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$, *whose level is not neat, with degrees defined through the cover* $\mathcal{A}_{g,K(3)}$ *(Lemma 9.14),* α_V *computed at the image of* $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$ *in* $\mathrm{PGSp}_{2g}(\mathbb{A}_f)$, *and constants given in Theorem 9.13(b).*

The dependence of the bounds on D, d and δ is given by formulas (through the largest solution Π_0 of an explicit inequality, Theorem 7.4); the constants of (G, X, K, ρ) in them are effectively computable (Table 1, Section 7.1, Corollary 9.10), given a fixed set of representatives of the connected components of the adjoint Shimura variety (Section 9); for $\mathcal{A}_g$ none is needed (Theorem 9.13). For adjoint G the constants of Theorem 8.1 are at most $\exp(O(d\log(d+1)))\, D^{O(1)}$, so for fixed d polynomial in D. In general the constant c of Theorem 1.1, through the recursion of Theorem 9.12, is exponential in a power of D (Remark 8.2). Call the smallest special subvariety containing Z its *special closure*. In the setting of Theorem 8.1, a special subvariety V maximal in Z that is not strongly special in the special closure of Z satisfies, under $\mathrm{QRH}(\delta; L_V)$ and for fixed d, $\alpha_V \le D^{2^z + o(1)}$ and $d_{L_V} \le D^{2^z/\mu + o(1)}$ as $D \to \infty$, where $z = \dim Z$ and μ is the exponent of d_{L_V} in the lower bound for the Galois orbit of V (Corollary 3.4 and Remark 8.2).

Turning these bounds into a list needs, in addition, an explicit model of the Shimura variety with a universal family of abelian varieties. No explicit model is known in general. Let

$g \geq 2$, let d_Υ be a positive multiple of 8 and $\Upsilon := (d_\Upsilon, \dots, d_\Upsilon)$, with g entries. Mumford's moduli space M_Υ of principally polarized abelian varieties with an Υ-marking is open in a closed subscheme of a projective space given by explicit equations over $\mathbb{Q}$, and $M_\Upsilon \otimes \mathbb{C}$ is a connected Shimura variety for GSp_{2g}, at a level K_Υ (Section 10.2). Let $\lambda\colon \mathrm{GSp}_{2g} \to \mathrm{PGSp}_{2g}$ be the map to the adjoint group.

Theorem 1.4. *Let $g \geq 2$, $8 \mid d_\Upsilon$, and $Z \subset M_\Upsilon \otimes \overline{\mathbb{Q}}$ closed, geometrically irreducible and defined over a number field. Given numbers that bound, for every special subvariety V maximal in Z, the degree of V for the Baily–Borel bundle $\mathcal{L}_{K_\Upsilon}$ at the level K_Υ of M_Υ, the discriminant d_{L_V}, and α_V at the level $\lambda(K_\Upsilon)$, and given a number in $(0, \bar{B}]$, where $\bar{B}$ is the constant B of Table 1 for PGSp_{2g} with which α_V is formed (Section 10.1), Algorithm 10.20 terminates, and its output is exactly the set of special subvarieties maximal in Z.*

Algorithm 10.20 is built on Urbanik's algorithm [87, Cor. 6.3] (Section 1.2). The bounds that Theorem 1.4 needs are given by Theorem 1.3(i) at the level $K(2d_\Upsilon)$, which lies in the level of M_Υ (Lemmas 10.2 and 10.3), and the positive number by the constant B of the adjoint datum itself, which enters the constants of Theorem 1.3(i) and is computable from g as they are (Section 10.9); so by Corollary 1.2 the listing needs no hypothesis. For a Shimura variety of abelian type at a neat level, every connected component receives a finite étale map from a connected Shimura variety of Hodge type that is a closed subvariety of some $M_\Upsilon \otimes \mathbb{C}$ (Proposition 10.19); given this map, the algorithm, run on each component of the preimage and followed by its last step on the union of the outputs, lists the special subvarieties maximal in a closed irreducible subvariety from equations for its preimage (Theorem 10.21(c)).

Most effective results concern products of modular curves: curves in $\mathbb{C}^2$ [52, 8, 31], subvarieties of $Y(1)^n$ [9, 10, 64] and linear relations among powers of singular moduli [32, 33]; beyond them, non-compact curves in Hilbert modular varieties [13] and fiber powers of the Legendre family [12]. Binyamini's bounds are effective up to one possible exceptional imaginary quadratic field in [9] and one constant from Siegel's theorem in [10, Thm. 7]; both exceptions disappear with the zero-free half-plane of [19]. For fixed n, the bounds of [10, Thm. 7] are polynomial in the degree and logarithmic height of Z and the degree of its field of definition; the height of Z does not enter the constants of Theorem 8.1.

Daw [21] bounded the degrees of strongly special subvarieties from below; Daw, Javanpeykar and Kühne extended this bound to all special subvarieties of positive dimension [22, Thm. 6.1] and bounded, with no arithmetic hypothesis, the degrees of maximal special subvarieties that are not of facteur type [22, Thm. 1.3], a class defined by Ullmo [84]. Binyamini and Daw bounded the degree of every weakly optimal subvariety of Z, on a connected Shimura variety $\Gamma\backslash X^+$ with Γ neat and $Z(G)(\mathbb{R})$ compact, hence of every special subvariety of positive dimension maximal in Z, by a polynomial in the degree of Z; the polynomial depends on equations for a Baily–Borel embedding and for the canonical connection [11, Thm. 5 and p. 2157].

1.2. **The method.** We follow the route of Edixhoven, Klingler, Ullmo and Yafaev. For a special subvariety $V \subset Z$ that is not strongly special, their Hecke step says that Z contains a larger special subvariety once a prime l as in [50, Thm. 9.2.1, (9.1)] exists. For a suitable prime l that splits completely in L_V and a Hecke correspondence T_m at l, the Galois orbit of V lies in $Z \cap T_m Z$; if $Z \not\subset T_m Z$, an upper bound for the degree of this intersection contradicts a lower bound for the Galois orbit, and if $Z \subset T_m Z$, then Z contains a larger special subvariety [50, Thm. 7.2.1]. Two inputs make this work: a lower bound for the Galois

orbit of V of the form $C(N)\alpha_V(\log d_{L_V})^N$ [85, Thm. 2.19], and the existence of a split prime l smaller than a fixed power of $\alpha_V \log d_{L_V}$ [50, Prop. 10.2.3]. GRH enters only here, both times to find primes that split completely in L_V [50, Prop. 10.1.2], [85, Prop. 2.13]. We make each step explicit and replace GRH by $\mathrm{QRH}(\delta; L_V)$.

Split primes. Under $\mathrm{QRH}(\delta; L)$ an explicit formula with a smoothing that removes the constants at $s = 0$ and $s = -1$ gives primes that split completely in L in every interval $(x/4, x]$ with x a fixed power of $\log d_L$, the power being $1/\delta$ instead of the $2 + o(1)$ of GRH (Lemmas 3.1 and 3.2). Frei, Sofos and Widmer obtain, under the same hypothesis, primes of every Frobenius class in $(x/2, x]$ for $x \geq c(\delta)(\log d_L)^{1/\delta}(\log\log d_L)^{2/\delta}$, with a constant $c(\delta)$ that is not made explicit [34, Cor. 1.2]; we need only the completely split case, but with explicit constants. For the Galois bound we rerun the argument of Ullmo and Yafaev for the torus T_V [86, §6], with the split primes taken up to a fixed power of the discriminant, as Tsimerman does under GRH for CM points of $\mathcal{A}_g$ [82, §7.3, Thm. 7.6]; so any $\delta > 0$ suffices and the exponent μ in the bound $\gg \alpha_V d_{L_V}^{\mu}$ does not depend on δ (Proposition 3.3 and Corollary 3.4). Klingler and Yafaev use this bound in the weaker form $\gg \alpha_V(\log d_{L_V})^N$; we use the power of d_{L_V} itself. Then the prime l of the Hecke step, of size a power of $\log d_{L_V}$ and $\log \alpha_V$, is set against an orbit of size $\alpha_V d_{L_V}^{\mu}$, so any $\delta > 0$ works there too, and for adjoint G and fixed $[F : \mathbb{Q}]$ the bounds for α_V and d_{L_V} are at most a power of the degree of Z (Lemma 6.8 and Remark 8.2).

Uniformity in the level. The Hecke step works at an Iwahori level I_l that varies with the prime l. Every constant must therefore be uniform in the level, including the constant in Bézout's inequality. We show that the complete linear system of $\mathcal{L}_J^{\otimes e}$ is base-point free and separates points for every $e \geq e^*(n)$, at every neat level J of every Shimura datum of dimension at most n (Theorem 1.5). So Bézout's inequality holds at every neat level with the constant $e^*(n)^{2n}$ (Lemma 5.2), and the Hecke step runs as in [50], with degrees taken for the Baily–Borel bundle at each level and the exponent 2^r, r the codimension of V in Z.

Theorem 1.5. *Let (G, X) be a Shimura datum, K neat, and $n := \dim \mathrm{Sh}_K(G, X)$. Put $e^*(n) := \lfloor \sum_{k=1}^{n} k\, 2^{1/k} \rfloor + 2$. For every integer $e \geq e^*(n)$ the complete linear system $|\mathcal{L}_K^{\otimes e}|$ on the Baily–Borel compactification is base-point free and separates points. Moreover $e^*(n) \leq \lfloor n(n+3)/2 \rfloor + 2$.*

The proof applies Fujino's Angehrn–Siu theorem for log canonical pairs [35] to the Baily–Borel compactification, which is log canonical with log canonical bundle $\mathcal{L}_K$ by Mumford's description of the canonical extension [62, Prop. 3.4]. For ball quotients, G. Di Cerbo and L. F. Di Cerbo show that on a smooth toroidal compactification $\widetilde{Y}$ of dimension $n \geq 2$ with boundary $\partial\widetilde{Y}$ the divisor $m(K_{\widetilde{Y}} + \partial\widetilde{Y})$ is base-point free for $m \geq \binom{n}{2} + 1$ and separates the points of $\widetilde{Y} \smallsetminus \partial\widetilde{Y}$ for $m \geq \binom{n}{2} + 2$ [26, Cor. 3.3 and 3.4].

Degrees. Daw, Javanpeykar and Kühne bound the degree of a maximal special subvariety that is not of facteur type by an iteration of Hecke correspondences [22, §10], which builds on the one Daw ran for strongly special subvarieties [21, Thm. 9.3] after [50, Thm. 9.2.1]. For adjoint G, the iteration of [22], with the uniform Bézout step and a fiber-saturation lemma, also bounds every positive-dimensional one by an effectively computable function of D that is $D^{2^{\dim Z}+o(1)}$ as $D \to \infty$ and whose constants depend only on (G, X, K, ρ). The bound of [11, Thm. 5] holds for every weakly optimal subvariety of Z and is polynomial in D of unspecified degree; Theorem 1.6 treats only the positive-dimensional special subvarieties maximal in Z, for adjoint G and a neat product level, and there gives $D^{2^{\dim Z}+o(1)}$. In Theorem 1.6, d_G is the factor by which degrees can grow from a Shimura subvariety to $\mathrm{Sh}_K(G, X)$, p^{f_2} and p^{k_2}

bound the index of the level and the degree of the Hecke correspondence used at p, K_p is hyperspecial for $p \geq N_K$, and Σ_V is the finite set of primes at which the derived group of H_V is not quasi-split, does not split over an unramified extension, or has a level that is not hyperspecial (Section 7.1).

Theorem 1.6. *Let (G, X) be a Shimura datum with G adjoint and K a neat product level, let $Z \subset \mathrm{Sh}_K(G, X)_{\mathbb{C}}$ be closed and irreducible, of degree D, and let V be a special subvariety maximal in Z with $\dim V \geq 1$. Put $r := \dim Z - \dim V$ and $n := \dim \mathrm{Sh}_K(G, X)$. Then, for every prime $p \geq N_K$ outside the finite set Σ_V of Section 7.1,*

$$\deg V \;\leq\; d_G\big(d_G\, c_B\, p^{k_2+2f_2}\big)^{2^r-1} D^{2^r}, \qquad c_B = e^*(n)^{2n},$$

where d_G, k_2, f_2 and N_K, which depend only on (G, X, K, ρ), are the maxima over the connected components of the constants of Section 7.1 at the levels $g_i K g_i^{-1}$ with the lattices $\rho(g_i)\widehat{\mathbb{Z}}^{n_\rho}$, and Σ_V is that of the translate of V (Section 2.2). Consequently $\deg V$ is at most the maximum over the connected components of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ of the constants $c^{\mathrm{ad}}_{\deg}(\dim Z, D)$ of Theorem 7.4 (Section 2.2), a function of (G, X, K, ρ), $\dim Z$ and D alone, effectively computable given the representatives g_i of Section 2.2, and at most $D^{2^{\dim Z}+o(1)}$ as $D \to \infty$. No arithmetic hypothesis is used.

Assembly. Maximal special subvarieties fall into three classes: strongly special ones, which Theorem 1.6 bounds; those that are not strongly special in the special closure of Z, which the Hecke step bounds; and those whose torus is the center of the generic Mumford–Tate group of Z, which are strongly special in the special closure but not in S. The last class needs a separate argument (Lemmas 8.5 and 8.8). For a general Shimura datum, the map to the adjoint Shimura variety need not be injective on connected components, and maximality can be lost under it; a recursion on dimension at the fixed level handles this (Section 9).

Listing. For $\mathcal{A}_g$ in Mumford's model, the bounds determine finitely many PEL data, each with an explicit locus of abelian varieties with the given endomorphisms (Lemma 10.1). Every special subvariety V is rigid in the normalization of one of these loci (Lemma 10.10), so it lies in the finite set of weakly special subvarieties of bounded degree, containing the rigid ones, that Urbanik's algorithm computes on the normalization [87, Cor. 6.3]; a test on the algebraic monodromy at one point, computed on a certified curve (Lemmas 10.13 and 10.14 and Proposition 10.15), decides which outputs are rigid, and rigid ones are special.

At a few points we argue differently from [50], and from [22], [21] and [85]. The proof of [50, Thm. 9.2.1] bounds the degree of $Z \cap T_m Z$, taken at an Iwahori level, by Bézout's inequality with constant 1 [50, (9.5)], and the proof of [22, Prop. 10.1] does the same at its Iwahori level. For an ample bundle this inequality can fail (Example 5.1); we use the constant c_B of Lemma 5.2, which does not affect the main theorem of [50]: the choice of l in [50, §10.2] absorbs it. The other differences are described where they occur: in the proofs of Theorem 6.1 and Proposition 6.6, at Lemmas 6.2, 6.3 and 6.5, and in Sections 7.1, 7.2 and 7.4.

1.3. **Plan.** Section 2 fixes notation and the complexity. Section 3 proves the split-prime bounds and the Galois bound under $\mathrm{QRH}(\delta; L_V)$. Section 4 proves Theorem 1.5. Section 5 proves Bézout's inequality at every neat level, and Section 6 the Hecke step built on it. Section 7 proves Theorem 1.6. Section 8 proves Theorem 8.1, the case of adjoint groups at levels with a neat factor (Proposition 8.9), and Section 9 proves Theorem 1.1 for every Shimura datum, with $\mathcal{A}_g$ in Section 9.6. Section 10 proves Theorem 1.4. Sections 3 and 4 are independent of each other; Section 6 uses both.

2. Setting, notation and the complexity

2.1. **Shimura varieties and degrees.** A Shimura datum (G, X) satisfies Deligne's axioms; $E(G, X)$ is its reflex field and X^+ a connected component of X. For a compact open subgroup $K \subset G(\mathbb{A}_f)$,

$$\mathrm{Sh}_K(G, X)_{\mathbb{C}} = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

with its canonical model over $E(G, X)$. We say that K is a *product level* if $K = \prod_p K_p$ with $K_p \subset G(\mathbb{Q}_p)$, and *neat* in the sense of Pink [68, 0.6]. We write $G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap \lambda^{-1}(G^{\mathrm{ad}}(\mathbb{R})^+)$, where $\lambda\colon G \to G^{\mathrm{ad}}$ is the adjoint map and $G^{\mathrm{ad}}(\mathbb{R})^+$ the identity component, and $G^{\mathrm{ad}}(\mathbb{Q})_+ := G^{\mathrm{ad}}(\mathbb{Q}) \cap G^{\mathrm{ad}}(\mathbb{R})^+$. For neat K every connected component of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ is a quotient $\Gamma\backslash X^+$, with $\Gamma \subset G^{\mathrm{ad}}(\mathbb{Q})_+$ the image of $G(\mathbb{Q})_+ \cap gKg^{-1}$ for some $g \in G(\mathbb{A}_f)$ [50, §4.1], which is neat [68, 0.5] and arithmetic (Lemma 2.1). We fix a faithful representation $\rho\colon G \to \mathrm{GL}_{n_\rho}$ with $\rho(K) \subset \mathrm{GL}_{n_\rho}(\widehat{\mathbb{Z}})$, and write $n := \dim \mathrm{Sh}_K(G, X)$.

Arithmetic subgroups are those commensurable with a congruence subgroup [68, 0.5]. We use the following form of the statement that images of arithmetic groups are arithmetic [56, Prop. 3.2].

Lemma 2.1. *Let G be a connected reductive $\mathbb{Q}$-group, $K \subset G(\mathbb{A}_f)$ and $U \subset G^{\mathrm{ad}}(\mathbb{A}_f)$ compact open subgroups with $\lambda(K) \subset U$, $h \in G(\mathbb{A}_f)$, $\Gamma := G(\mathbb{Q})_+ \cap hKh^{-1}$ and $\Gamma' := G^{\mathrm{ad}}(\mathbb{Q})_+ \cap \lambda(h)U\lambda(h)^{-1}$, with $G(\mathbb{Q})_+$ and $G^{\mathrm{ad}}(\mathbb{Q})_+$ as above. Then $\lambda(\Gamma) \subset \Gamma'$, the index $[\Gamma' : \lambda(\Gamma)]$ is finite, and $\lambda(\Gamma)$ and Γ' are arithmetic.*

Proof. The inclusion is clear. As $G^{\mathrm{ad}}(\mathbb{R})^+$ has finite index in $G^{\mathrm{ad}}(\mathbb{R})$ [56, p. 54], Γ' has finite index in the congruence subgroup $G^{\mathrm{ad}}(\mathbb{Q}) \cap \lambda(h)U\lambda(h)^{-1}$, so it is arithmetic. Likewise $\lambda^{-1}(G^{\mathrm{ad}}(\mathbb{R})^+)$ is open, hence of finite index, in $G(\mathbb{R})$, so Γ has finite index in the congruence subgroup $G(\mathbb{Q}) \cap hKh^{-1}$ and is arithmetic. Hence $\lambda(\Gamma)$ is arithmetic [56, Prop. 3.2]. Two arithmetic subgroups are commensurable, so $\lambda(\Gamma) \subset \Gamma'$ gives $[\Gamma' : \lambda(\Gamma)] < \infty$. □

Let $\overline{\mathrm{Sh}}_K(G, X)$ be the Baily–Borel compactification. For neat K the line bundle of holomorphic forms of top degree on $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ extends to an ample line bundle $\mathcal{L}_K$ on $\overline{\mathrm{Sh}}_K(G, X)$, given at the generic points of the boundary divisors by forms with logarithmic poles and defined over $E(G, X)$; for neat $K_2 \subset gK_1g^{-1}$ the morphism $\mathrm{Sh}_{K_2} \to \mathrm{Sh}_{K_1}$ induced by g extends to the compactifications and pulls $\mathcal{L}_{K_1}$ back to $\mathcal{L}_{K_2}$ [50, Prop. 5.3.2]. For a closed subvariety $Y \subset \mathrm{Sh}_K(G, X)_{\mathbb{C}}$ we put

$$\deg Y := \sum_{Y'} \deg_{\mathcal{L}_K} \overline{Y'},$$

the sum over the irreducible components Y' of Y of the degrees of their closures [50, §5.1 and Def. 5.3.3]. We use the projection formula in the form $\deg_{f^*\mathcal{L}} Y = (\deg f) \deg_{\mathcal{L}} f(Y)$ for f generically finite and proper [50, §5.1].

2.2. **Special subvarieties.** A *special subvariety* of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ is an irreducible component of the image of $\mathrm{Sh}_{K_H}(H, X_H)_{\mathbb{C}} \to \mathrm{Sh}_K(G, X)_{\mathbb{C}}$, $[x, h] \mapsto [x, hg]$, for a Shimura subdatum $(H, X_H) \subset (G, X)$, an element $g \in G(\mathbb{A}_f)$ and $K_H := H(\mathbb{A}_f) \cap gKg^{-1}$. We may and do take H to be the generic Mumford–Tate group on X_H, and write H_V for it. Irreducible components of preimages and images of special subvarieties under the morphisms of Shimura varieties used below are special [50, p. 14]. An irreducible subvariety is *Hodge generic* if it is not contained in a special subvariety of smaller dimension than the connected component containing it, and a subvariety defined and irreducible over a number field is Hodge generic if its geometric components are [50, Def. 7.1.1, Lemma 7.1.2]. Reflex fields decrease under

inclusion: $E(G,X)\subset E(H,X_H)$ [56, Remark 12.3(c)], and a morphism of Shimura varieties induced by $(H,X_H)\to(G,X)$ is defined over $E(H,X_H)$ [56, Remark 13.8].

Let V be special with generic Mumford–Tate group H_V. In Sections 2 to 8 the group G is adjoint wherever T_V appears, and we write T_V for the connected center of H_V; in Section 9 we pass to the adjoint group first, so that T_V is always the torus of the introduction. Let L_V be the splitting field of T_V, $d_{L_V}:=|\operatorname{disc} L_V|$, and

$$\beta_V:=\log d_{L_V}.$$

The subvariety V is *strongly special* if $T_V=1$ [50, Def. 2.1.5]. More generally, a special subvariety of $\mathrm{Sh}_{K_P}(P,X_P)_{\mathbb{C}}$ for a Shimura datum (P,X_P) is strongly special there if the connected center of the image of its generic Mumford–Tate group in P^{ad} is trivial [50, Def. 2.1.5]. If $T_V\neq 1$, then $T_V(\mathbb{R})$ is compact and T_V is not split over $\mathbb{Q}$ [50, Lemma 2.4.3], so $L_V\neq\mathbb{Q}$ and $\beta_V\geq\log 3$ by Minkowski's theorem.

Let $K_{T_V}^m$ be the maximal compact open subgroup of $T_V(\mathbb{A}_f)$ and, for V given by (H_V,X_{H_V}) and g as above, $K_{T_V}:=T_V(\mathbb{A}_f)\cap gKg^{-1}$. Replacing (H_V,X_{H_V},g) by $(qH_Vq^{-1},qX_{H_V},qgk)$ with $q\in G(\mathbb{Q})$ and $k\in K$ conjugates $K_{T_V}^m$ and K_{T_V} by q, and replacing g by hg with $h\in H_V(\mathbb{A}_f)$ changes neither, as T_V is central in H_V; so the indices $|K_{T_V,p}^m/K_{T_V,p}|$ depend only on V (compare [50, Remark 2.4.2]).

Definition 2.2 ([50, Def. 2.5.1])**.** For V special with $T_V\neq 1$,

$$\alpha_V:=\prod_{p:\,K_{T_V,p}^m\neq K_{T_V,p}}\max\bigl(1,B\,|K_{T_V,p}^m/K_{T_V,p}|\bigr),$$

with the constant B of Table 1. If $T_V=1$, we put $\alpha_V:=1$ and $d_{L_V}:=1$. The *complexity* of V is

$$\Delta(V):=\max\bigl(\deg V,\ d_{L_V},\ \alpha_V\bigr).$$

Right translation $[x,a]\mapsto[x,ag^{-1}]$ is an isomorphism $\mathrm{Sh}_K(G,X)\to\mathrm{Sh}_{gKg^{-1}}(G,X)$, defined over $E(G,X)$ [56, Thm. 13.6] and pulling $\mathcal{L}_{gKg^{-1}}$ back to $\mathcal{L}_K$ (Section 2.1), so it preserves degrees and fields of definition containing $E(G,X)$; it maps special subvarieties to special subvarieties with the same T_V, L_V and indices, and the image of $X^+\times\{g\}$ onto $\mathrm{Sh}_{gKg^{-1}}(G,X^+)_{\mathbb{C}}$, the image of $X^+\times\{1\}$. The level gKg^{-1} is again a neat product level, and if K satisfies (Lv) of Section 2.3, or (Lv′) of Section 8.6, so does gKg^{-1} for the lattice $\rho(g)\widehat{\mathbb{Z}}^{n_\rho}$.

So in Sections 6 to 8 we assume, as [50, p. 14] and [22, §9.1] do, that Z lies in $\mathrm{Sh}_K(G,X^+)_{\mathbb{C}}$. A special subvariety V of $\mathrm{Sh}_K(G,X^+)_{\mathbb{C}}$ is then the image of $X_{H_V}^+\times\{1\}$ with $X_{H_V}^+\subset X^+$ [85, Lemma 2.1], so that $K_{T_V}=T_V(\mathbb{A}_f)\cap K$; the same convention is used for a Shimura subdatum (P,X_P) with $X_P^+\subset X^+$ in place of (G,X), and for an F-irreducible Z (Section 6) it applies to a geometric component of Z containing V; when a proof chooses a component of a preimage in $\mathrm{Sh}_J(P,X_P)_{\mathbb{C}}$, for $J\subset K\cap P(\mathbb{A}_f)$ compact open, it is chosen in the image of $X_P^+\times\{1\}$.

For Z in the image of $X^+\times\{g_i\}$, with g_i in a fixed finite set of representatives of $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$, Theorems 7.4 and 8.1 hold with the constants of the level $g_iKg_i^{-1}$ and the lattice $\rho(g_i)\widehat{\mathbb{Z}}^{n_\rho}$ in place of those of K and $\widehat{\mathbb{Z}}^{n_\rho}$. The symbols $c_{\deg}^{\mathrm{ad}}(z,D)$ and $c^{\mathrm{ad}}(\delta,z,D,d)$ below are the constants of the level K; Section 9 uses their maxima $\bar{c}_{\deg}^{\mathrm{ad}}$ and $\bar{c}^{\mathrm{ad}}$ over the representatives. These maxima are effectively computable from (G,X,K,ρ) and the chosen representatives; the finiteness of the double coset sets [56, Lemma 5.12] makes them well defined, a finite subset of $G(\mathbb{A}_f)$ (resp. $G^{\mathrm{ad}}(\mathbb{A}_f)$) meeting every double coset may be used

in place of a set of representatives, and we do not discuss how to compute one in general (for $\mathcal{A}_g$ none is needed, proof of Theorem 9.13).

2.3. **The constants of the datum.** Klingler and Yafaev work under the following normalization of the level [50, Situation 9.1]:

(Lv) K is a neat product level, and K_3 lies in the principal congruence subgroup of level 3 of $\mathrm{GL}_{n_\rho}(\mathbb{Z}_3)$.

Lemma 9.1 removes it at an explicit cost. The constants in Table 1 depend only on (G, X, K, ρ) and, except μ and f, are the ones in [50, 85, 28]; each except C_0 is given by an explicit formula below (a, B, c_U, k_1, f, μ, s_ρ) or is bounded by the proof in its source through finiteness arguments that can be carried out effectively: finite subgroups of $\mathrm{GL}_r(\mathbb{Z})$, $r \le \operatorname{rank} G$, for n_T and R_G (both are at most $|\mathrm{GL}_{\operatorname{rank} G}(\mathbb{F}_3)|$), the local unit theorem for r_U (one may take $r_U = 2 \operatorname{rank} G$), and root systems, minuscule coweights and the Hodge types of ρ for A, t_ρ, B_ρ and C_Φ [85, Lemmas 2.4–2.8]; C_0 is any integer larger than every prime l at which the schematic closure of $\rho(G)$ in $\mathrm{GL}_{n_\rho,\mathbb{Z}}$ is not reductive over $\mathbb{Z}_l$ or $K_l \neq G(\mathbb{Z}_l)$; the primes of the first kind divide the integer M_0 of Section 9, which is found from equations for $\rho(G)$ by making effective the steps used there (the finiteness of the non-smooth locus and [18, Props. 3.1.9 and 3.1.12]), and those of the second kind are finitely many and are read off from K. None of the constants involves an analytic input. We supply one step for B_ρ: [85, Lemma 2.6] bounds the coordinates of the characters of T_V in ρ read in $X^*(\mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_\mathrm{m})$ through r_T (below), and in an LLL-reduced basis of the lattice $X^*(T_V) \subset X^*(\mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_\mathrm{m})$ the coordinates of a vector are at most a constant depending only on $\dim T_V$ times its length, since nonzero integer vectors have length at least 1. Put $c_U := 1/(2C_\Phi)$, where C_Φ bounds the orders of the component groups of the stabilizers in [28, Lemma 4.4.1] in the following form. For a special V let $T := T_V$, let $F_T \supset L_T$ and $r_T \colon \mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_\mathrm{m} \to T$ be the field and surjection of [85, p. 42], and for $d \ge 1$ and a vector v of the d-th exterior power of $\overline{\mathbb{Q}}^{n_\rho}$ let N be the subgroup of $X^*(\mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_\mathrm{m})$ generated by the differences $x - x'$ of the weights x, x' of $\mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_\mathrm{m}$, acting through $\rho \circ r_T$, in the support of v. The bound C_Φ satisfies

$$\left|\left(X^*(\mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_\mathrm{m})/N\right)_{\mathrm{tors}}\right| \;\le\; C_\Phi \qquad \text{for all special } V,\ d \ge 1 \text{ and } v \in \textstyle\bigwedge^d \overline{\mathbb{Q}}^{n_\rho}.$$

These come from the finitely many characters allowed by [85, Lemmas 2.5–2.6], so C_Φ, and with it c_U, is computable from (G, ρ).

We write $B' := Bc_U$. The bound n_T and the field degrees R_G are bounds over all special subvarieties at once, through the finitely many conjugacy classes of finite subgroups of $\mathrm{GL}_r(\mathbb{Z})$, $r \le \operatorname{rank} G$ [85, proof of Lemma 2.5].

2.4. **The analytic hypothesis.** For a number field L and $\delta \in (0, 1/2]$, $\mathrm{QRH}(\delta; L)$ is the statement that $\zeta_L(s) \neq 0$ for $\operatorname{Re} s > 1 - \delta$, $s \neq 1$. We put

$$\kappa(\delta) := \frac{2-\delta}{\delta(1+\delta)}, \qquad w(\delta) := (1 + 2^{-1-\delta})(1 + 2^{-\delta}) < 3.$$

3. Split primes and Galois orbits under a zero-free half-plane

This section proves the two analytic inputs of the Hecke step for one Galois field L under $\mathrm{QRH}(\delta; L)$: a count of primes that split completely in L from x a power of $\log d_L$ on, and the lower bound of Ullmo and Yafaev for Galois orbits with an explicit constant. The first is an explicit formula; the second reruns [86, §6] with it.

TABLE 1. Constants attached to (G, X, K, ρ), for G adjoint.

Constant	Meaning	Source
n_T	a bound for $[L_V : \mathbb{Q}]$ over all special V	[85, proof of Lemma 2.5]
R_G	a bound for the degree of the Galois closure of $E(H_V, X_{H_V})$ over all special V	[85, Lemma 2.5], [50, Def. 2.5.1]
C_0	every prime $l > C_0$ has $K_l = G(\mathbb{Z}_l)$ hyperspecial	[50, p. 53]
a, A, r_U	a bounds the indices $[K_H^m : K_H^{m'}]$ and $[K_H : K_H']$ of the auxiliary levels of [85, pp. 21–22], for instance $a = \lvert\mathrm{GL}_{n_\rho}(\mathbb{F}_3)\rvert$; A an integer with the property of [85, Prop. 2.9], fixed once and for all; r_U a bound, over all special V and primes p, for the number of cyclic factors of $K_{T_V,p}^m/K_{T_V,p}$	[85, pp. 20–22, 29]
B	$B = 1/(A^{r_U}(a!)^{r_U})$	[85, p. 31]
c_U	$\lvert K_{T_V,p}^m/K_{T_V,p}\rvert \geq c_U p$ for $p > C_0$ unramified in L_V with $K_{T_V,p}^m \neq K_{T_V,p}$	[85, Prop. 3.15], [28, §4.4], Lemma 6.2
k_1, f	$k_1 = 2An_\rho^3 B_\rho$ with B_ρ the bound B of [28, Prop. 7.4.3] for $\rho\lvert_{T_V}$, uniform in V by [85, Lemma 2.6] as in [50, §8.3.1] ([88, Prop. 2.6] for special points); $f := (\dim G + 1)^2$, which bounds the Iwahori indices of [50, (8.8)] as $< l^f$ (Lemma 6.3)	[28, Prop. 7.4.3], [50, (8.6)]
t_ρ, s_ρ	s_ρ, for instance $s_\rho = n_\rho$, bounds the number of characters of the representation of H_V/H_V^{der} induced by $t \mapsto \rho(t)^m$ (m the degree of $T_V \to H_V/H_V^{\mathrm{der}}$); t_ρ bounds their Hodge weights	[85, proof of Lemma 2.6]
μ	$\mu = 1/(16\, t_\rho s_\rho\, n_T(n_T - 1))$	Proposition 3.3

3.1. **Split primes.** Let $L/\mathbb{Q}$ be Galois of degree n_L and signature (r_1, r_2), $d_L := \lvert\operatorname{disc} L\rvert$, and put

$$A_L := \tfrac{3}{2} + \tfrac{1}{2}\log d_L, \qquad \vartheta_L(x) := \sum_{\substack{x/4 < p \leq x \\ p \text{ splits completely in } L}} \log p.$$

Lemma 3.1. *Let $\delta \in (0, 1/2]$ and assume $\mathrm{QRH}(\delta; L)$. For every $x \geq 8$,*

$$\vartheta_L(x) \;\geq\; \frac{3x}{8n_L}\Big(1 - \tfrac{16}{3}w(\delta)\kappa(\delta)A_L\, x^{-\delta}\Big) - 1.4262\sqrt{x} - \epsilon_L(x)\log\mathrm{rad}(d_L) - 0.694 - \frac{1.922}{x},$$

where $\epsilon_L(x) = 1$ if some prime dividing d_L lies in $(x/4, x]$ and $\epsilon_L(x) = 0$ otherwise.

Proof. Write $-\zeta_L'/\zeta_L(s) = \sum_n a(n)n^{-s}$, with $a(n) = \sum_{N\mathfrak{p}^k = n} \log N\mathfrak{p}$. As L is Galois, $a(p) = n_L \log p$ if p splits completely, $a(p) = 0$ if p is unramified and does not split completely, and $a(p^k) \leq n_L \log p$ for every prime p and every $k \geq 1$ (indeed, with e_p and f_p the ramification index and the residue degree of p in L, $a(p^k) = (n_L/e_p)\log p$ if $f_p \mid k$ and 0 otherwise). In particular $a(n) \leq n_L \log p$ when $n = p$ is a ramified prime and when $n = p^k$

with $k \geq 2$, and $a(n) = 0$ when n is not a prime power. By the functional equation every nontrivial zero ϱ of ζ_L satisfies $\delta \leq \operatorname{Re}\varrho \leq 1-\delta$.

Step 1 (a smoothed explicit formula). Let $F(x) := \sum_{n\leq x} a(n)(x-n)$. Then

$$F(x) = \frac{1}{2\pi i}\int_{(2)} \Big(-\frac{\zeta_L'}{\zeta_L}(s)\Big)\frac{x^{s+1}}{s(s+1)}\,ds.$$

Let $x \geq 2$. The kernel decays like $|s|^{-2}$, so the contour may be moved to the left and $F(x)$ is the absolutely convergent sum of the residues; the side integrals tend to 0, by the bounds of Lagarias and Odlyzko for ζ_L'/ζ_L [53, Lemmas 5.4, 5.6 and 6.2], along heights $T_k \to \infty$ chosen away from the zeros, which exist by [53, Lemma 5.4]; the computation is carried out in the proof of [20, Prop. 6], with decay exponent 1; the left sides $\operatorname{Re} s = -j - \frac{1}{2}$ avoid the additional pole of our kernel at $s = -1$. Let $o_0 := r_1 + r_2 - 1$ and $o_1 := r_2$ be the orders of vanishing of ζ_L at $s = 0$ and $s = -1$, and $o_m \in \{r_2, r_1 + r_2\}$ its order at $s = -m$ for $m \geq 2$. The residues are $x^2/2$ at $s = 1$ and $-x^{\varrho+1}/(\varrho(\varrho+1))$ at each nontrivial zero ϱ, with multiplicity. At $s = 0$ and $s = -1$ they are $-o_0 x(\log x - 1) + c_0 x$ and $o_1(\log x + 1) - c_1$, for some constants c_0, c_1 (both poles have order at most 2). At $s = -m$, $m \geq 2$, the residue is $-o_m x^{1-m}/(m(m-1))$.

Step 2 (removing c_0 and c_1). As $x \geq 8$, Step 1 applies to x, $x/2$ and $x/4$. Let $G(x) := F(x) - 3F(x/2) + 2F(x/4) = \sum_n a(n) b_x(n)$, where $b_x(n) = (x-n)_+ - 3(x/2-n)_+ + 2(x/4-n)_+$. Then $b_x(n) = 0$ for $n \leq x/4$, $b_x(n) = 2n - x/2$ on $(x/4, x/2]$ and $b_x(n) = x - n$ on $(x/2, x]$, so $0 \leq b_x \leq x/2$. In the combination the terms $c_0 x$ and c_1 cancel, $x(\log x - 1)$ becomes $(x \log 2)/2$, $\log x + 1$ becomes $-\log 2$, a zero ϱ acquires the factor $(1 - 2^{-\varrho-1})(1 - 2^{-\varrho})$, and the trivial zero at $-m$ the factor $(2^{m-1} - 1)(2^m - 1)$. Hence

$$G(x) = \frac{3x^2}{16} - \sum_{\varrho} \frac{x^{\varrho+1}(1-2^{-\varrho-1})(1-2^{-\varrho})}{\varrho(\varrho+1)} - \frac{o_0 \log 2}{2} x - o_1 \log 2 - R(x),$$

with $R(x) := \sum_{m\geq 2} o_m (2^{m-1}-1)(2^m-1)x^{1-m}/(m(m-1)) \geq 0$. For $x \geq 8$, $R(x) \leq 2n_L \sum_{k\geq 1} 2^{-k}/(k(k+1)) = 2n_L(1 - \log 2) < 0.614\, n_L$.

Step 3 (the zeros). For each ϱ, $|x^{\varrho+1}(1-2^{-\varrho-1})(1-2^{-\varrho})| \leq w(\delta)x^{2-\delta}$. By the Cauchy–Schwarz inequality $|\varrho||\varrho+1| \geq \operatorname{Re}\varrho(1 + \operatorname{Re}\varrho) + (\operatorname{Im}\varrho)^2$, and so, with $u := \operatorname{Re}\varrho$ and $v := (\operatorname{Im}\varrho)^2$,

$$\frac{|2-\varrho|^2}{(2-u)|\varrho||\varrho+1|} \leq \frac{(2-u)^2+v}{(2-u)(u(1+u)+v)} \leq \max\Big(\frac{2-u}{u(1+u)}, \frac{1}{2-u}\Big) \leq \kappa(\delta),$$

because $(2-u)/(u(1+u))$ decreases on $[\delta, 1]$ and $1/(2-u) \leq 1/(1+\delta) \leq \kappa(\delta)$. Hence $\sum_\varrho |\varrho(\varrho+1)|^{-1} \leq \kappa(\delta) \sum_\varrho \operatorname{Re}(2-\varrho)^{-1}$.

Step 4 (Hadamard's product at $s = 2$). Stark's identity [78, Lemma 1, (5)], applied to $\xi_L(s) = s(s-1)d_L^{s/2}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_L(s)$, gives

$$\sum_\varrho \operatorname{Re}\frac{1}{2-\varrho} = \frac{3}{2} + \frac{1}{2}\log d_L + r_1\Big(-\frac{\log\pi}{2} + \frac{\Gamma'(1)}{2\Gamma(1)}\Big) + r_2\Big(-\log 2\pi + \frac{\Gamma'(2)}{\Gamma(2)}\Big) + \frac{\zeta_L'}{\zeta_L}(2) \leq A_L,$$

since the coefficients of r_1 and r_2, $-(\gamma + \log\pi)/2$ and $1 - \gamma - \log 2\pi$ with γ Euler's constant, are negative and $\zeta_L'/\zeta_L(2) < 0$. With Steps 2 and 3, and $o_0 \leq n_L$, $o_1 \leq n_L/2$,

$$G(x) \geq \frac{3x^2}{16} - w(\delta)\kappa(\delta)A_L x^{2-\delta} - 0.347\, n_L x - 0.961\, n_L.$$

Step 5 (counting). Since $0 \le b_x \le x/2$ and b_x vanishes on $[1, x/4]$,

$$G(x) \le \frac{x}{2}\, n_L\Big(\vartheta_L(x) + \big(\psi(x) - \vartheta(x)\big) + \epsilon_L(x)\log \operatorname{rad}(d_L)\Big),$$

where ψ and ϑ are Chebyshev's functions, and $\psi(x) - \vartheta(x) < 1.42620\sqrt{x}$ for $x > 0$ [74, Thm. 13, (3.36)]. Dividing the two bounds for $G(x)$ by $xn_L/2$ gives the lemma. □

Lemma 3.2. *Let $\delta \in (0, 1/2]$ and assume* $\mathrm{QRH}(\delta; L)$. *Put*

$$x_1(L,\delta) := \max\Big(\big(\tfrac{32}{3} w(\delta)\kappa(\delta) A_L\big)^{1/\delta},\ (48n_L)^2,\ (\log d_L + 1)^2\Big).$$

For every $x \ge x_1(L,\delta)$ we have $\vartheta_L(x) \ge x/(8n_L)$, so at least $x/(8n_L \log x)$ primes in $(x/4, x]$ split completely in L. If $\beta := \log d_L \ge 1$, the condition $x \ge x_1(L,\delta)$ holds as soon as

$$x^\delta \ge 64\kappa(\delta)\beta, \qquad x \ge (48n_L)^2, \qquad \sqrt{x} \ge \beta + 1.$$

Proof. As $x \ge 8$, Lemma 3.1 applies, and $x^\delta \ge \frac{32}{3} w\kappa A_L$ makes the bracket at least $1/2$, so the main term is at least $3x/(16n_L)$. Bound $\epsilon_L \log \operatorname{rad}(d_L) \le \log d_L$ and $0.694 + 1.922/x < 1$. As $\sqrt{x} \ge 48n_L$, $x/(16n_L) \ge 3\sqrt{x}$, and as $\sqrt{x} \ge \log d_L + 1$, $3\sqrt{x} \ge 1.4262\sqrt{x} + \log d_L + 1$. So $\vartheta_L(x) \ge 3x/(16n_L) - x/(16n_L) = x/(8n_L)$, and each prime counted contributes at most $\log x$. For the last assertion, $w(\delta) < 3$ and $A_L \le 2\beta$ when $\beta \ge 1$, so $\frac{32}{3} w(\delta)\kappa(\delta)A_L \le 64\kappa(\delta)\beta$. □

3.2. **The Galois bound.** Let (G, X) be a Shimura datum with G adjoint, $(H, X_H) \subset (G, X)$ a Shimura subdatum with H the generic Mumford–Tate group on X_H, and $H^{\mathrm{ab}} := H/H^{\mathrm{der}}$, which is isogenous to the connected center T of H [85, Remark 2.20]; assume $T \ne 1$. Fix $x \in X_H$, let $(H^{\mathrm{ab}}, \{\bar{x}\})$ be the induced Shimura datum of CM type, $E := E(H^{\mathrm{ab}}, \{\bar{x}\})$, L the splitting field of H^{ab} (that of T), and $r_L := r \circ N_{L/E} \colon \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_{\mathrm{m}} \to H^{\mathrm{ab}}$, where r is the reciprocity morphism, which is surjective [85, proof of Prop. 2.14]. Let $K^m_{H^{\mathrm{ab}}}$ be the maximal compact open subgroup of $H^{\mathrm{ab}}(\mathbb{A}_f)$ and

$$\mathrm{Im}_L := \text{the image of } r_L\big((\mathbb{A}_f \otimes L)^\times\big) \text{ in } H^{\mathrm{ab}}(\mathbb{Q})\backslash H^{\mathrm{ab}}(\mathbb{A}_f)/K^m_{H^{\mathrm{ab}}}.$$

Let $\chi_1, \chi_2, \dots$ be the characters, at most s_ρ in number, of the representation of H^{ab} induced by $t \mapsto \rho(t)^m$, with m the degree of the isogeny $T \to H^{\mathrm{ab}}$ (which kills its kernel) [85, proof of Lemma 2.6]; they span $X^*(H^{\mathrm{ab}}) \otimes \mathbb{Q}$, and their coordinates in the canonical basis $(\chi_\sigma)_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})}$ of $X^*(\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_{\mathrm{m}})$ are Hodge weights, of absolute value at most t_ρ [85, proof of Lemma 2.6]. Put $\mathfrak{d} := 2t_\rho s_\rho n_T(n_T - 1)$, $\theta := 1/(4\mathfrak{d}) = 2\mu$, and

$$\beta_0(\delta) := \max\Big(\frac{128\kappa(\delta)}{\theta^2\delta^2},\ \Big(\frac{96}{\theta^3}\Big)^{1/2},\ 3n_T \log n_T + 1,\ \frac{64n_T}{\theta}\Big), \qquad c_*(\delta) := \exp(-\mu\beta_0(\delta)).$$

Proposition 3.3. *If* $\mathrm{QRH}(\delta; L)$ *holds, then* $|\mathrm{Im}_L| \ge c_*(\delta)\, d_L^{\mu}$. *The exponent $\mu = 1/(16t_\rho s_\rho n_T(n_T - 1))$ does not depend on δ.*

This is the bound to which Ullmo and Yafaev reduce their theorem [86, Thm. 6.1 and p. 33], which assumes GRH, with the split primes supplied by Lemma 3.2; their argument goes back to Amoroso and Dvornicich [2] and Yafaev [88, Thm. 2.15]. We write the proof out, with the primes taken up to a power of the discriminant, $x = d_L^\theta$, as in [82, proof of Thm. 7.6]; taking x a power of d_L is what lets any $\delta > 0$ suffice.

Proof. Let $p_1, \dots, p_\ell \le x$ be distinct primes that split completely in L. Fix places $v_j \mid p_j$ and let $P_j \in (\mathbb{A}_f \otimes L)^\times$ be p_j at v_j and 1 elsewhere.

(1) Suppose $a \in \mathbb{Z}^\ell \smallsetminus \{0\}$ and $r_L(\prod_j P_j^{a_j}) = \pi k$ with $\pi \in H^{\mathrm{ab}}(\mathbb{Q})$ and $k \in K^m_{H^{\mathrm{ab}}}$. Write $\chi_i \circ r_L = \sum_\sigma c_{i,\sigma}\chi_\sigma$ with $|c_{i,\sigma}| \le t_\rho$. Since χ_σ acts on $(\mathbb{A}_f \otimes L)^\times$ through $1 \otimes \sigma$ and the

decomposition group at v_j is trivial, $\pi_i := \chi_i(\pi) \in L^\times$ has valuation of absolute value at most $t_\rho|a_j|$ at the places above p_j and valuation 0 at every other finite place.

(2) As G is adjoint, $H^{\mathrm{ab}}(\mathbb{R})$ is compact, complex conjugation acts by -1 on $X^*(H^{\mathrm{ab}})$ and L is CM, so $|\sigma(\pi_i)| = 1$ for every embedding σ. With $Q := \prod_j p_j^{|a_j|}$, the numbers $\pi_i' := Q^{t_\rho}\pi_i$ are algebraic integers with $|\sigma(\pi_i')| = Q^{t_\rho}$.

(3) The π_i generate L [88, Lemma 2.16]: if $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ fixes every π_i, comparing valuations at the places above some p_j with $a_j \neq 0$ shows that σ fixes each $\chi_i \circ r_L$ in the regular module $X^*(\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_{\mathrm{m}})$. As r_L is surjective, σ acts trivially on $X^*(H^{\mathrm{ab}}) \otimes \mathbb{Q}$, so $\sigma = 1$.

(4) The monomials $\prod_i \pi_i'^{e_i}$ with $0 \le e_i \le n_T - 1$ span L over $\mathbb{Q}$; a basis $b_1, \dots, b_{n_L}$ among them satisfies $|\sigma(b_k)| \le Q^{t_\rho s_\rho(n_T-1)}$, and Hadamard's inequality gives $|\det(\mathrm{Tr}\, b_k b_{k'})| \le n_T^{3n_T/2} Q^{\mathfrak{d}}$. This is a multiple of d_L, so $\log d_L \le \frac{3}{2} n_T \log n_T + \mathfrak{d} \sum_j |a_j| \log x$.

(5) If $\ell \ge |\mathrm{Im}_L|$, two of the classes of the $r_L(P_j)$ coincide or one is trivial, which gives a as in (1) with $0 < \sum |a_j| \le 2$. So if $\log d_L - \frac{3}{2} n_T \log n_T > 2\mathfrak{d} \log x$, then $\ell < |\mathrm{Im}_L|$ [86, p. 33].

(6) Let $\beta := \log d_L \ge \beta_0(\delta)$ and $x := d_L^\theta$. Then $x^\delta = e^{\theta\delta\beta} \ge (\theta\delta\beta)^2/2 \ge 64\kappa(\delta)\beta$; $\theta\beta \ge 64 n_T \ge 2\log(48 n_T)$; and $\sqrt{x} = e^{\theta\beta/2} \ge (\theta\beta/2)^3/6 \ge 2\beta \ge \beta + 1$. By Lemma 3.2 there are at least $x/(8 n_T \log x)$ such primes in $(x/4, x]$. Also $\beta - \frac{3}{2} n_T \log n_T > \beta/2 = 2\mathfrak{d} \log x$. By (5), $|\mathrm{Im}_L| > x/(8 n_T \log x) = e^{2y}/(16 n_T y)$ with $y := \theta\beta/2 \ge 32 n_T$, and $e^y \ge y^2/2 \ge 16 n_T y$, so $|\mathrm{Im}_L| \ge e^y = d_L^\mu$. If $\beta < \beta_0(\delta)$, then $|\mathrm{Im}_L| \ge 1 \ge c_*(\delta) d_L^\mu$. □

For a special V with $T_V \neq 1$ let F_V be the Galois closure of $E(H_V, X_{H_V})$; it contains L_V [85, proof of Prop. 2.14] and $[F_V : \mathbb{Q}] \le R_G$. Put

$$C_* := \frac{c_*(\delta)}{R_G\, a\, (a!)^{r_U}}, \qquad \Omega_V := C_*\, \alpha_V\, d_{L_V}^\mu.$$

Corollary 3.4. *Let G be adjoint and K satisfy* (Lv). *Let $(H, X_H) \subset (G, X)$ be a Shimura subdatum with H the generic Mumford–Tate group on X_H and $T := Z(H)^\circ \neq 1$, $K_H := K \cap H(\mathbb{A}_f)$, V_H a geometric component of $\mathrm{Sh}_{K_H}(H, X_H)_{\mathbb{C}}$ and V its image in $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$, so that $T_V = T$. Let $F \supset E(H, X_H)$ be a number field whose Galois closure F^c has degree at most R_G (for instance $F = E(H', X_{H'})$ for a subdatum $(H', X_{H'}) \subset (H, X_H)$, or its Galois closure). Assume* $\mathrm{QRH}(\delta; L_V)$. *Then:*

(i) $\deg_{\mathcal{L}_{K_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot V_H) \ge \Omega_V$, *and hence* $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot V) \ge \Omega_V$ *in* $\mathrm{Sh}_K(G, X)$;

(ii) if $Y \subset V_H$ is closed, geometrically irreducible and defined over $\overline{\mathbb{Q}}$, $K_H^m := K_T^m K_H$, $\pi\colon \mathrm{Sh}_{K_H}(H, X_H) \to \mathrm{Sh}_{K_H^m}(H, X_H)$ is the natural map, $\Theta_A \subset K_T^m/K_T$ is the image of $x \mapsto x^A$, with A the integer of Table 1, and $\Theta_A \cdot Y \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y \cap \pi^{-1}\pi(Y)$, then $\deg_{\mathcal{L}_{K_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y) \ge \Omega_V$.

The same bounds hold for the orbits under any Galois group containing $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

Proof. The field F^c contains the Galois closure of $E(H, X_H)$, hence L_V [85, proof of Prop. 2.14]. We first prove (i) for $F = F_V$, the Galois closure of $E(H, X_H)$. The first bound of (i) is then [85, Thm. 2.19, (6)] (which is [50, Thm. 2.4.4]) with GRH replaced by $\mathrm{QRH}(\delta; L_V)$, and with $d_{L_V}^\mu$ in place of $(\log d_{L_V})^N$. In its proof GRH is used only in [85, Prop. 2.13] (see [85, pp. 30 and 33]), which bounds $|\mathrm{Im}_L|$ from below for $L = L_V$; this bound enters the proof of [85, Thm. 2.19] only as a factor, the number of Galois conjugates in [85, Prop. 2.14]. We replace it by Proposition 3.3. The image of $r_{(H^{\mathrm{ab}}, \tilde{x})}((\mathbb{A}_f \otimes F_V)^\times) = r_L(N_{F_V/L}((\mathbb{A}_f \otimes F_V)^\times))$ has index at most $[F_V : L] \le R_G$ in Im_L, because r_L maps $L^\times$ into $H^{\mathrm{ab}}(\mathbb{Q})$ and the unit

ideles into $K^m_{H^{\mathrm{ab}}}$, so factors through an idele class group of L, where the norm from F_V has image of index at most $[F_V : L]$ by class field theory. So the number of conjugates in [85, Prop. 2.14] is at least $c_*(\delta)d_L^{\mu}/R_G$. By [85, Remark 2.20] the constant of [85, Thm. 2.19] is that of Prop. 2.14 divided by $a\,(a!)^{r_U}$, and the factor α_V comes from [85, Lemmas 2.15–2.18], which use no GRH. This gives (6) of [85, Thm. 2.19], which is the first bound of (i); the second follows from the first by [85, Lemma 2.2] and [50, Cor. 5.3.10], as $Z(H)(\mathbb{R})$ is compact.

Form (7) of [85, Thm. 2.19] is proved in the same way for Y in place of V_H [85, pp. 29–31], at the auxiliary levels $K'_H \subset K_H^{m'}$ of [85, pp. 21–22], for components V' and Y' over V_H and Y and the map $\pi'\colon \mathrm{Sh}_{K'_H}(H, X_H) \to \mathrm{Sh}_{K_H^{m'}}(H, X_H)$ [85, Lemma 2.11]. By [85, Lemma 2.12] the degree of the Galois orbit of Y' is at least the degree of the union of its irreducible components mapping onto $\pi'(Y')$ times the number of Galois conjugates of $\pi'(Y')$. GRH enters only in this number, through [85, Prop. 2.14], whose assertion for Y' follows from that for V' because the Galois conjugates of $\pi'(V')$ are disjoint.

For general F we prove (ii), and then (i), by running this argument over F itself: [85, (5) and Lemmas 2.12, 2.16–2.18] use only that the map $\mathrm{Sh}_{K'_H}(H, X_H) \to \mathrm{Sh}_{K_H}(H, X_H)$, π and π' are defined over F, and the hypothesis on Θ_A is used with F; the remaining factor, the number of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-conjugates of $\pi'(Y')$, is at least the number of $\mathrm{Gal}(\overline{\mathbb{Q}}/F^c)$-conjugates, which the argument above bounds with $F^c \supset L_V$ in place of F_V and index at most $[F^c : L_V] \le R_G$. This is (ii). For (i), $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot V_H \supset \mathrm{Gal}(\overline{\mathbb{Q}}/F^c) \cdot V_H$, and the argument above applies verbatim to F^c. □

This is the form of [86, Thm. 6.1].

4. Point separation by the Baily–Borel bundle

The Bézout inequality of Lemma 5.2 needs one exponent e for which the complete linear system of $\mathcal{L}_K^{\otimes e}$ on the Baily–Borel compactification is base-point free and separates points; very ampleness is not needed. This section gives such an exponent for every Shimura variety, depending only on the dimension.

Proof of Theorem 1.5. Reduction to one component. $\overline{\mathrm{Sh}}_K(G, X)$ is the disjoint union of the Baily–Borel compactifications of its connected components $\Gamma_i\backslash X^+$, with Γ_i neat arithmetic (Section 2.1). Sections over a disjoint union are tuples of sections, so it suffices to prove base-point freeness and separation of points on one component $Y := \overline{\Gamma\backslash X^+}$, with interior $Y^\circ = \Gamma\backslash X^+$: two points on different components are then separated by a section vanishing on one component and not at the given point of the other.

Y is a log canonical pair with log canonical bundle $\mathcal{L}_K|_Y$. Let $\pi\colon \widetilde{Y} \to Y$ be a smooth toroidal compactification of Y° with reduced normal crossings boundary $\partial\widetilde{Y}$, which exists as Γ is neat [5]; π is an isomorphism over Y°. That Y with its boundary is log canonical is known [1, Thm. 3.2 and Cor. 3.3]; we recall the argument, which also identifies the bundle. By Mumford's description of the canonical extensions [62, Prop. 3.4(a), (b)], the extension of the cotangent bundle is $\Omega^1_{\widetilde{Y}}(\log \partial\widetilde{Y})$ and the extension of the canonical bundle is the pull-back of an ample bundle $\mathcal{O}_Y(1)$ on Y, whose high powers give the Baily–Borel embedding; taking top exterior powers,

$$K_{\widetilde{Y}} + \partial\widetilde{Y} \cong \pi^*\mathcal{O}_Y(1).$$

The bundles $\mathcal{O}_Y(1)$ and $\mathcal{L}_K|_Y$ agree on Y° and at the generic point of each boundary divisor of Y (both are given there by forms with logarithmic poles [50, Prop. 5.3.2(1)]), so they are isomorphic on the normal variety Y. Let $\Delta := \pi_*\partial\widetilde{Y}$, the reduced sum of the boundary

divisors of Y ($\Delta = 0$ if the boundary has codimension at least 2). Then $K_Y + \Delta = \pi_*(K_{\widetilde{Y}} + \partial\widetilde{Y})$ is linearly equivalent to $\mathcal{L}_K|_Y$, hence Cartier, and $K_{\widetilde{Y}} + \partial\widetilde{Y} - \pi^*(K_Y + \Delta)$ is a principal divisor with zero push-forward, hence zero, since a rational function with trivial divisor on the normal Y is a unit. So $K_{\widetilde{Y}} + \partial\widetilde{Y} = \pi^*(K_Y + \Delta)$. The pair $(\widetilde{Y}, \partial\widetilde{Y})$ is log canonical [51, Cor. 3.12], and therefore so is (Y, Δ) [51, Lemma 3.10].

Angehrn–Siu. Let $e \geq e^*(n)$ and $\sigma(n) := \sum_{k=1}^{n} k\, 2^{1/k}$. Then $\mathcal{L}_K^{\otimes e}|_Y \equiv K_Y + \Delta + N$ with $N := (e-1)\mathcal{L}_K|_Y$ ample. For every irreducible $W \subset Y$ of dimension $k \geq 1$, $(N^k \cdot W) = (e-1)^k \deg_{\mathcal{L}_K} W \geq (e-1)^k > \sigma(n)^k$, as $\deg_{\mathcal{L}_K} W$ is a positive integer and $e - 1 \geq \lfloor \sigma(n) \rfloor + 1$. With the constant choice $c(k) := \sigma(n)$ we have $\sum_{k \leq n} 2^{1/k} k/c(k) = 1$ and $\sum_{k \leq n} k/c(k) \leq 1$. Fujino's theorems [35, Thms. 1.1 and 1.2] now give, for every point of Y, a section of $\mathcal{L}_K^{\otimes e}|_Y$ not vanishing there, and, for every two points, sections separating them. Finally $k\, 2^{1/k} \leq k + 1$ because $(1 + 1/k)^k \geq 2$, so $\sigma(n) \leq n(n+3)/2$. □

The values are $e^*(1) = 4$, $e^*(2) = 6$, $e^*(3) = 10$. The bound depends only on n, and in particular not on the level, which is what the uniform Bézout step needs. In the compact case $\Delta = 0$ and $\mathcal{L}_K$ is the canonical bundle, and Theorem 1.5 is an Angehrn–Siu bound for it. Fujino's theorem gives separation of points, not of tangent vectors, and Lemma 5.2 shows that this is all we need.

5. Bézout's inequality uniformly in the level

In this section (G, X) is a Shimura datum and $K \subset G(\mathbb{A}_f)$ a neat compact open subgroup. The Hecke step works on Shimura varieties of subdata at levels that vary with a prime.

5.1. **An example.**

Example 5.1. Let C be a hyperelliptic curve of genus $g \geq 2$ with hyperelliptic involution ι, let $p \in C$, and let $\mathcal{L} := \mathcal{O}(p) \boxtimes \mathcal{O}(p)$ on $C \times C$, an ample line bundle. The diagonal Δ_C and the graph Γ_ι of ι both have $\mathcal{L}$-degree 2, but $\Delta_C \cap \Gamma_\iota$ consists of the $2g + 2$ fixed points of ι. So $\deg(V \cap W) \leq \deg V \cdot \deg W$ fails for an ample bundle; [36, Example 8.4.6], cited for this inequality in [50, (9.5)], is a statement about subvarieties of projective space, that is, for a very ample bundle.

5.2. **The inequality at every neat level.**

Lemma 5.2. *(a) Let $A, B \subset \mathbb{P}^N_{\mathbb{C}}$ be closed; the degree of a closed set is the sum of the degrees of its irreducible components, and* Irr *denotes the set of irreducible components, of all dimensions. Then $\sum_{C \in \mathrm{Irr}(A \cap B)} \deg C \leq \deg A \cdot \deg B$.*

(b) Let (H, X_H) be a Shimura datum, $J \subset H(\mathbb{A}_f)$ a neat compact open subgroup, $n_J := \dim \mathrm{Sh}_J(H, X_H)$ and $e \geq e^*(n_J)$ an integer. For closed $A, B \subset \overline{\mathrm{Sh}}_J(H, X_H)$, of dimensions a and b (maxima over components),

$$\sum_{C \in \mathrm{Irr}(A \cap B)} \deg_{\mathcal{L}_J} C \;\leq\; e^{a+b} \deg_{\mathcal{L}_J} A \cdot \deg_{\mathcal{L}_J} B.$$

The same holds for closed $A, B \subset \mathrm{Sh}_J(H, X_H)_{\mathbb{C}}$, with degrees of closures and C running over $\mathrm{Irr}(A \cap B)$.

(c) In particular, if (H, X_H) is a Shimura subdatum of (G, X) and $n := \dim \mathrm{Sh}_K(G, X)$, then (b) holds with $e := e^*(n)$ at every neat $J \subset H(\mathbb{A}_f)$, so with the constant $c_B := e^*(n)^{2n}$, which does not depend on (H, X_H) or J.

Proof. (a) is [37, Cor. 2, pp. 98–99] for irreducible A, B; in general each component of $A \cap B$ is a component of some $A_i \cap B_j$, and we sum over pairs of components.

(b) By Theorem 1.5 for (H, X_H) and J, the complete linear system $|\mathcal{L}_J^{\otimes e}|$ is base-point free and separates points of $\overline{\mathrm{Sh}}_J$. It defines $\varphi\colon \overline{\mathrm{Sh}}_J \to \mathbb{P}^N$ with $\varphi^*\mathcal{O}(1) \cong \mathcal{L}_J^{\otimes e}$; φ is finite, since $\mathcal{L}_J$ is ample and so φ contracts no curve, and injective. A finite injective morphism is a homeomorphism onto its closed image, so φ maps $\mathrm{Irr}(A \cap B)$ bijectively onto $\mathrm{Irr}(\varphi(A) \cap \varphi(B))$, and $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$. In characteristic 0, φ restricted to an irreducible closed Y is birational onto $\varphi(Y)$, so the projection formula gives $\deg \varphi(Y) = \deg_{\mathcal{L}_J^{\otimes e}} Y = e^{\dim Y} \deg_{\mathcal{L}_J} Y$. By (a),

$$\sum_C \deg_{\mathcal{L}_J} C \le \sum_C e^{\dim C} \deg_{\mathcal{L}_J} C = \sum_C \deg \varphi(C)$$
$$\le \deg \varphi(A) \cdot \deg \varphi(B) \le e^{a+b} \deg_{\mathcal{L}_J} A \cdot \deg_{\mathcal{L}_J} B,$$

using $e \ge 1$ and, for reducible A, B, the sum over pairs of components. For A, B in the open part, the components of $\overline{A} \cap \overline{B}$ that meet Sh_J are the closures of the components of $A \cap B$, and the others only add to the left side.

(c) As $X_H \subset X$, $n_J \le n$ for every neat J, so $e^*(n) \ge e^*(n_J)$, and $a, b \le n_J \le n$. □

Lemma 5.3. *(a) Let (H, X_H) be a Shimura datum, $I \subset H(\mathbb{A}_f)$ neat compact open, $m \in H(\mathbb{Q}_l)$, $I_m := I \cap mIm^{-1}$, and $T_m := \pi_2 \circ \pi_1^{-1}$ on $\mathrm{Sh}_I(H, X_H)$, with π_1 natural and π_2 induced by m. Then $\overline{\pi}_1^* \mathcal{L}_I \cong \mathcal{L}_{I_m} \cong \overline{\pi}_2^* \mathcal{L}_I$, and $\deg_{\mathcal{L}_I} T_m Y \le [I : I_m] \deg_{\mathcal{L}_I} Y$ for every closed Y.*

(b) If $(H, X_H) \subset (G, X)$, $K_H := K \cap H(\mathbb{A}_f)$ and $t \in Z(H)(\mathbb{A}_f)$, then $\theta_t\colon [x, h] \mapsto [x, ht]$ satisfies $\overline{\theta}_t^* \mathcal{L}_{K_H} \cong \mathcal{L}_{K_H}$.

Proof. [50, Prop. 5.3.2(1)] (Section 2.1) for the morphisms induced by 1, by m and by t; then the projection formula, π_1 being finite étale of degree at most $[I : I_m]$. □

6. The Hecke step with constants uniform in the level

Under $\mathrm{QRH}(\delta; L_V)$, a special subvariety $V \subset Z$ that is not strongly special lies in a larger special subvariety of Z once Ω_V exceeds an explicit bound in terms of $\deg Z$, the degree of the field of definition of Z, $\log d_{L_V}$ and $\log \alpha_V$ (Theorem 6.7); for maximal V the inequality fails, which bounds $\alpha_V d_{L_V}^{\mu}$ (Lemma 6.8). The argument is that of [50, §9], with constants uniform in the level of the Hecke correspondence and with Galois groups over the field of definition of Z: a criterion (Theorem 6.1), then a prime that meets it (Proposition 6.6).

In this section G is adjoint and K satisfies (Lv), and we put $n := \dim \mathrm{Sh}_K(G, X)$ and $c_B := e^*(n)^{2n}$ (Lemma 5.2(c)). Let $(P, X_P) \subset (G, X)$ be a Shimura subdatum, $K_P := K \cap P(\mathbb{A}_f)$, $F \supset E(P, X_P)$ a number field and $\bar{d} \ge 1$ an integer. For a special subvariety V put $E_V := E(H_V, X_{H_V})$; a special subvariety V of $\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$ and its image in $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$, which is special, have the same α_V, β_V and reflex field E_V [50, Remark 9.1.1]. All degrees on $\mathrm{Sh}_J(P, X_P)$ are taken for its own Baily–Borel bundle $\mathcal{L}_J$. Put $k_F := k_1 \bar{d} + f$ and, for $D' \ge 1$ and a prime l,

$$\Psi(D') := c_B\, D'^2, \qquad \Psi_l(D') := l^{k_F + 2f}\, \Psi(D') = c_B\, l^{k_F+2f} D'^2.$$

6.1. The criterion.

Theorem 6.1. *Let G be adjoint and K satisfy* (Lv)*, and let (P, X_P) and F be as above. Let $Z \subset \mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$ be Hodge generic and F-irreducible, $D_Z := \deg_{\mathcal{L}_{K_P}} Z$, and $V \subset Z$*

special with $r := \dim Z - \dim V > 0$. *Let* $\bar{d} \geq [FE_V : E_V]$ *be an integer and assume* $\mathrm{QRH}(\delta; L_V)$. *Suppose that*

(1) V *is not strongly special in* $\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$, *and*
(2) *there is a prime* $l > C_0$ *that splits completely in* L_V, *with* $l > 1/c_U$ *and* $K_{T_V,l} = K^m_{T_V,l}$, *such that*

$$\left(c_B\, l^{k_F+2f}\, \bar{d}\, D_Z\right)^{2^r} \;<\; \Omega_V. \tag{6.1}$$

Then Z *contains a special subvariety* V' *with* $V \subsetneq V'$, *in a geometric component of* Z *containing* V.

Lemma 6.2. *Let* V *be special,* $T := T_V$, *and let* $p > C_0$ *be a prime unramified in* L_V *with* $K^m_{T,p} \neq K_{T,p}$. *Then* $|K^m_{T,p}/K_{T,p}| \geq (p-1)/C_\Phi \geq c_U p$.

Proof. The proofs of [85, Lemma 3.17 and Prop. 3.15] assume p unramified in the field $F_T \supset L_T$ of [85, p. 42], and that of [85, Prop. 3.15] uses the torus $\mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_{\mathrm{m}}$ over $\mathbb{Z}_p$, while [85, Prop. 3.15] assumes only p unramified in L_T; we use it at every $p > C_0$ unramified in L_V, and there it follows from [28, §4.4] directly, with our c_U. Indeed, with $T = T_V$, as p is unramified in L_T, $T_{\mathbb{Q}_p}$ is the generic fiber of a torus $\mathcal{T}$ over $\mathbb{Z}_p$ [28, §3.3], and $\mathcal{T}(\mathbb{Z}_p) = K^m_{T,p}$, as a compact subgroup of $T(\mathbb{Q}_p)$ lies in $\mathcal{T}(\mathcal{O}_{\mathsf{k}}) \cap T(\mathbb{Q}_p) = \mathcal{T}(\mathbb{Z}_p)$ for an unramified splitting field k, and, as $K_p = G(\mathbb{Q}_p) \cap \mathrm{GL}_{n_\rho}(\mathbb{Z}_p)$ for $p > C_0$, $K_{T,p}$ is the stabilizer of $W := \mathbb{Z}_p^{n_\rho}$ in $\mathcal{T}(\mathbb{Z}_p)$, so $|K^m_{T,p}/K_{T,p}| = |\mathcal{T}(\mathbb{Z}_p) \cdot W|$, and $\mathcal{T}$ does not fix W if $K_{T,p} \neq K^m_{T,p}$. The argument of [28, §4.4] gives $|\mathcal{T}(\mathbb{Z}_p) \cdot W| \geq (p-1)/|\Phi|$, where Φ is the group of components of the stabilizer in $\mathcal{T}_{\overline{\mathbb{F}}_p}$ of a subspace; if the subspace has dimension d, the stabilizer is $\bigcap \ker(x - x')$ over the weights x, x' in the support of one vector of its d-th exterior power, which are weights of T under $X^*(\mathcal{T}_{\overline{\mathbb{F}}_p}) = X^*(T_{\overline{\mathbb{Q}}_p})$ [28, proof of Lemma 4.4.1]; so $|\Phi|$ is at most the order of the torsion of $X^*(T)/N$, with N the subgroup generated by these differences. As the map $r_T^*\colon X^*(T) \to X^*(\mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_{\mathrm{m}})$ dual to the surjection r_T of [85, p. 42] is injective, $X^*(T)/N$ embeds in $X^*(\mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_{\mathrm{m}})/r_T^*N$, whose torsion has order at most C_Φ. So $|K^m_{T,p}/K_{T,p}| \geq (p-1)/C_\Phi \geq c_U p$. □

Remark. [50, Thm. 8.1] requires $(T_V)_{\mathbb{F}_l}$ to be a torus, which [50, p. 55] draws from the converse in the proof of [85, Lemma 3.17]. The condition $l > 1/c_U$ in (2) of Theorem 6.1 is needed for the torus: a split torus whose level is maximal need not have a torus as its closure. Let $\mathbb{G}_{\mathrm{m}}$ act on $\mathbb{Q}^3$ by $t \mapsto \mathrm{diag}(1, t^2, t)$ with the lattice $\Lambda_0 := \mathbb{Z}(e_1+e_2) + 3\mathbb{Z}e_2 + \mathbb{Z}e_3$. In the basis $e_1+e_2, 3e_2, e_3$ of Λ_0, t sends e_1+e_2 to $(e_1+e_2) + \frac{t^2-1}{3}\,3e_2$, so $\mathbb{G}_{\mathrm{m}}(\mathbb{Q}_3) \cap \mathrm{GL}(\Lambda_0 \otimes \mathbb{Z}_3) = \mathbb{Z}_3^\times$ is maximal, while the closure is $\mathrm{Spec}\,\mathbb{Z}_3[t^{\pm 1}, (t^2-1)/3]$, whose special fiber is $\mathbb{Z}/2 \times \mathbb{G}_{\mathrm{a}}$. So the converse stated in the proof of [85, Lemma 3.17] (p. 43, for a torus Λ in $\mathrm{GL}_{n_\rho,\mathbb{Q}_p}$ with Zariski closure $\Lambda_{\mathbb{Z}_p}$ in $\mathrm{GL}_{n_\rho,\mathbb{Z}_p}$ and $K_{\Lambda,p} := \Lambda(\mathbb{Q}_p) \cap \mathrm{GL}_{n_\rho}(\mathbb{Z}_p)$: if $K_{\Lambda,p} = K^m_{\Lambda,p}$ is hyperspecial, then $\Lambda_{\mathbb{Z}_p}$ is a torus) holds once $p - 1 > C_\Phi$, by the argument of [28, §4.4], but not for all p: here the analogue of C_Φ is 2 and $p = 3$, and $1/c_U = 4 > 3$. The CM elliptic curve with endomorphism ring $\mathbb{Z}[\sqrt{-7}]$ gives the same phenomenon at $l = 2$ inside $(\mathrm{GL}_2, \mathcal{H}^\pm)$.

Compared with [50, (9.1)], which reads $l^{(k+2f)2^r}(\deg Z)^{2^r} < C(N)\alpha_V \beta_V^N$, the base gains the factor $c_B\,\bar{d}$ and has k_F in place of k and the f of Table 1 in place of theirs, the exponent stays 2^r, and the right side is the power form Ω_V of Corollary 3.4. As $\Psi_l^{\circ r}(x) = (c_B l^{k_F+2f})^{2^r-1} x^{2^r}$, (6.1) implies

$$\Omega_V \;>\; \Psi_l^{\circ r}(\bar{d} D_Z), \tag{6.2}$$

the form the proof uses; also $\bar{d}\,\Psi_l(D') \le \Psi_l(\bar{d}D')$ and $\Psi_l(x) \ge x$ for $x \ge 1$. The proof follows that of [50, Thm. 9.2.1], with the following differences. The Galois groups are those over F, which need not lie in E_V, at the cost of the factor $\bar{d}$ (Step 1, Lemma 6.4). The Galois lower bound is that of Corollary 3.4. Bézout's inequality, at [50, (9.5)] and in [50, Sublemma 9.2.5], has the constant of Lemma 5.2. In Step 3, as P need not be adjoint, the proof of [50, Thm. 7.2.1] is run on the image of $\widetilde{Z}$ (the union of the conjugates of a component of the preimage of Z at the Iwahori level, Step 2) in the Shimura variety $\mathrm{Sh}_{I^{\mathrm{ad}}}(P^{\mathrm{ad}}, X_P^{\mathrm{ad}})$ of the adjoint group at an adjoint level I^{ad}. Finally, $\widetilde{Y}_H$, the lift of the union $\widetilde{Y}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-conjugate components of $\widetilde{Z} \cap T_m\widetilde{Z}$ (Step 4) to the Shimura variety of the smallest special subvariety containing it, is chosen so as to be Hodge generic of dimension $\dim \widetilde{Y}$ (Lemma 6.5).

Step 1 of the proof of Theorem 6.1 follows [50, §8.3.2], which rests on [50, Lemma 8.1.6]. The proof of (a) there takes the point fixed by a maximal compact subgroup to lie in the apartment of a maximal split torus chosen in advance; this can fail for small l [81, 2.1 and 3.6.1] (for SL_2 over $\mathbb{Q}_3$ and the diagonal torus T, $T(\mathbb{Z}_3)$ fixes every vertex adjacent to the apartment of T). The proof of (b) reduces to a hyperspecial subgroup [81, 3.8.2], which $P(\mathbb{Q}_l)$ need not have, as (2) does not make P unramified at l. The following lemma gives what Step 1 uses. For a connected reductive $\mathbb{Q}_l$-group H we write $\mathcal{B}(H)$ for its extended Bruhat–Tits building [81, 1.2 and 2.1] and $H(\mathbb{Q}_l)^{\Omega}$ for the fixer of $\Omega \subset \mathcal{B}(H)$ in $H(\mathbb{Q}_l)$; as in [50, §8.1.1 and Remark 8.1.1], the fixers of chambers are called Iwahori subgroups.

Lemma 6.3. *Let l be a prime and H a connected reductive $\mathbb{Q}_l$-group, and put $d := \dim H$.*

(a) Let $T \subset H$ be a split torus such that no nonzero weight of T on $\mathrm{Lie}\,H$ lies in $(l-1)X^*(T)$, and let $\Theta \subset T(\mathbb{Q}_l)$ be the subgroup of elements of order dividing $l-1$. Then every point of $\mathcal{B}(H)$ fixed by Θ lies in the apartment of a maximal split torus $S \supset T$ of H, and the maximal compact subgroup of $S(\mathbb{Q}_l)$ fixes that apartment pointwise.

(b) If a point $y \in \mathcal{B}(H)$ lies in the closure of a chamber C, then $[H(\mathbb{Q}_l)^y : H(\mathbb{Q}_l)^C] \le l^{d(d+1)/2}$.

(c) Let (P, X_P), V and l be as in Theorem 6.1, with (2), and $K_{P,l} := K_l \cap P(\mathbb{Q}_l)$. Every maximal compact subgroup $K_l^1 \supset K_{P,l}$ of $P(\mathbb{Q}_l)$ is $P(\mathbb{Q}_l)^y$ for a point y of the apartment of a maximal split torus $S \supset T_{V,\mathbb{Q}_l}$ of $P_{\mathbb{Q}_l}$. Let C^1 be a chamber of this apartment with y in its closure, x_0 a special point in the closure of C^1, and C^2 a chamber of the apartment with x_0 in its closure. Then $I_l^1 := P(\mathbb{Q}_l)^{C^1} \subset K_l^1$ and $I_l^2 := P(\mathbb{Q}_l)^{C^2}$ are in good position with respect to S, $I_l := I_l^1 \cap I_l^2 \cap K_{P,l}$ is in good position with respect to T_V, and $[K_{P,l} : I_l] < l^f$ with $f := (\dim G + 1)^2$.

Proof. For a maximal split torus S of H, the group $Z_H(S)(\mathbb{Q}_l)$ acts on the apartment $\mathcal{A}(S)$ by translations through the homomorphism ν of [81, 1.2], which vanishes on bounded elements; so the maximal compact subgroup of $S(\mathbb{Q}_l)$ fixes $\mathcal{A}(S)$ pointwise. Also $h\mathcal{A}(S) = \mathcal{A}(hSh^{-1})$ for $h \in H(\mathbb{Q}_l)$ [81, 2.1], and fixers of points are compact open [81, 3.2 and 3.4.1].

(a) Let y be fixed by Θ. Choose a maximal split torus $S \supset T$ and $\lambda \in X_*(T)$ with $\langle \beta, \lambda\rangle \ne 0$ for every nonzero weight β of T on $\mathrm{Lie}\,H$. By [18, Thm. 4.1.7], $P_H(\lambda) = Z_H(\lambda) \ltimes U_H(\lambda)$, where $U := U_H(\lambda)$ is unipotent, $\mathrm{Lie}\,U$ is the sum of the weight spaces $(\mathrm{Lie}\,H)_\beta$ with $\langle \beta, \lambda\rangle > 0$, and $\lambda(t)u\lambda(t)^{-1} \to 1$ as $t \to 0$ for $u \in U(\mathbb{Q}_l)$; and $Z_H(\lambda) = Z_H(T) =: L$, as $Z_H(T) \subset Z_H(\lambda)$ and both are connected with Lie algebra $(\mathrm{Lie}\,H)_0$, by the choice of λ. Let $D \subset X_*(S) \otimes \mathbb{R}$ be a vector chamber whose closure contains λ, and U_D the group generated by the root groups $U_a(\mathbb{Q}_l)$ with a positive on D [81, 3.3]; λ has weights ≥ 0 on their Lie algebras, so $U_D \subset P_H(\lambda)(\mathbb{Q}_l) = U(\mathbb{Q}_l)L(\mathbb{Q}_l)$. The Iwasawa decomposition

$H(\mathbb{Q}_l) = U_D\, N_H(S)(\mathbb{Q}_l)\, P_x$, for $x \in \mathcal{A}(S)$ and P_x its stabilizer [15, Cor. 7.3.2(i)], which applies to $H(\mathbb{Q}_l)$ by [16, Thm. 5.1.20], gives $\mathcal{B}(H) = U_D \cdot \mathcal{A}(S)$; so $y = uu_2 \cdot x$ with $u \in U(\mathbb{Q}_l)$, $u_2 \in L(\mathbb{Q}_l)$ and $x \in \mathcal{A}(S)$. The point $z := u_2 \cdot x$ lies in $\mathcal{A}(u_2 S u_2^{-1})$ and $u_2 S u_2^{-1} \supset T$, so Θ fixes z; it remains to show $u \cdot z = z$.

The group $U_z := U(\mathbb{Q}_l) \cap H(\mathbb{Q}_l)^z$ is compact open in $U(\mathbb{Q}_l)$ and normalized by Θ. For n large, $\lambda(l)^n u \lambda(l)^{-n} \in U_z$, so u lies in the compact subgroup $\Gamma := \lambda(l)^{-n} U_z \lambda(l)^n$, which Θ normalizes, as it commutes with $\lambda(l)$. Compact subgroups of $U(\mathbb{Q}_l)$ are pro-l, as U has a composition series with quotients $\mathbb{G}_{\mathrm{a}}$. For $\theta \in \Theta$ put $c_\theta := u^{-1}\theta u \theta^{-1} \in \Gamma$; as $\theta u \theta^{-1} \cdot z = \theta \cdot y = y = u \cdot z$, c_θ lies in the pro-l group $\Delta := \Gamma \cap U_z$, and $\theta \mapsto c_\theta$ is a 1-cocycle of Θ in Δ. As $|\Theta|$ is prime to l, it is a coboundary (by the Schur–Zassenhaus theorem on the finite quotients of Δ by its open normal Θ-stable subgroups, and compactness): $c_\theta = u'^{-1}\theta u' \theta^{-1}$ for some $u' \in \Delta$ and all $\theta \in \Theta$. Then $v := uu'^{-1} \in U(\mathbb{Q}_l)$ commutes with Θ. The torus T is split, so $\Theta = \mathrm{Hom}(X^*(T), \mu_{l-1})$, and Θ acts on each $(\mathrm{Lie}\, H)_\beta \subset \mathrm{Lie}\, U$ through $\beta|_\Theta \neq 1$, as $\beta \notin (l-1)X^*(T)$; so $(\mathrm{Lie}\, U)^\Theta = 0$, and since $\log\colon U \to \mathrm{Lie}\, U$ is a T-equivariant isomorphism, $v = 1$. Hence $u = u' \in U_z$, and $y = u \cdot z = z \in \mathcal{A}(u_2 S u_2^{-1})$.

(b) Let S be a maximal split torus whose apartment contains C, hence y; put $B := H(\mathbb{Q}_l)^C \subset H(\mathbb{Q}_l)^y$, and let $\widetilde{W}$ be the group of affine transformations of $\mathcal{A}(S)$ induced by $N_H(S)(\mathbb{Q}_l)$ [81, 1.2], and $q_w := |BwB/B|$ for $w \in \widetilde{W}$ [81, 3.3.1]. By the Bruhat decomposition [81, 3.3.1], $H(\mathbb{Q}_l)^y$ is the disjoint union of the BwB with w in the stabilizer $\widetilde{W}_y$ of y in $\widetilde{W}$ (as $B \subset H(\mathbb{Q}_l)^y$, and $n \in N_H(S)(\mathbb{Q}_l)$ fixes y if and only if its image in $\widetilde{W}$ does), so $[H(\mathbb{Q}_l)^y : B] = \sum_{w \in \widetilde{W}_y} q_w$. An element of $\widetilde{W}_y$ with trivial image in the finite Weyl group $W_0 := N_H(S)(\mathbb{Q}_l)/Z_H(S)(\mathbb{Q}_l)$ is a translation fixing y, hence trivial; so $|\widetilde{W}_y| \leq |W_0| \leq 2^N$, where N is the number of reflections of W_0 (a Weyl chamber is determined by its signs on the N reflecting hyperplanes), and $2N \leq d$, as $\mathrm{Lie}\, H$ contains the nonzero root spaces of $2N$ distinct roots $\pm a$, one pair for each reflecting hyperplane. For $w \in \widetilde{W}_y$ write $w = r_1 \cdots r_j \omega$ with $\omega C = C$ and $r_1 \cdots r_j$ reduced, as in [81, 3.3.1], and let v_i be the vertex of the local Dynkin diagram representing r_i, with the integer $d(v_i)$ of [81, 1.8]; then $q_w = l^{d(v_1)+\cdots+d(v_j)}$, and j is the number of walls separating C from wC. Each of these walls contains y, which lies in the closures of C and of wC, and distinct walls through y have distinct directions, which are kernels of roots; so $j \leq N$. Each $d(v_i)$ is the dimension of a homogeneous variety of the special fiber of a smooth model of H [81, 3.5.4 and the paragraph after it], so $d(v_i) \leq d$. Hence $[H(\mathbb{Q}_l)^y : B] \leq 2^N l^{Nd} \leq l^{N(d+1)} \leq l^{d(d+1)/2}$.

(c) By [81, 3.2], $K_l^1 = P(\mathbb{Q}_l)^y$ for a point y. A nonzero weight of T_V on $\mathrm{Lie}\, P \subset \mathfrak{gl}_{n_\rho}$ is $\chi - \chi'$ for weights χ, χ' of T_V in ρ, and does not lie in $(l-1)X^*(T_V)$: otherwise the group of components of $\ker(\chi - \chi')$, which is the stabilizer in T_V of the line through $v_\chi + v_{\chi'}$ for weight vectors v_χ, $v_{\chi'}$ [28, proof of Lemma 4.4.1], would have order divisible by $l-1$, over $\overline{\mathbb{Q}}$ as over $\overline{\mathbb{F}}_l$, whereas these orders are at most C_Φ (Section 2.3 and Lemma 6.2) and $l - 1 > 2C_\Phi - 1 \geq C_\Phi$ by (2), as $c_U = 1/(2C_\Phi)$ and $C_\Phi \geq 1$. The torus $T_{V,\mathbb{Q}_l}$ is split, as l splits completely in L_V, and its elements of order dividing $l-1$ lie in $K^m_{T_V,l} = K_{T_V,l} = T_V(\mathbb{Q}_l) \cap K_l \subset K_{P,l} \subset K_l^1$, by (2) and Section 2.2; so they fix y, and (a), with $H = P_{\mathbb{Q}_l}$ and $T = T_{V,\mathbb{Q}_l}$, gives S. The maximal compact subgroup of $S(\mathbb{Q}_l)$ fixes the apartment of S, so it lies in I_l^1 and I_l^2; it contains that of $T_V(\mathbb{Q}_l)$, which also lies in $K_{P,l}$. As y and x_0 lie in the closure of C^1, and x_0 in that of C^2, we have $I_l^1 \subset K_l^1$ and $I_l^1, I_l^2 \subset P(\mathbb{Q}_l)^{x_0}$; so (b), applied to (y, C^1) and (x_0, C^2) with $d = \dim P \leq \dim G$, gives

$$[K_{P,l} : I_l] \leq [K_l^1 : I_l^1 \cap I_l^2] \leq [K_l^1 : I_l^1]\,[P(\mathbb{Q}_l)^{x_0} : I_l^2] \leq l^{d(d+1)} < l^f. \qquad \square$$

Lemma 6.4 (Lower bound at every level)**.** *Let $(P, X_P) \subset (G, X)$ be a Shimura subdatum, $K_P := K \cap P(\mathbb{A}_f)$, $J \subset K_P$ compact open, $V \subset \mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$ special with $T_V \neq 1$, defined by $(H_V, X_{H_V}) \subset (P, X_P)$, and $\widetilde{V}$ a component of the preimage of V in $\mathrm{Sh}_J(P, X_P)_{\mathbb{C}}$. If $\mathrm{QRH}(\delta; L_V)$ holds, then, with Ω_V as in Corollary 3.4,*

$$\deg_{\mathcal{L}_J}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot \widetilde{V}) \;\geq\; \deg_{\mathcal{L}_{K_P}}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot V) \;\geq\; \Omega_V,$$

and $\deg_{\mathcal{L}_J}(\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V) \cdot \widetilde{V}) \geq \Omega_V/\bar{d}$ when $\bar{d} \geq [FE_V : E_V]$.

Proof. The first inequality is the projection formula for the finite étale $\tau\colon \mathrm{Sh}_J \to \mathrm{Sh}_{K_P}$, defined over $E(P, X_P) \subset E_V$, with $\tau^*\mathcal{L}_{K_P} = \mathcal{L}_J$ [50, Prop. 5.3.2(1)]: each component C of the orbit has $\deg_{\mathcal{L}_J} C = \deg(\tau|_C) \deg_{\mathcal{L}_{K_P}} \tau(C)$, and τ maps the orbit of $\widetilde{V}$ onto that of V. For the second, let V_H be a geometric component of $\mathrm{Sh}_{K_{H_V}}(H_V, X_{H_V})$, $K_{H_V} := K_P \cap H_V(\mathbb{A}_f) = K \cap H_V(\mathbb{A}_f)$, mapping onto V. The natural map q is defined over E_V, injective on the Hodge generic locus [85, Lemma 2.2], and the components of $\mathrm{Sh}_{K_{H_V}}(H_V, X_{H_V})$ are Hodge generic, so distinct conjugates of V_H have distinct images and [50, Cor. 5.3.10] ($Z(H_V)(\mathbb{R})$ is compact) gives $\deg_{\mathcal{L}_{K_P}}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot V) \geq \deg_{\mathcal{L}_{K_{H_V}}}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot V_H)$, which is $\geq \Omega_V$ by Corollary 3.4(i) with $F = E_V$. For the last assertion, $\mathcal{L}_J$ and $\overline{\mathrm{Sh}}_J(P, X_P)$ are defined over $E(P, X_P) \subset E_V$, so $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$-conjugate subvarieties of $\mathrm{Sh}_J(P, X_P)_{\mathbb{C}}$ have equal degrees; and $\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V)$ has index $[FE_V : E_V] \leq \bar{d}$ in $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$, so the $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$-orbit of $\widetilde{V}$ is the union of at most $\bar{d}$ conjugates of $\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V) \cdot \widetilde{V}$. □

Descent. Let Z, V and $\bar{d}$ be as in Theorem 6.1, with (1) and $\mathrm{QRH}(\delta; L_V)$, and let $J \subset K_P$ be compact open, $\tau\colon \mathrm{Sh}_J(P, X_P) \to \mathrm{Sh}_{K_P}(P, X_P)$ the natural map, $\widetilde{V}$ a component of $\tau^{-1}(V)$ in the image of $X_P^+ \times \{1\}$, and $\widetilde{Y} \subset \mathrm{Sh}_J(P, X_P)_{\mathbb{C}}$ closed and F-irreducible with $\widetilde{V} \subset \widetilde{Y}$, $\tau(\widetilde{Y}) \subset Z$ and $r_Y := \dim \widetilde{Y} - \dim \widetilde{V} > 0$. Let $\widetilde{Y}^{(1)}$ be a geometric component of $\widetilde{Y}$ containing $\widetilde{V}$, and S_Y the smallest special subvariety of $\mathrm{Sh}_J(P, X_P)_{\mathbb{C}}$ containing $\widetilde{Y}^{(1)}$. It is the image of $X_H^+ \times \{1\}$ for a Shimura subdatum $(H, X_H) \subset (P, X_P)$ with H the generic Mumford–Tate group on X_H and $X_H^+ \subset X_P^+$ [85, Lemma 2.1]. Put $K_H := K_P \cap H(\mathbb{A}_f)$, $J_H := J \cap H(\mathbb{A}_f)$ and $F_H := E(H, X_H)$; then $F_H \subset E_V$, because a Hodge generic point of $\widetilde{V}$ lifts to X_H^+, so that (H, X_H) contains a $P(\mathbb{Q})$-conjugate of (H_V, X_{H_V}). Let q denote the natural maps $\mathrm{Sh}_{J_H}(H, X_H) \to \mathrm{Sh}_J(P, X_P)$ and $\mathrm{Sh}_{K_H}(H, X_H) \to \mathrm{Sh}_{K_P}(P, X_P)$, and τ also the map $\mathrm{Sh}_{J_H}(H, X_H) \to \mathrm{Sh}_{K_H}(H, X_H)$; these maps are finite and defined over F_H, and $q\tau = \tau q$. Let S_H be the image of $X_H^+ \times \{1\}$ in $\mathrm{Sh}_{J_H}(H, X_H)_{\mathbb{C}}$, so that $q(S_H) = S_Y$. As q is finite, some component C_1 of $q^{-1}(\widetilde{Y}^{(1)}) \cap S_H$ maps onto $\widetilde{Y}^{(1)}$, and some component $\widetilde{V}_H$ of $q^{-1}(\widetilde{V}) \cap C_1$ maps onto $\widetilde{V}$; so $\dim C_1 = \dim \widetilde{Y}$ and $\dim \widetilde{V}_H = \dim \widetilde{V}$. Let $\widetilde{Y}_H$ be the union of the $\mathrm{Gal}(\overline{\mathbb{Q}}/FF_H)$-conjugates of C_1, and put $V_H := \tau(\widetilde{V}_H)$ and $Y_H := \tau(\widetilde{Y}_H) \subset \mathrm{Sh}_{K_H}(H, X_H)_{\mathbb{C}}$.

Lemma 6.5. *In this situation Y_H is FF_H-irreducible and Hodge generic, $q(V_H) = V$ and V_H has torus T_V, and $\dim Y_H - \dim V_H = r_Y$* [50, pp. 48–49]*. Moreover:*

- *(a)* $\deg_{\mathcal{L}_{K_H}} Y_H \leq \deg_{\mathcal{L}_J} \widetilde{Y}$*;*
- *(b) if $\Psi^{\circ r_Y}(\bar{d} \deg_{\mathcal{L}_{K_H}} Y_H) < \Omega_V$, then V_H is not strongly special in $\mathrm{Sh}_{K_H}(H, X_H)_{\mathbb{C}}$;*
- *(c) if Y_H contains a special $V_H' \supsetneq V_H$, then Z contains a special $V' \supsetneq V$ in a geometric component of Z containing V.*

Proof. A special subvariety $S' \subsetneq S_H$ containing C_1 would give the special subvariety $q(S') \supset \widetilde{Y}^{(1)}$, of dimension less than $\dim S_Y$; so C_1 is Hodge generic in $\mathrm{Sh}_{J_H}(H, X_H)_{\mathbb{C}}$. Galois

conjugates of special subvarieties are special [50, proof of Lemma 7.1.2], so every component $\sigma(C_1)$ of $\widetilde{Y}_H$ is Hodge generic, of dimension $\dim \widetilde{Y}$, and $\widetilde{Y}_H$ is closed and FF_H-irreducible. Hence Y_H is closed, FF_H-irreducible and Hodge generic. As τ and q are finite, $\dim Y_H - \dim V_H = \dim \widetilde{Y} - \dim \widetilde{V} = r_Y$. The irreducible closed set $\widetilde{V}_H$ has the dimension of $q^{-1}(\widetilde{V})$, so it is a component of it, hence special, and so is V_H; and $q(V_H) = \tau(q(\widetilde{V}_H)) = \tau(\widetilde{V}) = V$. As V_H lies in the image of $X_H^+ \times \{1\}$, it is the image of $X_{H''}^+ \times \{1\}$ for a subdatum $(H'', X_{H''}) \subset (H, X_H)$ with H'' the generic Mumford–Tate group on $X_{H''}$ and $X_{H''}^+ \subset X_H^+$ [85, Lemma 2.1], and then so is $V = q(V_H)$ in $\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$. So $X_{H''}^+ = \gamma X_{H_V}^+$ for some $\gamma \in P(\mathbb{Q})_+ \cap K_P$, and we may take $(H'', X_{H''})$ for (H_V, X_{H_V}) (Section 2.2): V_H and V have the same torus T_V and the same $K_{T_V} = T_V(\mathbb{A}_f) \cap K$, hence the same L_V, E_V, α_V and Ω_V (compare [50, Remark 9.1.1]).

(a) Each geometric component of Y_H is $\tau(C)$ for a component C of $\widetilde{Y}_H$, and $\deg_{\mathcal{L}_{J_H}} C = \deg(\tau|_C) \deg_{\mathcal{L}_{K_H}} \tau(C)$ by the projection formula, so $\deg_{\mathcal{L}_{K_H}} Y_H \le \deg_{\mathcal{L}_{J_H}} \widetilde{Y}_H$. For a component $C = \sigma(C_1)$ of $\widetilde{Y}_H$, $q(C) = \sigma(\widetilde{Y}^{(1)})$ is a geometric component of $\widetilde{Y}$, because q is defined over F_H and $\widetilde{Y}$ over F. Distinct C give distinct components, as q is injective on the Hodge generic locus [85, Lemma 2.2], and $\deg_{\mathcal{L}_{J_H}} C \le \deg_{\mathcal{L}_J} q(C)$ [50, Cor. 5.3.10], as $Z(H)(\mathbb{R})$ is compact. So $\deg_{\mathcal{L}_{J_H}} \widetilde{Y}_H \le \deg_{\mathcal{L}_J} \widetilde{Y}$.

(c) $V' := q(V'_H)$ is special; it lies in $q(Y_H) = \tau(q(\widetilde{Y}_H)) \subset \tau(\widetilde{Y}) \subset Z$ and contains $q(V_H) = V$; and $\dim V' = \dim V'_H > \dim V_H = \dim V$, as q is finite. Being irreducible, V' lies in a geometric component of Z, which contains V.

(b) Suppose V_H is strongly special in $\mathrm{Sh}_{K_H}(H, X_H)_{\mathbb{C}}$, that is, $T_V \subset Z(H)$. Then $T_V = Z(H)^\circ$ [50, Lemma 6.2]. Put $K_H^m := K_{T_V}^m K_H$ and let $\pi\colon \mathrm{Sh}_{K_H}(H, X_H) \to \mathrm{Sh}_{K_H^m}(H, X_H)$ be the natural map. The group $K_{T_V}^m / K_{T_V}$ acts on $\mathrm{Sh}_{K_H}(H, X_H)$ by the maps θ_t of Lemma 5.3(b), which are defined over F_H and so commute with $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$, and its orbits are the fibers of π. Let $\Theta_A \subset K_{T_V}^m / K_{T_V}$ be the group of Corollary 3.4(ii). Then $\Theta_A \cdot V_H \subset \mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot V_H$ [50, Sublemma 9.2.4]. Write deg for $\deg_{\mathcal{L}_{K_H}}$ and $\mathcal{Y} := \mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot Y$ for geometrically irreducible Y; $\deg \theta\mathcal{Y} = \deg \mathcal{Y}$ by Lemma 5.3(b), and $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$-conjugate subvarieties have equal degrees, as $\mathcal{L}_{K_H}$ is defined over $F_H \subset E_V$. Let Y_1 be a geometric component of Y_H containing V_H. As Y_H is defined over $FF_H \subset FE_V$, it contains $\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V) \cdot Y_1$, and $\mathcal{Y}_1$ is the union of at most $\bar{d}$ conjugates of this set; so $\deg \mathcal{Y}_1 \le D_1 := \bar{d} \deg Y_H$.

Suppose Y_j is geometrically irreducible, $V_H \subset Y_j$, $\dim Y_j > \dim V_H$ and $\deg \mathcal{Y}_j \le D_j := \Psi^{\circ(j-1)}(D_1)$. If $\Theta_A \cdot Y_j \subset \mathcal{Y}_j$, stop. Otherwise choose $\theta \in \Theta_A$ with $\theta Y_j \not\subset \mathcal{Y}_j$. The sets $\mathcal{Y}_j$ and $\theta\mathcal{Y}_j$ are of pure dimension $\dim Y_j$, and they have no common component: the components of $\theta\mathcal{Y}_j$ are the $\sigma\theta Y_j$, and $\sigma\theta Y_j = \sigma' Y_j$ would give $\theta Y_j \subset \mathcal{Y}_j$. Also $V_H \subset \mathcal{Y}_j \cap \theta\mathcal{Y}_j$: as Θ_A is a group, $\theta^{-1} V_H = \sigma V_H \subset \sigma Y_j$ for some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$. Let Y_{j+1} be a geometric component of $\mathcal{Y}_j \cap \theta\mathcal{Y}_j$ containing V_H. This set is $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$-stable, so the conjugates of Y_{j+1} are among its components; $\dim Y_{j+1} < \dim Y_j$; and Lemma 5.2(c) at the neat level K_H gives

$$\deg \mathcal{Y}_{j+1} \le c_B D_j^2 = D_{j+1}.$$

The dimension drops at each step, so $j \le r_Y$ throughout, and $D_j \le D_{j+1} \le \Psi^{\circ r_Y}(D_1) < \Omega_V$, as $\Psi(x) \ge x$. If $Y_{j+1} = V_H$, then $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot V_H) \le D_{j+1} < \Omega_V$, contradicting Lemma 6.4 for V_H at the level K_H. Otherwise we continue. So the process stops at some Y' with $\Theta_A \cdot Y' \subset \mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot Y'$ and $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot Y') < \Omega_V$. As the orbits of Θ_A lie in the fibers of π, also $\Theta_A \cdot Y' \subset \pi^{-1}\pi(Y')$, and Y' lies in the geometric component of $\mathrm{Sh}_{K_H}(H, X_H)_{\mathbb{C}}$ containing V_H; so Corollary 3.4(ii), applied to this component, whose image

has torus $Z(H)^\circ = T_V$ and hence the same L_V, α_V and Ω_V, with $F = E_V \supset E(H, X_H)$ and $Y = Y'$, gives $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot Y') \geq \Omega_V$: a contradiction. □

Proof of Theorem 6.1. We prove the theorem with (6.2) in place of (6.1), by induction on r, for all subdata (P, X_P), fields F, and Z, V, $\bar{d}$ and l at once. Recall that V is the image of $X^+_{H_V} \times \{1\}$ with $X^+_{H_V} \subset X^+_P$ (Section 2.2).

Step 1 (the level I, the element m and σ). Put $K_{P,l} := K_l \cap P(\mathbb{Q}_l)$. As $l > C_0$, $K_l = G(\mathbb{Z}_l)$ is hyperspecial, and l splits T_V. Under (2) the Zariski closure of T_V in $\mathrm{GL}_{n_\rho,\mathbb{Z}_l}$ is a torus, by [28, Lemma 3.3.1 and §4.4] applied to the split torus model of $T_{V,\mathbb{Q}_l}$ and the lattice $\mathbb{Z}_l^{n_\rho}$, so $(T_V)_{\mathbb{F}_l}$ is a torus, as [50, Thm. 8.1] requires. So (P, X_P), V and l are as in [50, Thm. 8.1], whose proof [50, §8.3] we follow. By (1) the image of T_V in P^{ad} is not trivial, and [28, Prop. 7.4.3] gives $m_0 \in T_V(\mathbb{Q}_l)$ whose image in $P^{\mathrm{ad}}(\mathbb{Q}_l)$ lies in no compact subgroup, with $[K_{P,l} : K_{P,l} \cap m_0^{Aj} K_{P,l} m_0^{-Aj}] < l^{k_1 j}$ for every integer $j \geq 1$: for $j = 1$ this is [50, (8.5), (8.6)], and the proof of [28, Prop. 7.4.3], which bounds this index for the powers m_0^{Aj} of its element (m^e there, with $m = m_0$, $e = Aj$), chooses m_0 independently of j and gives an exponent linear in j. There is a compact open subgroup $I_l \subset K_{P,l}$, the intersection of $K_{P,l}$ with two Iwahori subgroups, one chosen with respect to m_0 [50, Def. 8.3.1], such that $[K_{P,l} : I_l] \leq l^f$ [50, (8.8)] and, for every $j \geq 1$ and all $u_1, u_2 \in I_l$, the image of $u_1 m_0^{Aj} u_2$ in $P^{\mathrm{ad}}(\mathbb{Q}_l)$ generates an unbounded subgroup [50, (8.7), Prop. 8.1.4 and §8.3.3]. Here [50, §8.3.2] is applied with the torus S, the chambers C^1, C^2 and the groups I_l^1, I_l^2, I_l of Lemma 6.3(c), with $K_l^m := P(\mathbb{Q}_l)^{x_0}$ and C^2 the chamber that [50, §8.3.2] then chooses with respect to m_0; this also gives [50, (8.8)], whether or not $P(\mathbb{Q}_l)$ has hyperspecial subgroups. Put $I := K_P^l I_l$, with $K_P^l := \prod_{p \neq l}(K_p \cap P(\mathbb{Q}_p))$, and let $\tau\colon \mathrm{Sh}_I(P, X_P) \to \mathrm{Sh}_{K_P}(P, X_P)$ be the natural map, which is finite étale of degree at most $[K_{P,l} : I_l] \leq l^f$ and defined over $E(P, X_P) \subset F$. Let $\widetilde{V}$ be the image of $X^+_{H_V} \times \{1\}$ in $\mathrm{Sh}_I(P, X_P)_{\mathbb{C}}$, a component of $\tau^{-1}(V)$.

The reciprocity map $r_{(H_V, X_{H_V})}$ is a homomorphism from $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$ to an abelian quotient of $H_V(\mathbb{A})$ [85, §2.1], and by [85, Prop. 2.9] the class of m_0^A there is $r_{(H_V, X_{H_V})}(\sigma_0)$ for some $\sigma_0 \in \mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$. The image of $\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V)$ has index $j \leq [FE_V : E_V] \leq \bar{d}$ in that of $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$, so the class of $m := m_0^{Aj}$ is $r_{(H_V, X_{H_V})}(\sigma)$ for some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/FE_V)$. Then $\sigma(\widetilde{V})$ is the image of $X^+_{H_V} \times \{m\}$ [50, proof of Prop. 8.2.1], so $\sigma(\widetilde{V}) \subset T_m \widetilde{V}$, with T_m the correspondence of Lemma 5.3(a) on $\mathrm{Sh}_I(P, X_P)$. As in [50, (8.9)],

$$[I_l : I_l \cap m I_l m^{-1}] \;\leq\; [K_{P,l} : K_{P,l} \cap m K_{P,l} m^{-1}] \cdot [K_{P,l} : I_l] \;<\; l^{k_1 j + f} \;\leq\; l^{k_F}.$$

So [50, Thm. 8.1] holds with $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/FE_V)$ and k_F in place of $k_1 + f$.

Step 2 (the intersection $\widetilde{Z} \cap T_m \widetilde{Z}$). Let $\widetilde{Z}$ be the union of the $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-conjugates of a geometric component of $\tau^{-1}(Z)$ containing $\widetilde{V}$. Then $\widetilde{Z}$ is F-irreducible and $\tau(\widetilde{Z}) = Z$; it is Hodge generic, as Z is; and the projection formula and Lemma 5.3(a) give, as in [50, (9.4)],

$$\deg_{\mathcal{L}_I} \widetilde{Z} \leq l^f D_Z, \qquad \deg_{\mathcal{L}_I} T_m \widetilde{Z} \leq [I_l : I_l \cap m I_l m^{-1}] \deg_{\mathcal{L}_I} \widetilde{Z} < l^{k_F + f} D_Z.$$

The set $T_m \widetilde{Z}$ is closed and defined over F. By Step 1, $\sigma \widetilde{V} \subset T_m \widetilde{V} \subset T_m \widetilde{Z}$, and σ fixes F, so $\widetilde{V} \subset T_m \widetilde{Z}$. Hence $\widetilde{V} \subset \widetilde{Z} \cap T_m \widetilde{Z}$, and as this set is defined over F, it contains $\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V) \cdot \widetilde{V}$. The level I is neat, as $I \subset K_P$, so Lemma 5.2(c) applies on $\mathrm{Sh}_I(P, X_P)$:

$$\sum_{C \in \mathrm{Irr}(\widetilde{Z} \cap T_m \widetilde{Z})} \deg_{\mathcal{L}_I} C \;\leq\; c_B \cdot l^f D_Z \cdot l^{k_F + f} D_Z \;=\; \Psi_l(D_Z). \tag{6.3}$$

Step 3 (the case $\widetilde{Z} \subset T_m\widetilde{Z}$). We use the notation of Section 7.2: $P^{\mathrm{ad}} = P_1 \times \cdots \times P_s$ is the decomposition into $\mathbb{Q}$-simple factors, with projections π_i, and $X_P^+ = X_1^+ \times \cdots \times X_s^+$ the corresponding decomposition; we write $\mathrm{pr}_{\neq i} := \mathrm{pr}_{\{i\}^c}$ for the projection to the factors other than X_i^+ and $x_{\neq i} := \mathrm{pr}_{\{i\}^c}(x)$, and a subset $\widehat{Y} \subset X_P^+$ is X_i^+-saturated if $\widehat{Y} = X_i^+ \times \mathrm{pr}_{\neq i}(\widehat{Y})$. We obtain the conclusion of [50, Thm. 7.2.1] for (P, X_P) at the level I, with $p_0 = 3$, by its proof run as described below: the factor $K_3 \cap P(\mathbb{Q}_3)$ of I is neat by (Lv), and $l \neq 3$, as the hyperspecial K_l is not neat. Like V, $\widetilde{V}$ is defined by (H_V, X_{H_V}) and is not strongly special, and the unboundedness condition is that of Step 1. Three points of the proof of [50, Thm. 7.2.1] need care here, as P need not be adjoint and $\widetilde{Z}$ is only F-irreducible: the element q, the adjoint image and saturation.

Step 3a. The element q and the group Γ can be taken in the level. In the proof of [50, Lemma 7.2.8], in its notation, the element q and every element of the group $\Gamma = \mathcal{K}_1 \cap G_1(\mathbb{Q})_+$ must lie in the level group at every prime other than l and have l-component in U_l, so that $[\gamma x, 1] = [x, \gamma_l^{-1}]$ lies in the image of $\widetilde{V} \cdot U_l$ for all $\gamma \in \Gamma_q$; q is found by strong approximation and Γ is replaced by its subgroup of finite index $\{\gamma : \gamma_l \in U_l\}$, as in Step 2 of the proof of Lemma 7.2.

Step 3b. The image of $\widetilde{Z}$ in the adjoint Shimura variety has Zariski dense monodromy. The reduction to the adjoint group in [50, Lemma 7.2.2] gives V' only in $\jmath^{-1}(\jmath(\widetilde{Z}))$, for the map $\jmath$ below, and [50, Lemmas 7.2.3–7.2.4] cannot be run on P itself, since the closure of the monodromy is not open in $K_{P,l}$ when $Z(P)^\circ \neq 1$. Let $\widetilde{Z}^{(1)}$ be a geometric component of $\widetilde{Z}$ containing $\widetilde{V}$, and $\jmath\colon \mathrm{Sh}_I(P, X_P) \to \mathrm{Sh}_{I^{\mathrm{ad}}}(P^{\mathrm{ad}}, X_P^{\mathrm{ad}})$ the finite morphism of the proof of [50, Lemma 7.2.2] with $p_0 = 3$, where, for the adjoint map $\lambda\colon P \to P^{\mathrm{ad}}$, $I_3^{\mathrm{ad}} = \lambda(I_3)$, $I_l^{\mathrm{ad}} = \lambda(I_l)$ and I_p^{ad} is a maximal compact subgroup containing $\lambda(I_p)$ for $p \neq 3, l$; I^{ad} is neat by (Lv), $\jmath(\widetilde{Z})$ is F-irreducible and Hodge generic, $\jmath(\widetilde{Z}) \subset T_{\lambda(m)}\jmath(\widetilde{Z})$, and the unboundedness condition holds for $\lambda(I_l)$ [50, pp. 32–33]. We therefore run [50, Lemmas 7.2.3–7.2.6] on $\jmath(\widetilde{Z}) \subset \mathrm{Sh}_{I^{\mathrm{ad}}}(P^{\mathrm{ad}}, X_P^{\mathrm{ad}})$, where the monodromy of $\jmath(\widetilde{Z}^{(1)})$, a geometric component of $\jmath(\widetilde{Z})$ containing $\jmath(\widetilde{V})$, is Zariski dense by the argument of Step 1 of the proof of Lemma 7.2.

Step 3c. $\widetilde{Z}$ contains a special subvariety $\widetilde{V}' \supsetneq \widetilde{V}$. The argument of Step 2 of the proof of Lemma 7.2, with $U_l := \langle K_l', (u_1\lambda(m)u_2)^t \rangle$ of [50, Cor. 7.2.5] in place of the group of Step 0, run on $X_P^{\mathrm{ad},+} = X_P^+$, shows that the preimage $\widehat{A} \subset X_P^+$ of $\jmath(\widetilde{Z}^{(1)})$ is X_i^+-saturated for a factor P_i with $\pi_i(T_V) \neq 1$. Let $\widehat{Z}'$ be an irreducible component of the preimage of $\widetilde{Z}^{(1)}$ containing $X_{H_V}^+$; as both preimages have pure dimension $\dim \widetilde{Z}^{(1)}$, $\widehat{Z}'$ is an irreducible component of $\widehat{A}$. A connected complex submanifold contained in a locally finite union of irreducible analytic sets lies in one of them (Baire and the identity theorem); applied to $X_i^+ \times \{x_{\neq i}\}$ at the points x of $\widehat{Z}'$ on no other component of $\widehat{A}$, and by closedness at all points of $\widehat{Z}'$, this shows that $\widehat{Z}'$ is X_i^+-saturated. So $\widetilde{Z}$ contains the image $\widetilde{V}'$ of $X_i^+ \times \mathrm{pr}_{\neq i}(X_{H_V}^+)$, which is special, being the image of $X_{M'}^+ \times \{1\}$ for the Shimura subdatum defined by the identity component M' of the preimage in P of $P_i \times \mathrm{pr}_{\neq i}(\lambda(H_V))$, and contains $\widetilde{V}$ properly [50, Lemma 7.2.9]. Then $V' := \tau(\widetilde{V}')$ is special, $V \subset V' \subset \tau(\widetilde{Z}) = Z$, and $\dim V' = \dim \widetilde{V}' > \dim V$; being irreducible, V' lies in a geometric component of Z, which contains V.

Step 4 (the case $\widetilde{Z} \not\subset T_m\widetilde{Z}$). As $T_m\widetilde{Z}$ is defined over F and $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ permutes the geometric components of $\widetilde{Z}$ transitively, no geometric component of $\widetilde{Z}$ lies in $T_m\widetilde{Z}$; so every component of $\widetilde{Z} \cap T_m\widetilde{Z}$ has dimension less than $\dim Z$. Suppose $\widetilde{V}$ is one of them, as it must be if $r = 1$. Then so are its $\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V)$-conjugates, by Step 2, and Lemma 6.4 at $J = I$ and (6.3) give $\Omega_V/\bar{d} \leq \deg_{\mathcal{L}_I}(\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V) \cdot \widetilde{V}) \leq \Psi_l(D_Z)$, against $\bar{d}\,\Psi_l(D_Z) \leq \Psi_l(\bar{d}D_Z) \leq$

$\Psi_l^{\circ r}(\bar{d}D_Z) < \Omega_V$. So this case does not occur for $r = 1$, and for $r \geq 2$ a geometric component $\widetilde{Y}^{(1)}$ of $\widetilde{Z} \cap T_m\widetilde{Z}$ contains $\widetilde{V}$ properly. Let $\widetilde{Y}$ be the union of the $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-conjugates of $\widetilde{Y}^{(1)}$, which are components of $\widetilde{Z} \cap T_m\widetilde{Z}$. It is closed and F-irreducible, $\tau(\widetilde{Y}) \subset Z$, $\deg_{\mathcal{L}_I} \widetilde{Y} \leq \Psi_l(D_Z)$ by (6.3), and $0 < r_Y := \dim \widetilde{Y} - \dim \widetilde{V} < r$.

Step 5 (descent and induction). Apply Lemma 6.5 with $J = I$, and let (H, X_H), F_H, Y_H and V_H be as there. By Lemma 6.5(a), $\deg_{\mathcal{L}_{K_H}} Y_H \leq \Psi_l(D_Z)$, so, as $\Psi \leq \Psi_l$, Ψ_l is increasing and $\bar{d}\,\Psi_l(D_Z) \leq \Psi_l(\bar{d}D_Z)$,

$$\Psi^{\circ r_Y}(\bar{d}\deg_{\mathcal{L}_{K_H}} Y_H) \;\leq\; \Psi_l^{\circ r_Y}(\bar{d}\deg_{\mathcal{L}_{K_H}} Y_H) \;\leq\; \Psi_l^{\circ (r_Y+1)}(\bar{d}D_Z) \;\leq\; \Psi_l^{\circ r}(\bar{d}D_Z) \;<\; \Omega_V.$$

So the subdatum (H, X_H), the field $FF_H \supset E(H, X_H)$, and $V_H \subset Y_H$ in $\mathrm{Sh}_{K_H}(H, X_H)_{\mathbb{C}}$ satisfy the hypotheses of the theorem with $r_Y < r$ in place of r: Y_H is Hodge generic and FF_H-irreducible; (1) holds by Lemma 6.5(b); and (6.2) holds by the display for the same l and the same $\bar{d}$, since the conditions on l concern only K and T_V, and $F_H \subset E_V$ gives $FF_HE_V = FE_V$. By induction Y_H contains a special $V'_H \supsetneq V_H$, and Lemma 6.5(c) gives V'. □

The uniformity comes from two places: Lemma 5.2 has a constant that depends only on n, by Theorem 1.5 at every neat level; and the varying level enters only through $[K_{P,l} : I_l] \leq l^f$ and $[I_l : I_l \cap mI_lm^{-1}] < l^{k_F}$.

6.2. **The prime.** The criterion (6.1) reads $l^{b_r}\Delta_r(D_Z) < \Omega_V$, where, for $r \geq 0$ and $D' \geq 1$,

$$b_r := (k_F + 2f)\,2^r, \qquad \Delta_r(D') := \left(c_B\,\bar{d}\,D'\right)^{2^r}.$$

For a special subvariety V with $T_V \neq 1$ put

$$x_V := \max\Big((64\kappa(\delta)\beta_V)^{1/\delta},\ (\beta_V+1)^2,\ (48n_T)^2,\ 4C_0,\ 64n_T^2(\max(e^2/B', \log\alpha_V)+2)^2\Big).$$

Proposition 6.6. *Let V be special with $T_V \neq 1$, and assume* $\mathrm{QRH}(\delta; L_V)$. *Then there is a prime l with $x_V/4 < l \leq x_V$ and $l > C_0$ that splits completely in L_V and has $K_{T_V,l} = K^m_{T_V,l}$. Moreover $l > x_V/4 > 1/c_U$.*

Proof. (1) As $T_V \neq 1$, $\beta_V \geq \log 3 > 1$ (Section 2.2), and $[L_V : \mathbb{Q}] \leq n_T$; so the first three terms of x_V give the three conditions of Lemma 3.2 at $x = x_V$ for $L = L_V$. Hence at least $x_V/(8n_T \log x_V) \geq \sqrt{x_V}/(8n_T)$ primes in $(x_V/4, x_V]$ split completely in L_V, as $x/\log x \geq \sqrt{x}$ for $x > 1$. By the last term of x_V there are at least $\max(e^2/B', \log\alpha_V) + 2$ of them, and by the fourth they exceed C_0.

(2) Let i_V be the number of primes $p > C_0$, unramified in L_V, with $K^m_{T_V,p} \neq K_{T_V,p}$. For these $K_p = G(\mathbb{Z}_p) = G(\mathbb{Q}_p) \cap \mathrm{GL}_{n_\rho}(\mathbb{Z}_p)$ by the choice of C_0, which is the hypothesis of [85, Prop. 3.15], so $|K^m_{T_V,p}/K_{T_V,p}| \geq c_U p$ (by [28, §4.4] as in Lemma 6.2, as the printed proof of [85, Prop. 3.15] assumes p unramified in its field F_T, which may be taken to be F_V [85, pp. 11–12]). As i_V distinct primes have product at least $i_V!$, $\alpha_V \geq B'^{i_V} i_V! \geq (B'i_V/e)^{i_V}$, which exceeds e^{i_V} if $i_V > e^2/B'$. Hence $i_V \leq \max(e^2/B', \log\alpha_V)$. (In [50, Lemma 10.2.5] the count runs over all primes unramified in L_V with $K^m_{T_V,p} \neq K_{T_V,p}$, including those where $K_p \neq G(\mathbb{Z}_p)$, which are outside the hypothesis of [85, Prop. 3.15]; we only need primes $l > C_0$.)

(3) The primes of (1) are unramified in L_V and exceed C_0, and there are more than i_V of them; so one of them has $K_{T_V,l} = K^m_{T_V,l}$.

(4) $l > x_V/4 > 1/c_U$, since $x_V/4 \geq 16n_T^2(e^2/B' + 2)^2 > 16e^4/c_U^2 > 1/c_U$, as $B' = Bc_U \leq c_U \leq 1/2$. □

Remark. Step (2) of the proof of Proposition 6.6 uses neither $\mathrm{QRH}(\delta; L_V)$, nor the prime l, nor (Lv), and with it the index of K_{T_V} in $K_{T_V}^m$ is bounded from above. Let $\mathcal{P}_V$ be the set of primes p with $K_{T_V,p}^m \neq K_{T_V,p}$. As K is a product level, $[K_{T_V}^m : K_{T_V}] = \prod_{p\in\mathcal{P}_V} |K_{T_V,p}^m/K_{T_V,p}|$, and as $0 < B \leq 1$, each factor is at least its factor $\max(1, B|K_{T_V,p}^m/K_{T_V,p}|)$ in Definition 2.2 and at most B^{-1} times it. A prime of $\mathcal{P}_V$ is at most C_0, or ramified in L_V (at most $\log_2 d_{L_V}$ primes), or one of the $i_V \leq \max(e^2/B', \log\alpha_V)$ primes of (2); so $|\mathcal{P}_V| \leq C_0 + \log_2 d_{L_V} + e^2/B' + \log\alpha_V$, and

$$\alpha_V \;\leq\; [K_{T_V}^m : K_{T_V}] \;\leq\; B^{-|\mathcal{P}_V|}\,\alpha_V \;\leq\; B^{-(C_0+e^2/B')}\,\alpha_V^{1+\log(1/B)}\,d_{L_V}^{\log_2(1/B)},$$

with B, $B' = Bc_U$ and C_0 of Table 1, which depend only on (G, X, K, ρ). The same argument gives this inequality at every neat product level K with $\rho(K) \subset \mathrm{GL}_{n_\rho}(\widehat{\mathbb{Z}})$, with the C_0 and c_U of that level, and with any $B \in (0, 1]$ in place of that of Table 1, both in α_V and in $B' = Bc_U$.

For $V = \{x\}$ a special point and G adjoint, Binyamini, Schmidt and Yafaev [14, Def. 1] take x in the image of $X^+ \times \{1\}$, where $K_T = K \cap T(\mathbb{A}_f)$, for the Mumford–Tate group $T = T_V$ of x, is the group K_{T_V} of Section 2.2, so that $[K_T^m : K_T] = [K_{T_V}^m : K_{T_V}]$ and $\mathrm{disc}(x) = [K_{T_V}^m : K_{T_V}]\, d_{L_V}$. Let α_V be computed at K with the constant B of Theorem 1.1; it is at most the α_V of Theorem 1.1 (Lemma 9.1 and the paragraph after Theorem 9.12). So for Z, D and d as in Theorem 1.1 and x maximal in Z and in the image of $X^+ \times \{1\}$, under $\mathrm{QRH}(\delta; L_V)$, the display and Theorem 1.1(ii) give $\mathrm{disc}(x) \leq B^{-(C_0+e^2/B')}\, c(\delta, D, d)^{2+\log(1/B)+\log_2(1/B)}$.

6.3. **The Hecke step.**

Theorem 6.7. *Let G be adjoint and K satisfy* (Lv)*, let $(P, X_P) \subset (G, X)$ be a Shimura subdatum and $F \supset E(P, X_P)$ a number field. Let $Z \subset \mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$ be Hodge generic and F-irreducible, of degree D_Z as in Theorem 6.1, and let $V \subset Z$ be special and not strongly special in $\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$, with $r := \dim Z - \dim V > 0$. Let $\bar{d} \geq [FE_V : E_V]$ be an integer and assume $\mathrm{QRH}(\delta; L_V)$. If*

$$\Omega_V \;>\; \Delta_r(D_Z)\, x_V^{b_r},$$

then Z contains a special subvariety $V' \supsetneq V$ in a geometric component of Z containing V.

Proof. The prime l of Proposition 6.6 satisfies $l^{b_r}\Delta_r(D_Z) \leq x_V^{b_r}\Delta_r(D_Z) < \Omega_V$, which is (6.1); so Theorem 6.1 gives V'. □

For maximal V the inequality of Theorem 6.7 fails, and the following lemma turns that into bounds for α_V and d_{L_V}. Put

$$\varsigma := \max\Big(\frac{64\kappa(\delta)+1}{\mu} + 8n_T,\; 1 + 48n_T + 4C_0 + 8n_T(e^2/B' + 2)\Big),$$

and for $\Lambda \geq 1$ and $b \geq 1$

$$\mathcal{W}(\Lambda, b) := 2\log\Lambda + 1 + \frac{2b}{\delta}\log\frac{2\varsigma b}{e\,\delta}, \qquad \mathcal{B}(\Lambda, b) := \Lambda\,\big(\varsigma\,(\mathcal{W}(\Lambda, b) + 1)\big)^{b/\delta}.$$

Lemma 6.8. *Let $\Lambda \geq 1$ and $b \geq 1$, and let V be special with $T_V \neq 1$. If $\alpha_V d_{L_V}^{\mu} \leq \Lambda\, x_V^b$, then $\alpha_V d_{L_V}^{\mu} \leq \mathcal{B}(\Lambda, b)$. The function $\mathcal{B}$ is non-decreasing in Λ and in b.*

Proof. Put $\omega := \log(\alpha_V d_{L_V}^{\mu}) = \log\alpha_V + \mu\beta_V \geq 0$ and $S := (64\kappa(\delta)+1)\beta_V + 8n_T\log\alpha_V + 1 + 48n_T + 4C_0 + 8n_T(e^2/B' + 2)$. Since $S \geq 1$ and $1/\delta \geq 2$, each of the five terms of x_V is

at most $S^{1/\delta}$, as $64\kappa(\delta)\beta_V$, β_V+1, $48n_T$, $4C_0$ and $8n_T(\max(e^2/B',\log\alpha_V)+2)$ are at most S. As $\beta_V\le\omega/\mu$ and $\log\alpha_V\le\omega$, we have $S\le\varsigma(\omega+1)$. So the hypothesis gives

$$\omega \;\le\; \log\Lambda+\frac{b}{\delta}\log(\varsigma(\omega+1)).$$

With $\log y\le y/s+\log s-1$ for $y=\omega+1$ and $s=2b/\delta$, this gives $\omega\le\mathcal{W}(\Lambda,b)$; putting this back into the right side gives $\omega\le\log\mathcal{B}(\Lambda,b)$. Both $\mathcal{W}$ and $\mathcal{B}$ are non-decreasing in Λ, and in b because $2\varsigma b/(e\delta)>1$. □

7. Degrees of maximal special subvarieties

In this section G is adjoint, K is a neat product level, $n:=\dim\mathrm{Sh}_K(G,X)$ and $c_B:=e^*(n)^{2n}$ (Lemma 5.2(c)); (Lv) is not assumed and no analytic hypothesis is used. The iteration of [22, §10] cuts Z by a Hecke translate at an Iwahori level and bounds the intersection by Bézout's inequality with constant 1 at that level [22, proof of Prop. 10.1, p. 24]. We run it with Lemma 5.2 at the level of the Hecke translate and at primes $p\ge N_K$, with the inputs (DJK1)–(DJK3) in the form given below, and treat the facteur case, which Daw, Javanpeykar and Kühne exclude [22, p. 2], with a saturation lemma (Lemma 7.2).

7.1. **Inputs.** Let $\mathcal{G}_{\mathbb{Z}_p}$ be the Zariski closure of $\rho(G)$ in $\mathrm{GL}_{n_\rho,\mathbb{Z}_p}$, and let $N_K>2D_{\mathrm{sp}}+1$ be an integer such that $K_p=\mathcal{G}_{\mathbb{Z}_p}(\mathbb{Z}_p)$ is hyperspecial for every prime $p\ge N_K$, where $D_{\mathrm{sp}}\ge1$ is an integer for which [22, Lemma 7.5] holds. For a reductive $\mathbb{Q}$-group H and a product level $K_H\subset H(\mathbb{A}_f)$, let $\Sigma(H,K_H)$ be the set of primes p at which $H_{\mathbb{Q}_p}$ is not quasi-split, or does not split over an unramified extension, or $K_{H,p}$ is not hyperspecial, and $\Pi(H,K_H)$ the product of these primes (1 if there are none) [22, §2.5]. For a Shimura subdatum $(P,X_P)\subset(G,X)$ write $K(P):=K\cap P(\mathbb{A}_f)$ (the K_P of Sections 6 and 8), and for a special subvariety V of $\mathrm{Sh}_K(G,X^+)_{\mathbb{C}}$, the image of $X_H^+\times\{1\}$, put $\Sigma_V:=\Sigma(H^{\mathrm{der}},K(H^{\mathrm{der}}))$ and $\Pi_V:=\Pi(H^{\mathrm{der}},K(H^{\mathrm{der}}))$; these depend only on V [22, Def. 3.5], and are the same for V in $\mathrm{Sh}_{K(P)}(P,X_P)$ and for its image in $\mathrm{Sh}_K(G,X)$, since $K(P)\cap H^{\mathrm{der}}(\mathbb{A}_f)=K\cap H^{\mathrm{der}}(\mathbb{A}_f)$. We use the following results of [22], all with effectively computable constants depending only on (G,X,K,ρ).

(DJK1) [22, Thm. 6.1] There are $c_1,\eta_1>0$ such that $\deg V>c_1\Pi_V^{\eta_1}$ for every special subvariety V of $\mathrm{Sh}_K(G,X^+)_{\mathbb{C}}$ of positive dimension.

(DJK2) [22, Lemmas 4.1 and 4.2] There is an integer $d_G\ge1$ such that for every Shimura subdatum (P,X_P) with a component $X_P^+\subset X^+$, every closed irreducible $Y_P\subset\mathrm{Sh}_{K(P)}(P,X_P^+)_{\mathbb{C}}$ and its image Y in $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$, we have $\deg Y\le d_G\deg_{\mathcal{L}_{K(P)}}Y_P$; and $\deg_{\mathcal{L}_{K(P)}}Y_P\le\deg Y$ for every Hodge generic closed irreducible $Y_P\subset\mathrm{Sh}_{K(P)}(P,X_P)_{\mathbb{C}}$.

(DJK3) [22, Thm. 7.1], for primes $p\ge N_K$. There are integers $k_2,f_2\ge1$ with the following property. Let (P,X_P) be a Shimura subdatum with a component $X_P^+\subset X^+$, $P^{\mathrm{ad}}=P_1\times\cdots\times P_s$ the decomposition into $\mathbb{Q}$-simple factors with projections π_i, $V\subset\mathrm{Sh}_{K(P)}(P,X_P^+)_{\mathbb{C}}$ a special subvariety of positive dimension, the image of $X_H^+\times\{1\}$ for $(H,X_H)\subset(P,X_P)$ and a component $X_H^+\subset X_P^+$, $\mathcal{I}$ the set of i with $\pi_i(H^{\mathrm{der}})\ne1$, and $p\ge N_K$ a prime with $p\notin\Sigma_V$. Then there are a compact open $J_p\subset K(P)_p:=K_p\cap P(\mathbb{Q}_p)$ and $\alpha\in P(\mathbb{Q}_p)$ such that, with $J:=K(P)^pJ_p$:

- (i) $[K(P)_p:J_p]\le p^{f_2}$;
- (ii) $[J_p:J_p\cap\alpha J_p\alpha^{-1}]\le p^{k_2}$;
- (iii) the image $\widetilde{V}$ of $X_H^+\times\{1\}$ in $\mathrm{Sh}_J(P,X_P)_{\mathbb{C}}$ satisfies $\widetilde{V}\subset T_\alpha\widetilde{V}$;

(iv) $\pi_i(u_1\alpha u_2)$ generates an unbounded subgroup of $P_i(\mathbb{Q}_p)$ for all $i \in \mathcal{I}$ and $u_1, u_2 \in J_p$.

On (DJK2): we use the display after "By definition" in the proof of [22, Lemma 4.2], p. 9, with the factor $d^{\dim Y_P}$ on the left in place of d, for the integer d of that proof; the proof then gives $\deg Y \le d^{\dim Y_P} \deg_{\mathcal{L}_{K(P)}} Y_P$, and we take $d_G := d^{\dim \mathrm{Sh}_K(G,X)}$.

The proofs of [22, Lemmas 6.10 and 6.11], on which that of (DJK1) rests, take from that of [21, Lemma 6.6], besides the passage from the simply connected cover of H^{der} to H^{der}, only the bound of Lemma 7.1. In that passage [21, p. 92], for the simply connected cover $\pi\colon \widetilde{H} \to H^{\mathrm{der}}$, let $\widetilde{\mathcal{H}}$ be the reductive model of $\widetilde{H}_{\mathbb{Q}_p}$ with $\widetilde{\mathcal{H}}(\mathbb{Z}_p) = K(\widetilde{H})^m_p$, where $K(\widetilde{H})^m \supset K(\widetilde{H})$ is the maximal compact open subgroup of $\widetilde{H}(\mathbb{A}_f)$ of [22, p. 14] and $p \notin \Sigma(\widetilde{H}, K(\widetilde{H})^m)$, as in [22, Lemma 6.10], so that $K(\widetilde{H})^m_p$ is hyperspecial; let $\mathcal{H}$ be the reductive model of $H^{\mathrm{der}}_{\mathbb{Q}_p}$ to which π extends, and $\mathcal{Z}$ the kernel of $\widetilde{\mathcal{H}} \to \mathcal{H}$; read $K' := K(H^{\mathrm{der}})_p \cap \mathcal{H}(\mathbb{Z}_p)$, the group K' of Lemma 7.1, for $K(H^{\mathrm{der}})_p$, which need not lie in $\mathcal{H}(\mathbb{Z}_p)$. As $K(\widetilde{H})_p = \pi^{-1}(K(H^{\mathrm{der}})_p)$ lies in $\widetilde{\mathcal{H}}(\mathbb{Z}_p)$ and contains $\ker\pi$, we have $[K(\widetilde{H})^m_p : K(\widetilde{H})_p] \ge [\mathcal{H}(\mathbb{Z}_p) : K']/|H^1(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p), \mathcal{Z}(\mathbb{Z}_p^{\mathrm{un}}))|$, and $K' \ne \mathcal{H}(\mathbb{Z}_p)$ if $K(\widetilde{H})_p \ne K(\widetilde{H})^m_p$, since otherwise $K(\widetilde{H})_p \supset \widetilde{\mathcal{H}}(\mathbb{Z}_p) = K(\widetilde{H})^m_p$. With this the proofs of [22, Lemmas 6.10 and 6.11] are unchanged, with $c_3 := \max\{4, B_Z, p_0\}$ and $c_4 := (3cB_Z^{2b+2})^{-1}$ in [22, Lemma 6.10], where B_Z and b are the constants of [22, Lemmas 6.4 and 6.5], c is that of [22, p. 19] (as in Lemma 7.1) and p_0 is the largest prime with $K_p \ne \mathcal{G}_{\mathbb{Z}_p}(\mathbb{Z}_p)$, as in the proof of [22, Lemma 6.10]. Lemma 7.5 (Section 7.4), in whose notation we now write, replaces [22, Lemma 6.7], and step (iii) of its proof replaces [22, Lemma 6.8]. In that notation $\widetilde{H} = \prod_i \mathrm{Res}_{K_i/\mathbb{Q}} H_i'$, with H_i' absolutely almost simple and simply connected over a number field K_i; ℓ_i and s_i are the field and the integer of [73, §§0.2 and 0.4] for the quasi-split inner form of H_i', $N_i = d_{\ell_i}/d_{K_i}^{[\ell_i:K_i]}$, $\varpi_i = d_{K_i}^{\dim H_i'/2} N_i^{s_i/2}$, r_G is the absolute rank of G, and c_2^* is the constant of the lemma. Its bound carries $\prod_i \varpi_i^{1/2}$, where $\prod_i \varpi_i$ is the discriminant factor of Prasad's formula [73, Thm. 3.7], from which [22, p. 16] draw the lower bound for the volume of $\Gamma(\widetilde{H})^m\backslash\widetilde{H}(\mathbb{R})$; in that bound the remaining factor $\prod_i \varpi_i^{1/2}$ suffices. Prasad's theorem applies to each H_i', with S the set of archimedean places of K_i, because $H_i'(K_i \otimes \mathbb{R})$ is not compact: as H is the generic Mumford–Tate group on X_H, the projection of $h \in X_H$ to each $\mathbb{Q}$-simple factor of H^{ad} is non-trivial, so that factor has non-compact real points [56, Lemma 4.7]. As $\dim H_i' \ge 3$, and $s_i \ge 5$ when $\ell_i \ne K_i$, we have $\varpi_i^{1/2} \ge (d_{K_i}N_i)^{1/2}$, which pays the primes of $\Sigma(\widetilde{H}, K(\widetilde{H})^m)$ (in the notation of [22, pp. 14 and 16]) that divide some $d_{K_i}N_i$. At every other prime p of that set, some H_i' at some place $v \mid p$ of K_i is not quasi-split, or splits over the maximal unramified extension of $K_{i,v}$ while $K(\widetilde{H})^m$ is not hyperspecial at v (otherwise, as K_i and ℓ_i are unramified above p, $\widetilde{H}_{\mathbb{Q}_p}$ would be quasi-split and split over an unramified extension, and $K(\widetilde{H})^m_p$ would be hyperspecial). Then the factor of Prasad's Euler product at v is at least $q_v^{r_v+1}/(q_v+1) \ge q_v^2/(q_v+1)$, where q_v is the order of the residue field and $r_v \ge 1$ the rank of H_i' over the maximal unramified extension of $K_{i,v}$ [73, §2.5 and Prop. 2.10(iv)]; this is at least $\sqrt{p}$ for $p \ge 3$ and $\frac{4}{3} = \frac{2\sqrt{2}}{3}\sqrt{2}$ for $p = 2$, and all the factors are greater than 1 [73, Rem. 3.10]. So $\prod_i \varpi_i^{1/2}$ times the Euler product is at least $\frac{2\sqrt{2}}{3}\,\Pi(\widetilde{H}, K(\widetilde{H})^m)^{1/2}$; dividing by the bound of Lemma 7.5 as on [22, pp. 14–16], the factors $\prod_i \varpi_i^{1/2}$ cancel, and [22, (7)] holds with $C := B_Z^{3r_G}$ and $c_2 := 3c_2^*/(2\sqrt{2})$. The rest of the proof of [22, Thm. 6.1] is unchanged.

Lemma 7.1. *Let V be a special subvariety of $\mathrm{Sh}_K(G,X^+)_{\mathbb{C}}$, the image of $X_H^+\times\{1\}$, let $p\geq 5$ be a prime with $K_p=\mathcal{G}_{\mathbb{Z}_p}(\mathbb{Z}_p)$, let $\mathcal{H}$ be a reductive model over $\mathbb{Z}_p$ of $H^{\mathrm{der}}_{\mathbb{Q}_p}$, and let K' be the stabilizer of $\mathbb{Z}_p^{n_\rho}$ in $\mathcal{H}(\mathbb{Z}_p)$. Then $K'=\mathcal{H}(\mathbb{Z}_p)$ or $[\mathcal{H}(\mathbb{Z}_p):K']\geq(p-1)/c$, with c the constant of* [22, p. 19].

Proof. The argument for this bound in the proof of [21, Lemma 6.6] (p. 93), which [22, p. 18] cite, seeks a maximal torus of $\mathcal{H}$ whose $\mathbb{Z}_p$-points are not in K' through a density argument, which gives an element of $\mathcal{H}(\mathbb{Z}_p)\smallsetminus K'$ that is semisimple in the generic fiber but no maximal torus of $\mathcal{H}$ over $\mathbb{Z}_p$ containing it; it then takes the Zariski closure of a conjugate of the generic fiber of the torus to be a torus because its $\mathbb{Z}_p$-points form the maximal compact subgroup, which does not suffice (the remark after Lemma 6.2). We argue as follows. Suppose $K'\neq\mathcal{H}(\mathbb{Z}_p)$. The kernel $\mathcal{H}_1$ of $\mathcal{H}(\mathbb{Z}_p)\to\mathcal{H}(\mathbb{F}_p)$ is a pro-p group; if $\mathcal{H}_1\not\subset K'$, the $\mathcal{H}_1$-orbit of $\mathbb{Z}_p^{n_\rho}$ has more than one element and a power of p as size, so $[\mathcal{H}(\mathbb{Z}_p):K']\geq p$. Otherwise K' is the preimage of a proper subgroup of $\mathcal{H}(\mathbb{F}_p)$. As $p\geq 5$, $\mathcal{H}(\mathbb{F}_p)$ is generated by the groups $\bar{T}(\mathbb{F}_p)$ of its maximal tori $\bar{T}$ defined over $\mathbb{F}_p$: for $\mathcal{H}_{\mathbb{F}_p}$ simply connected because every semisimple element of $\mathcal{H}(\mathbb{F}_p)$ lies in such a group, so they generate a normal subgroup with quotient a p-group, and $\mathcal{H}(\mathbb{F}_p)$, a product of groups $\mathcal{H}_i(\mathbb{F}_{q_i})$ with $\mathcal{H}_i$ absolutely almost simple and simply connected, is perfect [55, Thm. 24.17], and in general because $\mathcal{H}(\mathbb{F}_p)=\bar{T}(\mathbb{F}_p)\,\pi(\widetilde{\mathcal{H}}(\mathbb{F}_p))$ for the simply connected cover $\pi\colon\widetilde{\mathcal{H}}\to\mathcal{H}_{\mathbb{F}_p}$, by Lang's theorem. So some $\bar{T}(\mathbb{F}_p)$ is not in the image of K', and $\bar{T}$ lifts to a maximal torus $\mathcal{T}$ of $\mathcal{H}$ [18, Thm. 3.2.6], by Hensel's lemma, which also makes $\mathcal{T}(\mathbb{Z}_p)\to\bar{T}(\mathbb{F}_p)$ onto; so $\mathcal{T}(\mathbb{Z}_p)\not\subset K'$. Then $\mathcal{T}$ does not fix $\mathbb{Z}_p^{n_\rho}$ in the sense of [28, §3.3], and the proof of [28, Prop. 4.3.9] in [28, §4.4], applied to $\mathcal{T}$, to the largest sublattice W' of $\mathbb{Z}_p^{n_\rho}$ that $\mathcal{T}$ fixes [28, proof of Lemma 3.3.1], and to $\mathbb{Z}_p^{n_\rho}$, gives $[\mathcal{T}(\mathbb{Z}_p):\mathcal{T}(\mathbb{Z}_p)\cap K']\geq(p-1)/|\pi_0(S)(\overline{\mathbb{F}}_p)|$, where $S\subset\mathcal{T}_{\mathbb{F}_p}$ is the stabilizer of the image L' of $p\mathbb{Z}_p^{n_\rho}\cap W'$ in W'/pW' [28, p. 633]. Over $\overline{\mathbb{F}}_p$, S is the common kernel of the differences $\chi-\chi'$ of the characters lying in one block of the finest partition of the weights of $W'\otimes\overline{\mathbb{F}}_p$ along which $L'\otimes\overline{\mathbb{F}}_p$ decomposes, and these characters are bounded by [22, Lemma 6.9]; so $|\pi_0(S)(\overline{\mathbb{F}}_p)|\leq c$, and $[\mathcal{H}(\mathbb{Z}_p):K']\geq[\mathcal{T}(\mathbb{Z}_p):\mathcal{T}(\mathbb{Z}_p)\cap K']\geq(p-1)/c$. □

Proof of (DJK3). The proof of [22, Thm. 7.1] follows that of [21, Thm. 7.1], which takes $J_p\subset P(\mathbb{Q}_p)$ from [50, §8.3.2] and bounds its index by [50, Lemma 8.1.6(b)] applied in $P(\mathbb{Q}_p)$ [21, p. 98]; [22, p. 22] makes this bound explicit by the proof of [50, Lemma 8.1.6(b)], with the Weyl group of $H^{\mathrm{der}}_{\mathbb{Q}_p}$. That proof first reduces to a hyperspecial subgroup of the group containing the Iwahori subgroups, here $P(\mathbb{Q}_p)$, which need not have one, as $p\notin\Sigma_V$ constrains only H^{der}; the proof goes through with Lemma 6.3 in place of [50, Lemma 8.1.6]. Let T be a maximal split torus of $H^{\mathrm{der}}_{\mathbb{Q}_p}$ whose apartment contains the point fixed by the hyperspecial subgroup $K_p\cap H^{\mathrm{der}}(\mathbb{Q}_p)$ [81, 3.8.1]; the maximal compact subgroup of $T(\mathbb{Q}_p)$ fixes that apartment pointwise (proof of Lemma 6.3), so it lies in $K_p\cap H^{\mathrm{der}}(\mathbb{Q}_p)\subset K(P)_p$. Let $\Theta\subset T(\mathbb{Q}_p)$ be the subgroup of elements of order dividing $p-1$. Two distinct characters χ,χ' of T intervening in ρ have coordinates of absolute value at most D_{sp} in the basis of [22, Lemma 7.5], so $\chi-\chi'\notin(p-1)X^*(T)$, as $p-1>2D_{\mathrm{sp}}$, and $\chi\neq\chi'$ on Θ. As $\Theta\subset\mathrm{GL}_{n_\rho}(\mathbb{Z}_p)$ has order prime to p and its characters have values in $\mathbb{Z}_p^\times$, the projectors onto the weight spaces of T preserve $\mathbb{Z}_p^{n_\rho}$; so the Zariski closure $T_{\mathbb{Z}_p}$ of T in $\mathrm{GL}_{n_\rho,\mathbb{Z}_p}$ is a torus [28, Lemma 3.3.1], as [28, Prop. 7.4.3] requires. (The proof of [22, Lemma 7.4], which is that of [21, Lemma 7.3] (p. 96), infers this from the maximality of $T(\mathbb{Q}_p)\cap\mathrm{GL}_{n_\rho}(\mathbb{Z}_p)$, which does not suffice; see the remark after Lemma 6.2.) A nonzero weight of T on $\mathrm{Lie}\,P\subset\mathfrak{gl}_{n_\rho}$ is such a difference $\chi-\chi'$, so it does not lie in $(p-1)X^*(T)$; and $\Theta\subset K(P)_p$. So Lemma 6.3(a),

with $H = P_{\mathbb{Q}_p}$, and Lemma 6.3(b) give the torus, chambers and groups of Lemma 6.3(c) with p, T and $K(P)_p$ in place of l, $T_{V,\mathbb{Q}_l}$ and $K_{P,l}$, the chamber C^2 being chosen with respect to α as in [50, §8.3.2]; hence (i) with $f_2 := (\dim G + 1)^2$; then (ii) holds with $k_2 := k' + f_2$, for the integer k' of [22, p. 21], and (iv) follows from [50, Prop. 8.1.4] in each $P_{i,\mathbb{Q}_p}$, $i \in \mathcal{I}$. □

7.2. **Saturation.** Let $(P, X_P) \subset (G, X)$ be a Shimura subdatum with P the generic Mumford–Tate group on X_P, $P^{\mathrm{ad}} = P_1 \times \cdots \times P_s$ as in (DJK3), and $X_P^+ = X_1^+ \times \cdots \times X_s^+$ the corresponding decomposition. For $\mathcal{A} \subset \{1, \dots, s\}$ write $P_{\mathcal{A}} := \prod_{i \in \mathcal{A}} P_i$, $X_{\mathcal{A}}^+ := \prod_{i \in \mathcal{A}} X_i^+$ and $\mathcal{A}^c$ for the complement. A subset $\widehat{Y} \subset X_P^+$ is *$X_{\mathcal{A}}^+$-saturated* if $\widehat{Y} = X_{\mathcal{A}}^+ \times \mathrm{pr}_{\mathcal{A}^c}(\widehat{Y})$.

Lemma 7.2. *Let $Y \subset \mathrm{Sh}_{K(P)}(P, X_P^+)_{\mathbb{C}}$ be closed, irreducible and Hodge generic, containing a special point, and let $\widehat{Y} \subset X_P^+$ be its preimage. Let p be a prime, $J_p \subset K(P)_p$ compact open, $J := K(P)^p J_p$, $\widetilde{Y}$ a component of the preimage of Y in $\mathrm{Sh}_J(P, X_P)_{\mathbb{C}}$, $\alpha \in P(\mathbb{Q}_p)$ and $\mathcal{I} \subset \{1, \dots, s\}$. Suppose that $\widetilde{Y} \subset T_\alpha \widetilde{Y}$, and that $\pi_i(u_1 \alpha u_2)$ generates an unbounded subgroup of $P_i(\mathbb{Q}_p)$ for all $i \in \mathcal{I}$ and $u_1, u_2 \in J_p$. Then $\widehat{Y}$ is $X_{\mathcal{I}}^+$-saturated.*

Proof. This is the proof of [50, Thm. 7.2.1], applied to $\widetilde{Y}$ itself in place of the preimage of a special subvariety, with the monodromy step of [50, p. 33] argued from a special point of Y (Step 1) and the last step (Step 2) carried out on X_P^+ rather than at the adjoint product level of [50, Lemma 7.2.8].

Step 0. Let $\widetilde{Y}^\infty$ be a component of the preimage of $\widetilde{Y}$ in the pro-étale cover of $\mathrm{Sh}_J(P, X_P)$ with group J_p [50, §4.1.3]. As Y is geometrically irreducible, the argument of [50, (7.2)] gives $u_1, u_2 \in J_p$ with $\widetilde{Y}^\infty = \widetilde{Y}^\infty \cdot (u_1 \alpha u_2)$; the Galois step of [50, Lemma 7.2.4] is not needed. Let $\widehat{Y}'$ be an analytic component of $\widehat{Y}$.

Step 1 (the monodromy). Let M_Y be the identity component of the Zariski closure of the monodromy of the variation of Hodge structure over the smooth locus Y^{sm} of Y. Let $r\colon Y' \to Y$ be a resolution of singularities that is an isomorphism over Y^{sm}. The pull-back to Y' of the variation on $\mathrm{Sh}_{K(P)}(P, X_P^+)_{\mathbb{C}}$ is polarizable; its Mumford–Tate group is P at the points over the Hodge generic points of Y, and a torus at the points over a special point of Y. Hence its connected monodromy group is P^{der} [4, Prop. 2]; this group is M_Y, as $\pi_1(Y^{\mathrm{sm}}) \to \pi_1(Y')$ is onto. So $M_Y = P^{\mathrm{der}}$, and the closure K_p' in $P(\mathbb{Q}_p)$ of the monodromy of $\widetilde{Y}$ at the lift $\widehat{Y}'$ contains an open subgroup of $M_Y(\mathbb{Q}_p)$ [50, Prop. 4.2.1]. (Without a special point this fails: for $\sigma \in Y(1)$ algebraic and not CM, $Y(1) \times \{\sigma\} \subset Y(1)^2$ is Hodge generic with monodromy $\mathrm{SL}_2 \times 1$.)

Step 2. Steps 0 and 1 hold for every choice of $\widetilde{Y}^\infty$ and $\widehat{Y}'$; translating by some $k \in K(P)_p$, which replaces J_p and α by $k^{-1} J_p k$ and $k^{-1} \alpha k$ and keeps the hypotheses, we may assume that $\widetilde{Y}$ is the image of $\widehat{Y}' \times \{1\}$, and we take for $\widetilde{Y}^\infty$ the component containing the image of $\widehat{Y}' \times \{1\}$; then K_p' is the closure of the stabilizer of $\widehat{Y}'$ in $P(\mathbb{Q})_+ \cap J$. By Steps 0 and 1, $\widetilde{Y}^\infty$ is stabilized by $U_p := \langle K_p', u_1 \alpha u_2 \rangle$ [50, Cor. 7.2.5], and $M_Y = P^{\mathrm{der}}$, so K_p' contains an open subgroup of $P^{\mathrm{der}}(\mathbb{Q}_p)$ and the image $\bar{U}_p$ of U_p in $P^{\mathrm{ad}}(\mathbb{Q}_p)$ is open. Fix $i \in \mathcal{I}$. Let $P_i^\natural$ be the $\mathbb{Q}$-simple factor of P^{der} over P_i, $\varpi\colon \widetilde{P}_i \to P_i^\natural$ its simply connected cover, H a $\mathbb{Q}_p$-simple factor of $P_{i,\mathbb{Q}_p}$ in which the image of $u_1 \alpha u_2$ is unbounded, and $\widetilde{\mathsf{H}}$ the factor of $\widetilde{P}_{i,\mathbb{Q}_p}$ over H. By [50, Lemma 7.2.6] for $\bar{U}_p$, $\varpi(\widetilde{\mathsf{H}}(\mathbb{Q}_p)) \subset U_p\, Z(P)(\mathbb{Q}_p)$, so commutators of elements of $\varpi(\widetilde{\mathsf{H}}(\mathbb{Q}_p))$ lie in U_p. The group $\widetilde{\mathsf{H}}(\mathbb{Q}_p)$ is perfect: it is $\widetilde{\mathsf{H}}'(\mathsf{k})$ for a finite $\mathsf{k}/\mathbb{Q}_p$ and $\widetilde{\mathsf{H}}'$ simply connected and absolutely almost simple, isotropic because $\mathsf{H}(\mathbb{Q}_p)$ is not compact [72], so $\widetilde{\mathsf{H}}'(\mathsf{k}) = \widetilde{\mathsf{H}}'(\mathsf{k})^+$ [69], [70, Thm. 7.6], which is perfect [38, Fait 4.1]. Hence $\varpi(\widetilde{\mathsf{H}}(\mathbb{Q}_p)) \subset U_p$.

Let $\Gamma_i := \{\gamma \in P_i^\natural(\mathbb{Q}) \cap P(\mathbb{Q})_+ \cap K(P)^p : \gamma_p \in K_p'\}$, an arithmetic subgroup of $P_i^\natural$ with closure $\overline{\Gamma}_i \subset K_p'$ in $P(\mathbb{Q}_p)$, and fix $g \in \widetilde{\mathsf{H}}(\mathbb{Q}_p)$ lying in no compact subgroup. By strong approximation for $\widetilde{P}_i$ with respect to the infinite place ($P_i(\mathbb{R})$ is not compact), $\widetilde{P}_i(\mathbb{Q})$ is dense in $\widetilde{P}_i(\mathbb{A}_f)$ [56, Thm. 4.16]. So $\overline{\Gamma}_i$ contains $\varpi(\widetilde{C}_p)$, with $\widetilde{C}_p$ the preimage of K_p' in $\widetilde{P}_i(\mathbb{Q}_p)$ (the elements of $\widetilde{P}_i(\mathbb{Q})$ in $\widetilde{C}_p$ whose image lies in $K(P)^p$ are dense in $\widetilde{C}_p$, and their images lie in Γ_i), and is therefore open in $P_i^\natural(\mathbb{Q}_p)$; and there is $q' \in \varpi(\widetilde{P}_i(\mathbb{Q}))$ that lies in $K(P)_\ell$ for every prime $\ell \neq p$ and has $q_p' \in \varpi(g)\overline{\Gamma}_i$; so $q' \in P(\mathbb{Q})_+$ and $q_p' \in U_p$. The closure of $\Gamma_i' := \langle \Gamma_i, q' \rangle$ in $P(\mathbb{Q}_p)$ contains $\varpi(g)$, so Γ_i has infinite index in Γ_i'. The image $\pi_i(\Gamma_i')$ is dense in $P_i(\mathbb{R})^+$. Indeed, let C be its closure in $P_i(\mathbb{R})$ and $\mathfrak{c}$ the Lie algebra of C. The group $\pi_i(\Gamma_i)$ is arithmetic in P_i [56, Prop. 3.2], hence a lattice in $P_i(\mathbb{R})$ and Zariski dense in P_i [56, Thms. 3.3 and 3.19], and it normalizes C; so $\mathfrak{c}$ is an ideal of $\operatorname{Lie} P_i(\mathbb{R})$, and $P_i(\mathbb{R}) = \mathsf{N} \times \mathsf{N}'$ with N, N' products of $\mathbb{R}$-simple factors and $\operatorname{Lie} \mathsf{N} = \mathfrak{c}$. Suppose $\mathsf{N}' \neq 1$. As C is closed with the identity component of N as its identity component, and N has finitely many components, the projection of C to N' is discrete; and $P_i(\mathbb{Q}) \to \mathsf{N}'$ is injective, as P_i is $\mathbb{Q}$-simple. Hence a sequence in $\pi_i(\Gamma_i')$ that tends to 1 is eventually 1, and $\pi_i(\Gamma_i')$ is discrete; it then contains the lattice $\pi_i(\Gamma_i)$ with finite index, and $[\Gamma_i' : \Gamma_i]$ is finite, as $P_i^\natural \to P_i$ has finite kernel. Thus $\mathsf{N}' = 1$ and $\mathsf{C} \supset P_i(\mathbb{R})^+$.

Each $\gamma \in \Gamma_i'$ lies in $P(\mathbb{Q})_+ \cap K(P)^p$, has $\gamma_p \in U_p$ and moves only the i-th coordinate of X_P^+; for $x \in \widehat{Y}'$, $[\gamma x, 1] = [x, \gamma_p^{-1}] \in \widetilde{Y}^\infty U_p = \widetilde{Y}^\infty$, so $\gamma x \in \widehat{Y}$. As $\widehat{Y}$ is closed, $X_i^+ \times \{x_{\neq i}\} \subset \widehat{Y}$ for every $x \in \widehat{Y}'$, hence for every $x \in \widehat{Y} = (K(P) \cap P(\mathbb{Q})_+) \cdot \widehat{Y}'$, since $P(\mathbb{Q})_+$ acts factorwise. Applying this for each $i \in \mathcal{I}$ in turn gives the lemma. □

Lemma 7.3. *In Lemma 7.2, let $V \subset Y$ be a special subvariety defined by (H, X_H), and suppose that the image of H^{der} in P^{ad} is $P_{\mathcal{I}}$. Then $\deg_{\mathcal{L}_{K(P)}} V \leq \deg_{\mathcal{L}_{K(P)}} Y$.*

Proof. Let $\Gamma \subset P^{\mathrm{ad}}(\mathbb{Q})$ be the image of $K(P) \cap P(\mathbb{Q})_+$, so that $\mathrm{Sh}_{K(P)}(P, X_P^+)_{\mathbb{C}} = \Gamma \backslash X_P^+$, and put $\Gamma_{\mathcal{I}} := \pi_{\mathcal{I}}(\Gamma)$ and $\Gamma_c := \pi_{\mathcal{I}^c}(\Gamma)$. These are arithmetic, as images of an arithmetic group under surjective homomorphisms [56, Prop. 3.2], and neat [68, 0.5], [50, §4.1.4], so Γ has finite index e in $\Gamma_{\mathcal{I}} \times \Gamma_c$, and $f \colon \Gamma \backslash X_P^+ \to S' := \Gamma_{\mathcal{I}} \backslash X_{\mathcal{I}}^+ \times \Gamma_c \backslash X_{\mathcal{I}^c}^+$ is finite étale of degree e. The Baily–Borel compactification of S' is the product of those of the factors, and $f^*(\mathcal{L}_{\mathcal{I}} \boxtimes \mathcal{L}_c) = \mathcal{L}_{K(P)}$, where $\mathcal{L}_{\mathcal{I}}$, $\mathcal{L}_c$ are the extensions of the canonical bundles of the factors. Since $Z(H)(\mathbb{R})$ fixes the points of X_H and H^{der} maps onto $P_{\mathcal{I}}$, $X_H^+ = X_{\mathcal{I}}^+ \times \{x_2\}$, so $f(V) = S_{\mathcal{I}}' \times \{w_0\}$ with $S_{\mathcal{I}}' := \Gamma_{\mathcal{I}} \backslash X_{\mathcal{I}}^+$. By Lemma 7.2, $f(Y) = S_{\mathcal{I}}' \times W$ with W closed and irreducible, of dimension b.

Every fiber $f^{-1}(S_{\mathcal{I}}' \times \{w\})$ is irreducible: for $w = \Gamma_c y$ it is the image of $X_{\mathcal{I}}^+ \times \Gamma_c y$, and Γ permutes the slices $X_{\mathcal{I}}^+ \times \{\gamma y\}$ transitively, as $\pi_{\mathcal{I}^c}(\Gamma) = \Gamma_c$. Every component of $f^{-1}(S_{\mathcal{I}}' \times W)$ maps onto $S_{\mathcal{I}}' \times W$, as f is finite and flat, so it contains the fiber over a general point of W; hence $f^{-1}(S_{\mathcal{I}}' \times W)$ is irreducible, and $Y = f^{-1}(S_{\mathcal{I}}' \times W)$, $V = f^{-1}(S_{\mathcal{I}}' \times \{w_0\})$, each of degree e over its image. With $a := \dim X_{\mathcal{I}}^+$ the projection formula gives $\deg_{\mathcal{L}_{K(P)}} V = e \deg_{\mathcal{L}_{\mathcal{I}}} S_{\mathcal{I}}'$ and $\deg_{\mathcal{L}_{K(P)}} Y = e\binom{a+b}{a} \deg_{\mathcal{L}_{\mathcal{I}}} S_{\mathcal{I}}' \deg_{\mathcal{L}_c} W \geq \deg_{\mathcal{L}_{K(P)}} V$, since $\deg_{\mathcal{L}_c} W \geq 1$. □

The choice of the groups $\Gamma_{\mathcal{I}}$, Γ_c rather than the projections of $K(P)$ is what makes the fibers irreducible: with $\pi_{\mathcal{I}^c}(K(P)) \cap P_{\mathcal{I}^c}(\mathbb{Q})_+$ in place of Γ_c the fiber would have as many components as the index of Γ_c in that group.

7.3. **The degree bound.** For $\Pi \geq 1$ put $y(\Pi) := \max(6,\, N_K + \log_2 \Pi + 1)$ and $p^*(\Pi) := 2y(\Pi)\log y(\Pi)$. For integers $0 \leq z \leq n$ and $D \geq 1$ put

$$R_{z,D}(\Pi) := d_G\big(d_G\, c_B\, p^*(\Pi)^{k_2+2f_2}\big)^{2^z-1} D^{2^z},$$

let $\Pi_0(z,D) := \sup\{\Pi \geq 1 : c_1\Pi^{\eta_1} \leq R_{z,D}(\Pi)\}$, which is finite because $R_{z,D}$ grows like a power of $\log\Pi$ ($\Pi_0 := 1$ if the set is empty), and

$$c^{\mathrm{ad}}_{\deg}(z,D) := R_{z,D}(\Pi_0(z,D)).$$

Theorem 7.4. *Let $Z \subset \mathrm{Sh}_K(G,X)_{\mathbb{C}}$ be closed and irreducible, $D := \deg Z$, and V a special subvariety maximal in Z with $\dim V \geq 1$. Put $r := \dim Z - \dim V$. For every prime $p \geq N_K$ with $p \notin \Sigma_V$,*

$$\deg V \;\leq\; d_G\big(d_G\, c_B\, p^{k_2+2f_2}\big)^{2^r-1} D^{2^r}, \qquad c_B = e^*(n)^{2n}. \tag{7.1}$$

Consequently $\deg V \leq c^{\mathrm{ad}}_{\deg}(\dim Z, D)$.

Proof. Fix p. We build closed irreducible $Y_0 := Z \supset Y_1 \supset \dots$, all containing V, with $\dim Y_{j+1} < \dim Y_j$ and $\deg Y_j \leq D_j$, where $D_0 := D$ and $D_{j+1} := d_G\, c_B\, p^{k_2+2f_2} D_j^2$. Since $Y_j \subset Z$, V is maximal in every Y_j.

If $Y_j = V$, we stop. Otherwise Y_j is not special. Let (P, X_P) be the Shimura subdatum, with P the generic Mumford–Tate group on X_P, of the smallest special subvariety containing Y_j, let $\psi\colon \mathrm{Sh}_{K(P)}(P,X_P)_{\mathbb{C}} \to \mathrm{Sh}_K(G,X)_{\mathbb{C}}$ be the natural map, Y_P a component of $\psi^{-1}(Y_j)$ with $\psi(Y_P) = Y_j$, which is Hodge generic, and $V_P \subset Y_P$ a component of $\psi^{-1}(V)$ with $\psi(V_P) = V$ [22, proof of Thm. 1.3]. Then V_P is special, of positive dimension and maximal in Y_P, since ψ is finite and maps special subvarieties to special subvarieties, and $\Sigma_{V_P} = \Sigma_V$. By (DJK2), $\deg_{\mathcal{L}_{K(P)}} Y_P \leq \deg Y_j \leq D_j$. Let J_p, α and $\mathcal{I}$ be as in (DJK3) for V_P and p, $J := K(P)^p J_p$, $\psi_J\colon \mathrm{Sh}_J(P,X_P)_{\mathbb{C}} \to \mathrm{Sh}_K(G,X)_{\mathbb{C}}$ the natural map, $\widetilde{V}$ the component of the preimage of V_P given by (DJK3)(iii), and $\widetilde{Y}$ a component of the preimage of Y_P containing $\widetilde{V}$. By the projection formula and Lemma 5.3(a),

$$\deg_{\mathcal{L}_J} \widetilde{Y} \leq p^{f_2} D_j, \qquad \deg_{\mathcal{L}_J} T_\alpha \widetilde{Y} \leq p^{k_2+f_2} D_j.$$

Cut case: $\widetilde{Y} \not\subset T_\alpha \widetilde{Y}$. By (DJK3)(iii), $\widetilde{V} \subset T_\alpha \widetilde{V} \subset T_\alpha \widetilde{Y}$, so $\widetilde{V} \subset \widetilde{Y} \cap T_\alpha \widetilde{Y}$, a proper closed subset of the irreducible $\widetilde{Y}$. Let C be a component of $\widetilde{Y} \cap T_\alpha \widetilde{Y}$ containing $\widetilde{V}$ and $Y_{j+1} := \psi_J(C)$, closed and irreducible as ψ_J is finite, containing V, with $\dim Y_{j+1} = \dim C < \dim Y_j$. The level $J \subset K(P)$ is neat, so Lemma 5.2(c) gives $\deg_{\mathcal{L}_J} C \leq c_B\, p^{k_2+2f_2} D_j^2$, and by (DJK2) and the projection formula for $\mathrm{Sh}_J \to \mathrm{Sh}_{K(P)}$, $\deg Y_{j+1} \leq d_G \deg_{\mathcal{L}_J} C \leq D_{j+1}$.

Stable case: $\widetilde{Y} \subset T_\alpha \widetilde{Y}$. The variety Y_P contains a special point, as the special subvariety $V_P \subset Y_P$ does [56, Lemma 13.3]. By (DJK3)(iv) and Lemma 7.2, the preimage of Y_P in X_P^+ is $X_{\mathcal{I}}^+$-saturated. By saturation, Y_P contains the image V' of $X_{\mathcal{I}}^+ \times \mathrm{pr}_{\mathcal{I}^c}(X_H^+)$, which contains V_P; the second factor is a point, as $Z(H)(\mathbb{R})$ fixes the points of X_H and $\pi_i(H^{\mathrm{der}}) = 1$ for $i \notin \mathcal{I}$, and V' is special, as in Step 3 of the proof of Theorem 6.1 with $P_{\mathcal{I}}$ and $\mathcal{I}^c$ in place of P_i and $\{i\}^c$. If $V' = V_P$, then $X_H^+ = X_{\mathcal{I}}^+ \times \mathrm{pr}_{\mathcal{I}^c}(X_H^+)$ by dimension, so H is the generic Mumford–Tate group on this set; its image in P^{ad} lies in the product of $P_{\mathcal{I}}$ with the image of H in $P_{\mathcal{I}^c}$, a torus, and projects onto $P_{\mathcal{I}}$, as P is the generic Mumford–Tate group on X_P; so its derived group, the image of H^{der}, is $P_{\mathcal{I}}$. The rest of the proof of [22, Thm. 8.1], which is that of [21, Thm. 8.1], which follows the proof of [50, Thm. 7.2.1] closely [21, p. 99], restricted to the factors in $\mathcal{I}$ [22, p. 22], shows that either Y_P contains a special subvariety

strictly containing V_P, or the image of H^{der} in P^{ad} is $P_{\mathcal{I}}$, where (H, X_H) defines V_P. The first is excluded by the maximality of V_P. In the second case Lemma 7.3 and (DJK2) give

$$\deg V \le d_G \deg_{\mathcal{L}_{K(P)}} V_P \le d_G \deg_{\mathcal{L}_{K(P)}} Y_P \le d_G \deg Y_j \le d_G D_j,$$

and we stop.

Each cut lowers the dimension, so there are at most r cuts. The recursion gives

$$D_j = (d_G\, c_B\, p^{k_2+2f_2})^{2^j-1} D^{2^j},$$

which is increasing in j, so every D_j that occurs is at most D_r. With the factor d_G of the stable case this is (7.1).

For the last assertion let p be the least prime $p \ge N_K$ not dividing Π_V; then $p \notin \Sigma_V$. At most $\log_2 \Pi_V$ primes divide Π_V. With $y := y(\Pi_V)$ and $x := p^*(\Pi_V) \ge 17$ we have $x/\log x \ge y$, since $\log(2y \log y) \le 2 \log y$, and $\pi(x) \ge x/\log x$ [74, (3.5)]; so $[N_K, x]$ contains at least $y - N_K + 1 > \log_2 \Pi_V$ primes, and $p \le p^*(\Pi_V)$. As the bases in (7.1) are at least 1 and $r \le \dim Z$, (DJK1) and (7.1) give $c_1 \Pi_V^{\eta_1} < \deg V \le R_{\dim Z, D}(\Pi_V)$. Hence $\Pi_V \le \Pi_0(\dim Z, D)$, and $\deg V \le c^{\mathrm{ad}}_{\deg}(\dim Z, D)$, as $R_{z,D}$ is non-decreasing. □

Theorem 1.6 is Theorem 7.4, applied at the level $g_i K g_i^{-1}$, with the lattice $\rho(g_i)\widehat{\mathbb{Z}}^{n_\rho}$, of the component containing Z (Section 2.2), with $c_B = e^*(n)^{2n}$ (Lemma 5.2(c)). The bound holds for facteur and non-facteur V alike. Facteur special subvarieties of bounded degree come in infinite families [22, p. 2], so a bound for the degree alone leaves infinitely many candidates; Theorem 8.1 bounds the other two terms d_{L_V} and α_V of $\Delta(V)$.

7.4. **An index bound.** The following lemma replaces [22, Lemma 6.7] in the proof of (DJK1) (Section 7.1).

Lemma 7.5. *Let V be a special subvariety of $\mathrm{Sh}_K(G, X^+)_{\mathbb{C}}$ of positive dimension, the image of $X_H^+ \times \{1\}$, and let $\pi\colon \widetilde{H} \to H^{\mathrm{der}}$, $\Gamma(H)^+$ and $\Gamma(\widetilde{H})$ be as in* [22, pp. 13–14]. *Write $\widetilde{H} = \prod_i \mathrm{Res}_{K_i/\mathbb{Q}} H_i'$ with K_i a number field and H_i' absolutely almost simple and simply connected over K_i* [22, §5]. *For each i let ℓ_i and s_i be the field and the integer that* [73, §§0.2 and 0.4] *attach to the quasi-split inner form of H_i', and put*

$$N_i := d_{\ell_i}/d_{K_i}^{[\ell_i:K_i]}, \qquad \varpi_i := d_{K_i}^{\dim H_i'/2} N_i^{s_i/2},$$

where d_L is the absolute value of the discriminant of a number field L (Prasad writes D_L). Let r_G be the absolute rank of G, and B_Z the constant of [22, Lemma 6.4]. *There is an effectively computable c_2^*, depending only on r_G and B_Z, such that*

$$[\Gamma(H)^+ : \pi(\Gamma(\widetilde{H}))] \le c_2^*\, (B_Z^{3r_G})^{|\Sigma_V|} \prod_i \varpi_i^{1/2}.$$

Proof. By the proof of [22, Lemma 6.4], its constant B_Z bounds $|Z(\widetilde{H})(\mathbb{C})|$. Let $Z_\pi := \ker \pi$, and let Σ be the union of Σ_V and the set of primes dividing $|Z_\pi(\mathbb{C})|$, so that $|\Sigma| \le |\Sigma_V| + \log_2 B_Z$. A class in the first Galois cohomology of a number field is *unramified* at a finite place if its restriction to an inertia group there is 0. Step (i) follows [22, proof of Lemma 6.7, p. 14]; steps (ii) and (iii) take the place of their restriction to a field that splits Z_π and of their Lemma 6.8 [22, p. 15].

(i) As Z_π is central, the connecting map $\partial\colon H^{\mathrm{der}}(\mathbb{Q}) \to H^1(\mathbb{Q}, Z_\pi)$ is a homomorphism: if $h, h' \in \widetilde{H}(\overline{\mathbb{Q}})$ and $\gamma := \pi(h)$ and $\gamma' := \pi(h')$ lie in $H^{\mathrm{der}}(\mathbb{Q})$, then $(hh')^{-1}\sigma(hh') = h^{-1}\sigma(h) \cdot h'^{-1}\sigma(h')$ for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, since $h^{-1}\sigma(h) \in Z_\pi(\overline{\mathbb{Q}})$ is central. Its kernel is $\pi(\widetilde{H}(\mathbb{Q}))$, and $\Gamma(\widetilde{H})$ is the full preimage of $\Gamma(H)^+$ in $\widetilde{H}(\mathbb{Q})$; so ∂ embeds $\Gamma(H)^+/\pi(\Gamma(\widetilde{H}))$ in $H^1(\mathbb{Q}, Z_\pi)$.

Let M be the image. For $p \notin \Sigma$ every class in M is unramified at p, since $\Gamma(H)^+ \subset K(H^{\mathrm{der}})$ [22, §3.4] and the image of $K(H^{\mathrm{der}})_p$ in $H^1(\mathbb{Q}_p, Z_\pi)$ lies in $H^1(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p), \mathcal{Z}(\mathbb{Z}_p^{\mathrm{un}}))$ for a smooth model $\mathcal{Z}$ of $Z_{\pi,\mathbb{Q}_p}$ over $\mathbb{Z}_p$ [22, p. 14], [21, p. 88].

(ii) The center of $\widetilde{H}$ is $\prod_i \mathrm{Res}_{K_i/\mathbb{Q}} A_i$ with $A_i := Z(H_i')$, and it contains Z_π. The kernel of $H^1(\mathbb{Q}, Z_\pi) \to H^1(\mathbb{Q}, Z(\widetilde{H}))$ is the image of $(Z(\widetilde{H})/Z_\pi)(\mathbb{Q})$, of order at most B_Z. By Shapiro's lemma [59, II.1.11], $H^1(\mathbb{Q}, \mathrm{Res}_{K_i/\mathbb{Q}} A_i) \cong H^1(K_i, A_i)$, through restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}/K_i)$ and evaluation at 1; as the inertia groups of K_i are the intersections of those of $\mathbb{Q}$ with $\mathrm{Gal}(\overline{\mathbb{Q}}/K_i)$, classes unramified at p go to classes unramified at every place of K_i above p. Let $M_i \subset H^1(K_i, A_i)$ be the image of M. Then $|M| \le B_Z \prod_i |M_i|$, and the classes in M_i are unramified at every place above a prime $p \notin \Sigma$.

(iii) Inner twisting acts trivially on the center, so A_i is isomorphic over K_i to the center of the quasi-split inner form $\mathcal{G}_i$ of H_i', and $\mathrm{Gal}(\overline{K_i}/K_i)$ acts on $X^*(A_i)$, the weight lattice modulo the root lattice, through its action on the Dynkin diagram of $\mathcal{G}_i$. If H_i' is not a triality form, $\mathcal{G}_i$ splits over ℓ_i (and $\ell_i = K_i$ for inner forms), so $A_i \cong \prod_{k=1}^{q_i} \mu_{n_{i,k}}$ over ℓ_i, with $\prod_k n_{i,k} = |A_i(\mathbb{C})|$; inflation and restriction [59, II.1.34] map $H^1(K_i, A_i)$ to $H^1(\ell_i, A_i) = \prod_k \ell_i^\times / \ell_i^{\times n_{i,k}}$ [59, II.1.22] with kernel $H^1(\mathrm{Gal}(\ell_i/K_i), A_i(\ell_i))$, of order at most $|A_i(\mathbb{C})|$ as $[\ell_i : K_i] \le 2$. If H_i' is a triality form, of type ${}^3\mathrm{D}_4$ or ${}^6\mathrm{D}_4$, then $X^*(A_i) \cong (\mathbb{Z}/2)^2$, and its three non-zero elements, the classes of the fundamental weights of the three outer vertices, are permuted as these vertices; as ℓ_i is the field fixed by the stabilizer in $\mathrm{Gal}(\overline{K_i}/K_i)$ of one outer vertex, $A_i \cong \ker(N\colon \mathrm{Res}_{\ell_i/K_i}\mu_2 \to \mu_2)$, whose character group is $(\mathbb{Z}/2)^{\mathrm{Hom}_{K_i}(\ell_i, \overline{K_i})}$ modulo the diagonal. Since $N(-1) = -1$, the map $H^1(K_i, A_i) \to H^1(K_i, \mathrm{Res}_{\ell_i/K_i}\mu_2) = \ell_i^\times/\ell_i^{\times 2}$ is injective; here put $q_i := 1$ and $n_{i,1} := 2$. In both cases these maps keep classes unramified, and a class in $\ell_i^\times/\ell_i^{\times m}$ that is unramified at a finite place w is an m-th power over the maximal unramified extension of $\ell_{i,w}$, so its valuation at w is divisible by m. For a number field L, a set S of finite places of L and $m \ge 1$ put $L(S,m) := \{a \in L^\times/L^{\times m} : v_w(a) \equiv 0 \bmod m \text{ for every finite place } w \notin S\}$, and let S_i be the set of places of ℓ_i above Σ. Then

$$|M_i| \le |A_i(\mathbb{C})| \prod_{k=1}^{q_i} |\ell_i(S_i, n_{i,k})|.$$

The valuations at the places in S map $L(S,m)$ to $(\mathbb{Z}/m)^S$ with kernel $L(\emptyset, m)$, and $|L(\emptyset,m)| \le m^{r_1(L)+r_2(L)} h_L$ (proof of Proposition 9.6); as $|S_i| \le [\ell_i : \mathbb{Q}]\,|\Sigma|$, we get $|\ell_i(S_i,m)| \le m^{[\ell_i:\mathbb{Q}](|\Sigma|+1)} h_{\ell_i}$.

(iv) By Corollary 9.10, $h_L \le \max(1, B_L(1+\log B_L)^{n_L-1})$ with $B_L \le (4/\pi)^{n_L/2} d_L^{1/2}$; so $h_L \le c_h(n_L)\, d_L^{3/4}$ with $c_h(n) := (4/\pi)^{n/2} \sup_{y \ge 0} (n+y/2)^{n-1} e^{-y/4}$, which is non-decreasing in n and at least 1. As $d_{\ell_i} = d_{K_i}^{[\ell_i:K_i]} N_i$, with $N_i = |N_{K_i/\mathbb{Q}}(\mathfrak{d}(\ell_i/K_i))|$ [73, §0.2],

$$h_{\ell_i}^{q_i} \le c_h([\ell_i:\mathbb{Q}])^{q_i}\, d_{K_i}^{3q_i[\ell_i:K_i]/4} N_i^{3q_i/4} \le c_h([\ell_i:\mathbb{Q}])^{q_i}\, \varpi_i^{1/2},$$

because $3q_i[\ell_i : K_i] \le \dim H_i'$ for every form, and $3q_i \le s_i$ for outer and triality forms (for inner forms $N_i = 1$). Indeed, $q_i \le 1$ except for forms of type D_r with r even that are not triality forms, where $q_i = 2$, $\dim H_i' \ge 28$ and, for outer forms, $s_i = 2r-1 \ge 7$; $\dim H_i' \ge 3$, $\dim H_i' \ge 8$ if $[\ell_i : K_i] = 2$, and $\dim H_i' = 28$ if $[\ell_i : K_i] = 3$; and $s_i \ge 5$ for outer and triality forms [73, §0.4].

(v) Each H_i' has rank at least 1, and $\sum_i [K_i : \mathbb{Q}] \operatorname{rank} H_i' = \operatorname{rank} H^{\mathrm{der}} \le r_G$; so there are at most r_G indices i, and $[\ell_i : \mathbb{Q}] \le 3[K_i : \mathbb{Q}] \le 3r_G$. Also $\prod_{i,k} n_{i,k} \le \prod_i |A_i(\mathbb{C})| \le |Z(\widetilde{H})(\mathbb{C})| \le$

B_Z. By (i)–(iv),

$$[\Gamma(H)^+ : \pi(\Gamma(\widetilde{H}))] = |M| \le B_Z^2\, B_Z^{3r_G(|\Sigma|+1)}\, c_h(3r_G)^{2r_G} \prod_i \varpi_i^{1/2},$$

and $|\Sigma| \le |\Sigma_V| + \log_2 B_Z$ gives the lemma with $c_2^* := B_Z^{2+3r_G(1+\log_2 B_Z)}\, c_h(3r_G)^{2r_G}$. □

8. The main theorem for adjoint groups

In this section G is adjoint, K satisfies (Lv), $n := \dim \mathrm{Sh}_K(G,X)$ and $e_0 := e^*(n)$. Let $F \supset E(G,X)$ be a number field, $d := [F:\mathbb{Q}]$, and $Z \subset \mathrm{Sh}_K(G,X)$ a closed subvariety, geometrically irreducible and defined over F, with $D := \deg Z$ and $z := \dim Z$. Let $(P, X_P) \subset (G,X)$ be the Shimura subdatum, with P the generic Mumford–Tate group on X_P, that defines the smallest special subvariety containing Z [50, p. 14], the special closure of Z; put $K_P := K \cap P(\mathbb{A}_f)$, $E_P := E(P,X_P)$, $T_Z := Z(P)^\circ$, and let $\psi\colon \mathrm{Sh}_{K_P}(P,X_P) \to \mathrm{Sh}_K(G,X)$ be the natural morphism, defined over E_P [56, Remark 13.8].

8.1. **Statement.** A special subvariety V maximal in Z belongs to exactly one of three classes:

(S) $T_V = 1$;
(N) $T_V \ne 1$ and $T_V \not\subset Z(P)$;
(C) $T_V \ne 1$ and $T_V \subset Z(P)$.

Here V is regarded inside the special subvariety defined by (P,X_P) (Lemma 8.3). Class (N) says that V is not strongly special in $\mathrm{Sh}_{K_P}(P,X_P)_{\mathbb{C}}$, since the image of T_V in P^{ad} is the connected center of the image of H_V; so the Hecke step applies to it. Class (C) consists of special subvarieties that are strongly special in $\mathrm{Sh}_{K_P}(P,X_P)_{\mathbb{C}}$ but not in $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$, and the Hecke step says nothing about them; the bounds come instead from the Galois orbit of the component containing the lift Z_P of Z (Lemma 8.3), for d_{L_V} (Lemma 8.5), and from the translates of Z_P by the center, for α_V (Lemma 8.8).

With b_r, Δ_r of Section 6 and $\mathcal{B}$ of Lemma 6.8, computed with $\bar{d} = d$, and $c_{\deg}^{\mathrm{ad}}$ of Section 7, put

$$c_{\mathrm{N}}(\delta,z,D,d) := \mathcal{B}\big(\Delta_z(D)/C_*,\ b_z\big),$$

$$c^{\mathrm{ad}}(\delta,z,D,d) := \max\Big(c_{\deg}^{\mathrm{ad}}(z,D),\ c_{\mathrm{N}}(\delta,z,D,d)^{1/\mu},\ e^{\beta_0(\delta)}(R_G d)^{1/\mu},\ a^4(a!)^{r_U}\, d\,(e_0^z D)^{2(z+1)}\Big).$$

Theorem 8.1. *In this setting let V be a special subvariety maximal in Z.*

(a) $\deg V \le c_{\deg}^{\mathrm{ad}}(z,D)$.

(b) If V is of class (S), then $\Delta(V) = \deg V \le c_{\deg}^{\mathrm{ad}}(z,D)$.

(c) If V is of class (N) and $\mathrm{QRH}(\delta; L_V)$ *holds, then* $\alpha_V\, d_{L_V}^{\mu} \le c_{\mathrm{N}}(\delta,z,D,d)$*; hence* $\alpha_V \le c_{\mathrm{N}}(\delta,z,D,d)$ *and* $d_{L_V} \le c_{\mathrm{N}}(\delta,z,D,d)^{1/\mu}$.

(d) If V is of class (C), then $T_V = T_Z$ and $\alpha_V \le a^4(a!)^{r_U}\, d\,(e_0^z D)^{2(z+1)}$*; if moreover* $\mathrm{QRH}(\delta; L_V)$ *holds, then* $d_{L_V} \le e^{\beta_0(\delta)}(R_G d)^{1/\mu}$.

In particular, if $\mathrm{QRH}(\delta; L_V)$ *holds, then* $\Delta(V) \le c^{\mathrm{ad}}(\delta,z,D,d)$*. No hypothesis is used in (a), (b) and in the bound for α_V in (d).*

The constants are non-decreasing in z, D and d. This holds because b_z, $\Delta_z(D)$ and $\Delta_z(D)/C_* \ge 1$ are non-decreasing, $\mathcal{B}$ is non-decreasing in both arguments (Lemma 6.8), and $R_{z,D}$ is non-decreasing in z and D.

Remark 8.2. For adjoint G the constant $c^{\mathrm{ad}}(\delta, z, D, d)$ is at most $\exp(O(d\log(d+1)))\, D^{O(1)}$, as $d\log\log(D+2) \le d\log d + \log(D+2)$. Indeed, let $\Lambda := \Delta_z(D)/C_* = (c_B dD)^{2^z}/C_*$ and $b := b_z = (k_1 d + 3f)2^z$. Then $\log c_{\mathrm{N}} = \log\Lambda + (b/\delta)\log(\varsigma(\mathcal{W}+1))$ and $\mathcal{W} = O(\log\Lambda + (b/\delta)\log(b/\delta))$, so for V of class (N)

$$\alpha_V\, d_{L_V}^{\mu} \;\le\; \frac{(c_B dD)^{2^z}}{C_*}\,\exp\Big(O\big(d\log(d+1) + d\log\log(D+2)\big)\Big),$$

the implied constants depending on (G, X, K, ρ) and δ. So, for fixed d, $\alpha_V \le D^{2^z+o(1)}$ and $d_{L_V} \le D^{2^z/\mu+o(1)}$ as $D\to\infty$. As $\log\Pi_0(z,D) = O(\log D)$ (Theorem 7.4), $c^{\mathrm{ad}}_{\deg}(z,D)$ is at most $D^{2^z+o(1)}$ as well. With the form $C(N)\alpha_V(\log d_{L_V})^N$ of the Galois bound used in [50] and a fixed $N > b/\delta$, the same argument bounds only $\alpha_V(\log d_{L_V})^{N-b/\delta}$. So d_{L_V} would be exponential in a power of D, and for $N = \lfloor b/\delta\rfloor + 2$, so that $N - b/\delta \ge 1$, doubly exponential in d. For a general Shimura datum, Theorem 9.12 evaluates these constants at (D_n, d_n), where $D_n \le (e_0^{2n} D_0)^{2^n}$ and $d_n \le d_0(e_0^{2n}D_0)^{2^{n+1}}$. The recursion on dimension lets the field of definition grow, and the constant of Theorem 1.1 is exponential in a power of D. At $\delta = 1/32$ the factor $1/C_*$ is large: $\log(1/c_*(\delta)) = \mu\beta_0(\delta) = 32\kappa(\delta)/(\mu\delta^2)$, about $2.0\cdot 10^6/\mu$. (The first term of $\beta_0(\delta)$ is the largest, as $\kappa(\delta)/\delta^2 \ge 8$ and $1/\theta \ge 8n_T \ge 16$.)

8.2. **The lift to the special closure.**

Lemma 8.3. *(a) Exactly one irreducible component Z_P of $\psi^{-1}(Z)$ satisfies $\psi(Z_P) = Z$. It is Hodge generic, geometrically irreducible and defined over FE_P, $\psi|_{Z_P}\colon Z_P \to Z$ is finite and birational, and $\deg_{\mathcal{L}_{K_P}} Z_P \le D$.*

(b) If $V \subset Z$ is special and maximal in Z, some component V_P of $\psi^{-1}(V)\cap Z_P$ has $\psi(V_P) = V$. It is special in $\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$, maximal among the special subvarieties of Z_P, and has the same T_V, α_V, β_V and reflex field as V.

Proof. (a) Let $z_0 = [h_0, 1] \in Z$, $h_0 \in X_P^+$, have Mumford–Tate group P; such points form the complement in Z of a countable union of proper closed subsets. If $[h, g] \in \psi^{-1}(z_0)$, then $h = qh_0$ for some $q \in G(\mathbb{Q})$, so the Mumford–Tate group of h is qPq^{-1}; it is contained in P, as $h \in X_P$, hence equal to P. So every point over z_0 is Hodge generic, and $\psi^{-1}(z_0)$ is one point by the injectivity of ψ on the Hodge generic locus [85, Lemma 2.2]. As ψ is finite and Z lies in its image, the smallest special subvariety containing Z being the image of $X_P^+ \times \{1\}$ (Section 2), some component of $\psi^{-1}(Z)$ maps onto Z. Two such components C_1, C_2 both contain the points over the Hodge generic points of Z, which are dense in each by Baire's theorem, as $\psi|_{C_i}$ is finite onto Z; so $C_1 = C_2$. The group $\mathrm{Gal}(\overline{\mathbb{Q}}/FE_P)$ permutes the components of $\psi^{-1}(Z)$ that map onto Z, so it fixes Z_P. Finally $\psi|_{Z_P}$ is finite and injective on a dense set, so of degree 1, and $\deg_{\mathcal{L}_{K_P}} Z_P \le \deg Z = D$ by [50, Cor. 5.3.10], as Z_P is Hodge generic and $Z(P)(\mathbb{R})$ is compact.

(b) Some component of $\psi^{-1}(V)\cap Z_P$ maps onto V, as $\psi|_{Z_P}$ is finite onto Z; it has dimension $\dim V$, so it is a component of $\psi^{-1}(V)$, hence special [50, p. 14]. A special V' with $V_P \subsetneq V' \subset Z_P$ would give the special subvariety $\psi(V') \subset Z$ of dimension larger than $\dim V$. Finally V_P and V are defined by the same Shimura datum $(H, X_H) \subset (P, X_P)$, V_P being the image of $X_H^+ \times \{1\}$ (Section 2), and $K_P \cap T(\mathbb{A}_f) = K \cap T(\mathbb{A}_f)$ for every torus $T \subset P$. □

Lemma 8.4. *If V is special and maximal in Z, with $T_V \neq 1$ and $T_V \subset Z(P)$, then $T_V = T_Z$. If moreover V is a point, then $Z = V$.*

Proof. Let $\nu\colon P \to P^{\mathrm{ab}} := P/P^{\mathrm{der}}$. All $h \in X_P$ have the same image $h^{\mathrm{ab}} := \nu \circ h$, as X_P is one $P(\mathbb{R})$-conjugacy class and P^{ab} is commutative. The Mumford–Tate group of h^{ab} is the image of that of a Hodge generic h, so it is P^{ab}. For $h \in X_{H_V}$ the Mumford–Tate group of h lies in H_V and maps onto that of h^{ab}, so $\nu(H_V) = P^{\mathrm{ab}}$; as $H_V = T_V \cdot H_V^{\mathrm{der}}$ with $H_V^{\mathrm{der}} \subset P^{\mathrm{der}}$, $\nu(T_V) = P^{\mathrm{ab}}$. If $T_V \subset Z(P)$, then $T_V \subset Z(P)^\circ$, and $Z(P)^\circ \to P^{\mathrm{ab}}$ is an isogeny, so $\dim T_V \geq \dim Z(P)^\circ$ and $T_V = T_Z$. If V is a point h, then $H_V = T_V$ is central in P, so $X_P = \{h\}$ and $Z = V$. □

8.3. **Class (C): the discriminant.**

Lemma 8.5. *Suppose $T_Z \neq 1$ and let L be its splitting field. If $\mathrm{QRH}(\delta; L)$ holds, then $c_*(\delta) d_L^\mu \leq R_G d$, that is, $d_L \leq e^{\beta_0(\delta)} (R_G d)^{1/\mu}$.*

Proof. Let S_P^+ be the connected component of $\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$ containing Z_P. Every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/FE_P)$ fixes Z_P (Lemma 8.3(a)), hence S_P^+; so the $\mathrm{Gal}(\overline{\mathbb{Q}}/E_P)$-orbit of S_P^+ has at most $[FE_P : E_P] \leq d$ members.

Let $C := P^{\mathrm{ab}}$, $\bar{x} := h^{\mathrm{ab}}$, $E_C := E(C, \bar{x}) \subset E_P$, K_C^m the maximal compact open subgroup of $C(\mathbb{A}_f)$ and $\mathrm{Cl}_C := C(\mathbb{Q}) \backslash C(\mathbb{A}_f) / K_C^m$. The map $\pi_0(\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}) \to \mathrm{Cl}_C$, $[x, g] \mapsto [\nu(g)]$, is well defined, as $\nu(K_P) \subset K_C^m$, and it is $\mathrm{Gal}(\overline{\mathbb{Q}}/E_P)$-equivariant when σ acts on Cl_C by multiplication by $r_C(N_{E_P/E_C}(s_\sigma))$, where s_σ is a finite idele of E_P with Artin symbol $\sigma|_{E_P^{\mathrm{ab}}}$ and r_C is the reciprocity morphism of $(C, \bar{x})$. This follows from Shimura reciprocity at a special point x of X_P and $\nu \circ r_x = r_C \circ N_{E(x)/E_C} = r_C \circ N_{E_P/E_C} \circ N_{E(x)/E_P}$, as in [56, §13, (64) and footnote 73]; that argument does not use that P^{der} is simply connected. Hence the orbit of S_P^+ has at least $|\mathrm{Im}_{E_P}|$ members, where Im_{E_P} is the image of $r_C \circ N_{E_P/E_C}$ in Cl_C.

The splitting field of C is L, as C and T_Z are isogenous [85, Remark 2.20], and $E_C \subset L$ [56, Remark 12.3(a)]. Let $L' := LE_P$. Since $N_{E_P/E_C} \circ N_{L'/E_P} = N_{L/E_C} \circ N_{L'/L}$, Im_{E_P} contains the image of $N_{L'/L}((\mathbb{A}_f \otimes L')^\times)$ under $r_L := r_C \circ N_{L/E_C}$. As in the proof of Corollary 3.4, this image has index at most $[L' : L] \leq [E_P : \mathbb{Q}] \leq R_G$ in Im_L. Proposition 3.3, applied to $(H, X_H) = (P, X_P)$, gives $|\mathrm{Im}_L| \geq c_*(\delta) d_L^\mu$. So $c_*(\delta) d_L^\mu / R_G \leq d$. □

8.4. **Class (C): the level.** For the next two lemmas compare [45]; we include short proofs, which give the constants used.

Lemma 8.6. *Let $Y \subset \mathbb{P}^N_{\mathbb{C}}$ be closed, reduced and of pure dimension, of degree e (the sum of the degrees of its components). Then Y is the common zero set of the forms of degree e vanishing on Y.*

Proof. Let $Y = Y_1 \cup \cdots \cup Y_t$, with $\deg Y_i = a_i$ and $\dim Y =: m < N$ (the case $Y = \mathbb{P}^N$ is trivial). Let $y \notin Y$; it suffices to find forms P_i of degree at most a_i vanishing on Y_i with $P_i(y) \neq 0$, and to multiply them and a power of a linear form not vanishing at y. If $m = N - 1$, take P_i an equation of Y_i. If $m \leq N - 2$, a general $(N - m - 1)$-plane Π through y misses Y_i, since projection from y maps Y_i onto an m-dimensional subvariety of $\mathbb{P}^{N-1}$. Let $\Lambda \subset \Pi$ be a hyperplane of Π not containing y. The projection $\pi_\Lambda\colon \mathbb{P}^N \smallsetminus \Lambda \to \mathbb{P}^{m+1}$ is finite on Y_i, its image is a hypersurface $V(F_i)$ of degree at most a_i, and the fiber of π_Λ through y is $\Pi \smallsetminus \Lambda$, which misses Y_i; take $P_i := F_i \circ \pi_\Lambda$. □

Lemma 8.7. *Let $X_0 \subset \mathbb{P}^N$ be closed and irreducible, of dimension m and degree at most Δ, and let $(Y_i)_{i \in I}$ be any family of closed subsets of $\mathbb{P}^N$, each of pure dimension and of degree at most Δ. Then the irreducible components C of $X_0 \cap \bigcap_i Y_i$ satisfy $\sum_C \deg C \leq \Delta^{m+1}$.*

Proof. By Lemma 8.6, each Y_i is the common zero set of forms of degree $\deg Y_i \le \Delta$; multiplying a form of degree $\Delta' < \Delta$ by $x_0^{\Delta-\Delta'}, \dots, x_N^{\Delta-\Delta'}$ does not change the zero set. So $X_0 \cap \bigcap_i Y_i = X_0 \cap V(f_1, \dots, f_s)$ for finitely many forms f_k of degree Δ. Build a tree with root X_0: a node X at stage k has the single child X if $X \subset V(f_{k+1})$, and otherwise the components of $X \cap V(f_{k+1})$, of dimension $\dim X - 1$ and total degree at most $\Delta \deg X$ (Lemma 5.2(a)). Along each path the dimension drops at most m times, so the leaves after stage s have total degree at most $\Delta^m \deg X_0 \le \Delta^{m+1}$. They cover $X_0 \cap V(f_1, \dots, f_s)$ and lie in it, so its components are among them. □

Let a, A and r_U be as in Table 1. For a torus T with $K_T := K \cap T(\mathbb{A}_f)$ and maximal compact open $K_T^m \subset T(\mathbb{A}_f)$, let $\Theta_A \subset K_T^m/K_T$ be the image of $x \mapsto x^A$.

Lemma 8.8. *If V is a special subvariety maximal in Z of class (C), then*

$$\alpha_V \le a^4 (a!)^{r_U}\, d\, (e_0^z D)^{2(z+1)}.$$

Proof. Put $\Delta := e_0^z D$ and $T := T_Z$, so $T_V = T$ by Lemma 8.4. Let $H := H_V$, $K_H := K \cap H(\mathbb{A}_f)$, and let V_P be as in Lemma 8.3(b), the image of a component V_H of $\mathrm{Sh}_{K_H}(H, X_H)_{\mathbb{C}}$.

Step 1 (translates stay in Z_P). Let F_V be the Galois closure of $E_V = E(H, X_H)$, of degree at most R_G. By [85, Lemma 2.15], with A from [85, Prop. 2.9], neither of which uses GRH, $\Theta_A \cdot V_H \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F_V) \cdot V_H$. Translation by $\theta \in T(\mathbb{A}_f)$ is defined on $\mathrm{Sh}_{K_H}(H, X_H)$ and on $\mathrm{Sh}_{K_P}(P, X_P)$, as T is central in H and in P; it commutes with the natural map and with $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$. So Θ_A permutes the at most $[FF_V : F_V] \le d$ orbits of $\mathrm{Gal}(\overline{\mathbb{Q}}/FF_V)$ on $\mathrm{Gal}(\overline{\mathbb{Q}}/F_V) \cdot V_H$, and the stabilizer $\Theta^{(F)} \subset \Theta_A$ of the orbit of V_H has index at most d. For $\theta \in \Theta^{(F)}$, $\theta V_H = \sigma V_H$ with $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/FF_V)$. As $E_P \subset E_V$ [56, Remark 12.3(c)], σ fixes FE_P, so $\sigma Z_P = Z_P$ and $\theta V_P = \sigma V_P \subset Z_P$.

Step 2 (cutting by all translates). Each $\theta \in \Theta^{(F)}$ has $V_P \subset \theta^{-1} Z_P \cap Z_P$, so θ stabilizes S_P^+, the component containing Z_P. We have $S_P^+ = \Gamma \backslash X_P^+$ with Γ the image in $P^{\mathrm{ad}}(\mathbb{Q})$ of $P(\mathbb{Q})_+ \cap g K_P g^{-1}$ for some g, which is arithmetic (Lemma 2.1) and neat [68, 0.6], [50, §4.1.4]. Translation by θ extends to the Baily–Borel compactification $\overline{S}_P^+$ and preserves $\mathcal{L} := \mathcal{L}_{K_P}|_{\overline{S}_P^+}$, the extension of the canonical bundle. By Theorem 1.5 for (P, X_P) and K_P, restricted to the component $\overline{S}_P^+$, and as e^* is increasing and $\dim S_P^+ \le n$, $|\mathcal{L}^{\otimes e_0}|$ defines a finite injective $\varphi\colon \overline{S}_P^+ \to \mathbb{P}^N$ with $\varphi^* \mathcal{O}(1) = \mathcal{L}^{\otimes e_0}$, birational on irreducible closed subsets, as in the proof of Lemma 5.2(b). Each $Y_\theta := \varphi(\theta^{-1} \overline{Z}_P)$ is irreducible of dimension z and degree $e_0^z \deg_{\mathcal{L}} Z_P \le \Delta$, since $\deg_{\mathcal{L}} Z_P \le D$ by (DJK2) (Z_P is Hodge generic, Lemma 8.3(a)). As φ is injective, $\varphi(\bigcap_\theta \theta^{-1} \overline{Z}_P) = \bigcap_\theta Y_\theta$, and Lemma 8.7 with $X_0 = Y_1$ shows that $\overline{W} := \bigcap_{\theta \in \Theta^{(F)}} \theta^{-1} \overline{Z}_P$ has at most Δ^{z+1} components, each of $\mathcal{L}$-degree at most Δ^{z+1}. The set $\overline{W}$ contains V_P and is $\Theta^{(F)}$-stable. Let W_0 be a component of $\overline{W} \cap S_P^+$ containing V_P and Θ_1 its stabilizer in $\Theta^{(F)}$; then $[\Theta^{(F)} : \Theta_1] \le \Delta^{z+1}$ and $\deg_{\mathcal{L}_{K_P}} W_0 \le \Delta^{z+1}$.

Step 3 (a stable subvariety is at least as large as its stabilizer). We use the level change of [85, pp. 21–22] for P in place of H. Put $K_P^m := K_T^m K_P$, compact open as T is central; choose $\gamma_3 \in \mathrm{GL}_{n_\rho}(\mathbb{Q}_3)$ with $K_{P,3}^m \subset \gamma_3 \mathrm{GL}_{n_\rho}(\mathbb{Z}_3) \gamma_3^{-1}$, let $K_P^{m'} := (K_{P,3}^m \cap \gamma_3 K(3)_3 \gamma_3^{-1}) \times \prod_{p \ne 3} K_{P,p}^m$, with $K(3)_3$ the level-3 congruence subgroup of $\mathrm{GL}_{n_\rho}(\mathbb{Z}_3)$, and $K_P' := K_P \cap K_P^{m'}$. Then $K_P^{m'}$ is neat, K_P' is normal in $K_P^{m'}$, and $[K_P^m : K_P^{m'}]$, $[K_P : K_P'] \le a$. The map $\pi'\colon \mathrm{Sh}_{K_P'}(P, X_P) \to \mathrm{Sh}_{K_P^{m'}}(P, X_P)$ is finite étale, and $\mathfrak{K} := K_P^{m'}/K_P'$ acts simply transitively on its fibers [85, Lemma 2.11 and its proof]; $f\colon \mathrm{Sh}_{K_P'}(P, X_P) \to \mathrm{Sh}_{K_P}(P, X_P)$ is finite étale of degree at most a; both pull the Baily–Borel bundle back to the Baily–Borel bundle [50, Prop. 5.3.2].

Let $\widetilde{\Theta}_1 \subset K_T^m$ be the preimage of Θ_1 and $\widetilde{\Theta}'_1 := \widetilde{\Theta}_1 \cap K_P^{m'}$, of index at most a. It acts on $\mathrm{Sh}_{K'_P}(P, X_P)$ by central translations commuting with f, so it permutes the at most a components of $f^{-1}(W_0)$. The stabilizer $\widetilde{\mathfrak{S}}$ of one of them, W', has index at most a in $\widetilde{\Theta}'_1$, so its image in K_T^m/K_T has at least $|\Theta_1|/a^2$ elements, and since $K'_P \cap T(\mathbb{A}_f) \subset K_T$, so does its image $\mathfrak{S}$ in $\mathfrak{K}$. As $\mathfrak{S}$ acts freely on W', $\deg(\pi'|_{W'}) \geq |\mathfrak{S}|$, and the projection formula gives

$$|\Theta_1|/a^2 \leq |\mathfrak{S}| \leq \deg_{\mathcal{L}_{K'_P}} W' \leq \deg_{\mathcal{L}_{K'_P}} f^{-1}(W_0) \leq a \deg_{\mathcal{L}_{K_P}} W_0 \leq a\Delta^{z+1}.$$

Step 4 (from Θ_A to α_V). Steps 1–3 give $|\Theta_A| \leq d \cdot \Delta^{z+1} \cdot a^3\Delta^{z+1}$. Let $K_H^{m'}$ and K'_H be the levels of [85, pp. 21–22] for H, $K_T^{m'} := K_H^{m'} \cap T(\mathbb{A}_f)$, $K'_T := K'_H \cap T(\mathbb{A}_f)$, and Θ'_e the image of $x \mapsto x^e$ on $K_T^{m'}/K'_T$. By [85, Lemma 2.18] with $e = a!$, $\alpha_V \leq (a!)^{r_U}|\Theta'_{A\cdot a!}|$. The map $K_T^{m'}/K'_T \to K_T^m/K_T$ sends $\Theta'_{A\cdot a!}$ into Θ_A, with kernel $(K_T^{m'} \cap K_T)/K'_T$ of order at most $[K_H : K'_H] \leq a$. Hence $\alpha_V \leq (a!)^{r_U} \cdot a \cdot |\Theta_A| \leq a^4(a!)^{r_U} d\Delta^{2(z+1)}$. □

8.5. **Proof of Theorem 8.1.** (a) If $\dim V \geq 1$, Theorem 7.4 applied to Z gives $\deg V \leq c^{\mathrm{ad}}_{\deg}(z, D)$; points have degree $1 \leq c^{\mathrm{ad}}_{\deg}(z, D)$.

(b) For class (S), $\alpha_V = d_{L_V} = 1$ by Definition 2.2.

(c) Let Z_P and V_P be as in Lemma 8.3. Class (N) says that V_P is not strongly special in $\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$, and then $r := z - \dim V \geq 1$: if $V = Z$, then $P = H_V$ and $T_V = Z(P)^\circ$. Apply Theorem 6.7 to (P, X_P), the field $F_1 := FE_P \supset E(P, X_P)$, the variety Z_P, which is Hodge generic, geometrically irreducible and defined over F_1 with $D_Z \leq D$ (Lemma 8.3(a)), and to V_P. Since $E_P \subset E_V$ [56, Remark 12.3(c)], $[F_1E_V : E_V] = [FE_V : E_V] \leq d$, so $\bar{d} = d$ is allowed. If $\Omega_V > \Delta_r(D)x_V^{b_r}$, then $\Omega_V > \Delta_r(D_Z)x_V^{b_r}$, as Δ_r is increasing, and Theorem 6.7 gives a special V' with $V_P \subsetneq V' \subset Z_P$, and $\psi(V') \subset Z$ is special of dimension larger than $\dim V$, against maximality. So $\alpha_V d_{L_V}^{\mu} \leq (\Delta_r(D)/C_*)\, x_V^{b_r}$, and Lemma 6.8 gives $\alpha_V d_{L_V}^{\mu} \leq \mathcal{B}(\Delta_r(D)/C_*, b_r) \leq c_{\mathrm{N}}(\delta, z, D, d)$, as $r \leq z$, $\Delta_r(D)/C_* \geq 1$ and b_r, $\Delta_r(D)$ are non-decreasing in r. As $\alpha_V \geq 1$ and $d_{L_V} \geq 1$, both bounds follow.

(d) $T_V = T_Z$ by Lemma 8.4, so L_V is the splitting field of T_Z; Lemmas 8.5 and 8.8 give the bounds.

The last assertion follows from (a)–(d), since the three classes exhaust the maximal V. □

8.6. **One neat factor suffices.** Klingler and Yafaev's normalization (Lv) asks for a neat factor at 3; for $\mathcal{A}_g$ at the level $K(N)$ the adjoint level $\lambda(K(N))$ has one at a prime dividing N, which need not be 3 (Theorem 9.13(a)). Only a neat factor at some prime is used:

(Lv′) K is a neat product level, $\rho(K) \subset \mathrm{GL}_{n_\rho}(\widehat{\mathbb{Z}})$, and K_{p_0} is neat for some prime p_0.

Proposition 8.9. *Theorem 8.1 holds with (Lv′) in place of (Lv), with the constants of Section 2.3 at the level K itself.*

Proof. We go through the uses of (Lv) in the proof of Theorem 8.1.

(1) The Hecke step (Theorem 6.1) uses the proof of [50, Thm. 7.2.1] when $\widetilde{Z} \subset T_m\widetilde{Z}$, at a prime l with K_l hyperspecial. That theorem assumes only that K_{p_0} is neat for some prime p_0, and its proof, like Step 3 of the proof of Theorem 6.1 run with p_0 in place of 3, uses p_0 only through the neatness of $K_{p_0} \cap P(\mathbb{Q}_{p_0})$ for Shimura subdata (P, X_P) [50, p. 21] and of its images under $\lambda\colon P \to P^{\mathrm{ad}}$ (the factor $I_{p_0}^{\mathrm{ad}}$ of that Step 3) and under the projections to the $\mathbb{Q}$-simple factors of P^{ad} [50, p. 32 and Lemma 7.2.8], which are neat as images of a subgroup of the neat group K_{p_0} [68, 0.6]; and $l \neq p_0$, since a hyperspecial subgroup is not neat.

(2) The other results of [50, 85, 28] that we use (the proofs of [50, Thm. 2.4.4] and [85, Thm. 2.19], used through Corollary 3.4; [85, Prop. 2.9, Lemmas 2.11, 2.15 and 2.18], used in Lemma 8.8; [50, Cor. 5.3.10, Lemma 6.2, Prop. 8.2.1, Thm. 8.1, Sublemma 9.2.4 and pp. 46–51] and [28, Prop. 7.4.3]), Lemma 8.3, Theorem 1.5, Lemma 5.2, Theorem 7.4 and Corollary 3.4 need only that K, or the arithmetic groups of its components, are neat [85, p. 20].

(3) The constant a and Step 3 of the proof of Lemma 8.8 use the auxiliary level of [85, pp. 21–22], which is neat because of its own factor at 3, whatever K_3 is.

(4) Lemmas 3.1 and 3.2, Propositions 6.6 and 3.3, and Lemmas 8.5 and 8.4 use no condition on the level beyond neatness.

□

9. Every Shimura datum

Let (G, X) be any Shimura datum, $K = \prod_p K_p$ a neat product level, $\rho\colon G \to \mathrm{GL}_{n_\rho}$ faithful with $\rho(K) \subset \mathrm{GL}_{n_\rho}(\widehat{\mathbb{Z}})$, $\lambda\colon G \to G^{\mathrm{ad}}$ the adjoint map and $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ the adjoint datum. The morphism $f\colon \mathrm{Sh}_K(G,X) \to \mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, for a level $K^{\mathrm{ad}} \supset \lambda(K)$, maps special subvarieties to special subvarieties, and a subvariety is special if and only if its image is [28, Prop. 2.2]. But f need not be injective on connected components, and then a special subvariety maximal in Z can have an image that is not maximal in $f(Z)$. On each component f is the quotient by a finite group of automorphisms, and we recurse on the dimension at the fixed level.

9.1. The level.

Lemma 9.1. *Let $K'_p := K_p$ for $p \neq 3$, $K'_3 := K_3 \cap \ker(\mathrm{GL}_{n_\rho}(\mathbb{Z}_3) \to \mathrm{GL}_{n_\rho}(\mathbb{F}_3))$, $K' := \prod_p K'_p$ and $g_3 := |\mathrm{GL}_{n_\rho}(\mathbb{F}_3)|$. Then K' satisfies (Lv), $[K : K'] \leq g_3$, and $\pi\colon \mathrm{Sh}_{K'}(G,X) \to \mathrm{Sh}_K(G,X)$ is finite étale of degree at most $[K : K']$ with $\pi^*\mathcal{L}_K = \mathcal{L}_{K'}$. Let $Z \subset \mathrm{Sh}_K(G,X)$ be closed and geometrically irreducible, defined over $F \supset E(G,X)$, of degree D, and V special and maximal in Z. Then there are a closed, geometrically irreducible $Z' \subset \mathrm{Sh}_{K'}(G,X)$, defined over a field of degree at most $\deg\pi \cdot [F : \mathbb{Q}]$, with $\deg_{\mathcal{L}_{K'}} Z' \leq \deg\pi \cdot D$, and a special V' maximal in Z' with $\pi(V') = V$, $\deg V \leq \deg_{\mathcal{L}_{K'}} V'$ and $T_{V'} = T_V$.*

Proof. K/K' acts transitively on the fibers of π, and for neat K the stabilizers are the image of $Z(G)(\mathbb{Q}) \cap K$, which acts trivially; so π is finite étale of degree at most $[K : K']$, and $\pi^*\mathcal{L}_K = \mathcal{L}_{K'}$ [50, Prop. 5.3.2(1)]. Choose a component V' of $\pi^{-1}(V)$ and a component Z' of $\pi^{-1}(Z)$ containing it. The set $\pi^{-1}(Z)$ is defined over F and has at most $\deg\pi$ components, which gives the field of Z'; and $\deg_{\mathcal{L}_{K'}} Z' \leq \deg_{\mathcal{L}_{K'}} \pi^{-1}(Z) = \deg\pi \cdot D$ by the projection formula. If $V' \subsetneq V'' \subset Z'$ with V'' special, $\pi(V'')$ would be a special subvariety of Z of dimension larger than $\dim V$. □

Since $gK'g^{-1} \cap T(\mathbb{A}_f) \subset gKg^{-1} \cap T(\mathbb{A}_f)$ for every torus T and every $g \in G(\mathbb{A}_f)$, each index $|K'^m_{T,p}/K_{T,p}|$ can only grow when K is replaced by K', so α computed at K' is at least α computed at K.

For the rest of this section we replace K by K', so that K_3 lies in the level-3 congruence subgroup for ρ.

An integral model. Let $\mathcal{G}$ be the schematic closure of $\rho(G)$ in $\mathrm{GL}_{n_\rho,\mathbb{Z}}$. For some integer M_0 it is reductive over $\mathbb{Z}[1/M_0]$: its non-smooth locus lies over finitely many primes, as $\mathcal{G} \to \operatorname{Spec}\mathbb{Z}$ is of finite presentation with smooth generic fiber [80, Tag 054K], and then [18,

Props. 3.1.9 and 3.1.12] apply. Let $\mathcal{S}$ be the finite set of primes consisting of 3, the primes dividing M_0, and the p with $K_p \neq \mathcal{G}(\mathbb{Z}_p)$. For $p \notin \mathcal{S}$ put $\mathcal{G}_p := \mathcal{G} \otimes \mathbb{Z}_p$, let $\mathcal{Z}_p$ be its center, a group of multiplicative type with generic fiber $Z(G)$ [18, Thm. 3.3.4], and $\mathcal{G}_p^{\mathrm{ad}} := \mathcal{G}_p/\mathcal{Z}_p$, an adjoint semisimple $\mathbb{Z}_p$-group with generic fiber G^{ad}; the sequence $1 \to \mathcal{Z}_p \to \mathcal{G}_p \to \mathcal{G}_p^{\mathrm{ad}} \to 1$ is exact as fppf sheaves [18, Lemma 3.3.1(2), Cor. 3.3.5].

The adjoint level. Put

$$K^{\mathrm{ad}} := \prod_{p \in \mathcal{S}} \lambda(K_p) \times \prod_{p \notin \mathcal{S}} \mathcal{G}_p^{\mathrm{ad}}(\mathbb{Z}_p) \ \subset G^{\mathrm{ad}}(\mathbb{A}_f).$$

When $Z(G)$ is not connected, $\lambda(K)$ itself need not be open: for $W = M_2(\mathbb{Q})$ with the quadratic form det, $\mathrm{SO}(W) = \{(A,B) \in \mathrm{GL}_2^2 : \det A = \det B\}/\mathbb{G}_{\mathrm{m}}$ has center μ_2 and adjoint group PGL_2^2, and the image of $\mathrm{SO}(W)(\mathbb{Z}_p)$ has index 2 in $\mathrm{PGL}_2(\mathbb{Z}_p)^2$ for odd p.

Lemma 9.2. *(i) K^{ad} is a compact open subgroup of $G^{\mathrm{ad}}(\mathbb{A}_f)$ containing $\lambda(K)$, and it normalizes K for the action of G^{ad} on G by inner automorphisms.*

(ii) Let $\rho^{\mathrm{ad}} := \mathrm{Ad}\colon G^{\mathrm{ad}} \to \mathrm{GL}(\mathrm{Lie}\, G^{\mathrm{der}})$ and $\Lambda := \mathrm{Lie}\, G^{\mathrm{der}} \cap \mathrm{End}(\mathbb{Z}^{n_\rho})$, via $d\rho$. Then $\rho^{\mathrm{ad}}(K^{\mathrm{ad}})$ preserves $\Lambda \otimes \widehat{\mathbb{Z}}$, K_3^{ad} lies in the level-3 congruence subgroup for $(\rho^{\mathrm{ad}}, \Lambda)$, and K^{ad} is neat. So K^{ad} satisfies (Lv) for $(\rho^{\mathrm{ad}}, \Lambda)$.

(iii) $\lambda^{-1}(K^{\mathrm{ad}}) = Z(G)(\mathbb{A}_f) \cdot K$.

(iv) If $Z(G)$ is connected, then $K^{\mathrm{ad}} = \lambda(K)$.

Proof. (i) Each $\lambda(K_p)$ is compact open, as λ is smooth, and for $p \notin \mathcal{S}$ the factor is $\mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_p)$ for the model $\mathcal{G}/Z(\mathcal{G})$ of G^{ad} over $\mathbb{Z}[1/M_0]$; so K^{ad} is compact open. At $p \notin \mathcal{S}$, $\lambda(\mathcal{G}_p(\mathbb{Z}_p)) \subset \mathcal{G}_p^{\mathrm{ad}}(\mathbb{Z}_p)$. At $p \in \mathcal{S}$, $\mathrm{Int}(\lambda(k)) = \mathrm{Int}(k)$ preserves K_p; at $p \notin \mathcal{S}$ the conjugation action of $\mathcal{G}_p$ on itself factors through $\mathcal{G}_p^{\mathrm{ad}}$ [18, proof of Cor. 3.3.5], so $\mathcal{G}_p^{\mathrm{ad}}(\mathbb{Z}_p)$ preserves $\mathcal{G}_p(\mathbb{Z}_p) = K_p$.

(ii) At $p \in \mathcal{S}$, $\mathrm{Ad}(\lambda(k))$ is conjugation by $\rho(k) \in \mathrm{GL}_{n_\rho}(\mathbb{Z}_p)$, which preserves $\mathrm{End}(\mathbb{Z}_p^{n_\rho})$ and $\mathrm{Lie}\, G^{\mathrm{der}} \otimes \mathbb{Q}_p$. At $p \notin \mathcal{S}$, $\Lambda \otimes \mathbb{Z}_p$ is the Lie algebra of the derived group $\mathcal{G}_p^{\mathrm{der}}$ [18, Thm. 5.3.1], a smooth closed subgroup of $\mathrm{GL}_{n_\rho, \mathbb{Z}_p}$, whose Lie algebra is saturated in $\mathrm{End}(\mathbb{Z}_p^{n_\rho})$; and $\mathcal{G}_p^{\mathrm{ad}}$ acts on $\mathcal{G}_p^{\mathrm{der}}$ by (i). For $k \in K_3$ write $\rho(k) = 1 + 3M$ and $\rho(k)^{-1} = 1 + 3M'$ with M, M' integral; then $\rho(k) Y \rho(k)^{-1} - Y = 3(MY + YM' + 3MYM')$ lies in $3\,\mathrm{End}(\mathbb{Z}_3^{n_\rho}) \cap (\mathrm{Lie}\, G^{\mathrm{der}} \otimes \mathbb{Q}_3) = 3(\Lambda \otimes \mathbb{Z}_3)$. For neatness: the eigenvalues of $\mathrm{Ad}(\lambda(k_3))$ are ratios of eigenvalues of $\rho(k_3)$, which are $\equiv 1$ modulo 3 in $\overline{\mathbb{Z}}_3$, so every element ξ of the group they generate satisfies $\xi \equiv 1 \bmod 3$, and a root of unity $\equiv 1 \bmod 3$ is 1 [68, p. 5]. So every element of K^{ad} is neat in the sense of [68, 0.6], and so is every conjugate of K^{ad} under $G^{\mathrm{ad}}(\mathbb{A}_f)$.

(iii) "$\supset$" is clear, and at $p \in \mathcal{S}$ the kernel of λ on $G(\mathbb{Q}_p)$ is $Z(G)(\mathbb{Q}_p)$. Let $p \notin \mathcal{S}$, $x \in G(\mathbb{Q}_p)$ with $y := \lambda(x) \in \mathcal{G}_p^{\mathrm{ad}}(\mathbb{Z}_p)$. By Lemma 9.3 below, applied over $\mathbb{Z}_p$ and $\mathbb{Q}_p$, the class of y in $H^1(\mathbb{Z}_p, \mathcal{Z}_p)$ maps to 0 in $H^1(\mathbb{Q}_p, Z(G))$, as y lifts to x. We show that $H^1(\mathbb{Z}_p, \mathcal{Z}_p) \to H^1(\mathbb{Q}_p, Z(G))$ is injective; then $y = \lambda(k)$ with $k \in \mathcal{G}_p(\mathbb{Z}_p)$, and $x \in Z(G)(\mathbb{Q}_p) K_p$. Let $\mathcal{T} \subset \mathcal{G}_p$ be a maximal torus over $\mathbb{Z}_p$; it exists by [18, Thm. 3.2.6], Lang's theorem and Hensel's lemma, and contains $\mathcal{Z}_p$ [18, Cor. 3.3.5]. Let $\mathcal{Z}_p^\circ$ be the maximal subtorus of $\mathcal{Z}_p$ [18, App. B], $\mathcal{T}_1 := \mathcal{T}/\mathcal{Z}_p^\circ$ and $\mathcal{T}^{\mathrm{ad}} := \mathcal{T}/\mathcal{Z}_p$, so that $\lambda|_{\mathcal{T}} = \varkappa \circ \pi$ with $\varkappa\colon \mathcal{T}_1 \to \mathcal{T}^{\mathrm{ad}}$ an isogeny. Torsors under a smooth affine $\mathbb{Z}_p$-group with connected special fiber are trivial (Lang's theorem and Hensel's lemma), so $H^1(\mathbb{Z}_p, \mathcal{T}) = H^1(\mathbb{Z}_p, \mathcal{Z}_p^\circ) = 0$ and $H^1(\mathbb{Z}_p, \mathcal{Z}_p) = \mathcal{T}^{\mathrm{ad}}(\mathbb{Z}_p)/\lambda(\mathcal{T}(\mathbb{Z}_p))$, which maps to $T^{\mathrm{ad}}(\mathbb{Q}_p)/\lambda(T(\mathbb{Q}_p)) \subset H^1(\mathbb{Q}_p, Z(G))$. Let $u \in \mathcal{T}^{\mathrm{ad}}(\mathbb{Z}_p)$ with $u = \lambda(t)$, $t \in T(\mathbb{Q}_p)$. For every character χ of $\mathcal{T}_1$ over an unramified splitting field, some multiple $m\chi$ is a character χ' of $\mathcal{T}^{\mathrm{ad}}$, so $|\chi(\pi(t))|^m = |\chi'(u)| = 1$; hence $\pi(t) \in \mathcal{T}_1(\mathbb{Z}_p)$. As $H^1(\mathbb{Z}_p, \mathcal{Z}_p^\circ) = 0$,

$\pi(t)=\pi(t')$ with $t'\in\mathcal{T}(\mathbb{Z}_p)$, so $t'^{-1}t\in Z(G)(\mathbb{Q}_p)$ and $u=\lambda(t')\in\lambda(\mathcal{T}(\mathbb{Z}_p))$. Adelically, if $x=(z_pk_p)_p$, then $z_p\in Z(G)(\mathbb{Q}_p)\cap\mathcal{G}_p(\mathbb{Z}_p)$ for almost all p, so $(z_p)\in Z(G)(\mathbb{A}_f)$.

(iv) If $Z(G)$ is connected, $\mathcal{Z}_p$ is a torus for $p\notin\mathcal{S}$, $H^1(\mathbb{Z}_p,\mathcal{Z}_p)=0$, and $\lambda(\mathcal{G}_p(\mathbb{Z}_p))=\mathcal{G}_p^{\mathrm{ad}}(\mathbb{Z}_p)$ by Lemma 9.3. □

Lemma 9.3. *Let R be a ring and $1\to\mathcal{Z}\to\mathcal{G}\to\mathcal{G}'\to 1$ an exact sequence of fppf sheaves of groups over R with $\mathcal{Z}$ central. Then $\mathcal{G}(R)\to\mathcal{G}'(R)\xrightarrow{\partial}H^1_{\mathrm{fppf}}(R,\mathcal{Z})$ is exact, and ∂ is a homomorphism.*

Proof. For $y\in\mathcal{G}'(R)$ the sheaf P_y of local lifts of y is a $\mathcal{Z}$-torsor: $\mathcal{Z}$ acts simply transitively, and P_y has sections locally, as $\mathcal{G}\to\mathcal{G}'$ is an epimorphism. Let $\partial(y)$ be its class [80, Tag 03AJ]; it vanishes if and only if $P_y(R)\neq\emptyset$. The map $(a,b)\mapsto ab$ identifies $P_y\wedge^{\mathcal{Z}}P_{y'}$ with $P_{yy'}$, which is well defined because $\mathcal{Z}$ is central, and the class of a contracted product is the sum of the classes: the group law on H^1 is the contracted product, since on a cover trivializing both torsors the contracted product has the product Čech cocycle. □

9.2. **The adjoint map on one component.** Let $S_i=[X^+\times g_i]$ be a connected component of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$, $\Gamma_i:=G(\mathbb{Q})_+\cap g_iKg_i^{-1}$, so that $S_i=\lambda(\Gamma_i)\backslash X^+$, and $\Gamma_i':=G^{\mathrm{ad}}(\mathbb{Q})_+\cap\lambda(g_i)K^{\mathrm{ad}}\lambda(g_i)^{-1}$, so that the component of $\mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}},X^{\mathrm{ad}})_{\mathbb{C}}$ containing $f(S_i)$ is $\Gamma_i'\backslash X^+$. Put $\mathfrak{G}_i:=\Gamma_i'/\lambda(\Gamma_i)$,

$$h_K:=\max_i|\mathfrak{G}_i|,$$

and let $\mathrm{Cl}(K):=Z(G)(\mathbb{Q})\backslash Z(G)(\mathbb{A}_f)/(Z(G)(\mathbb{A}_f)\cap K)$, which is finite.

Proposition 9.4. *(a) $\lambda(\Gamma_i)$ is a normal subgroup of finite index of Γ_i', both are neat and arithmetic, and $f|_{S_i}\colon S_i\to\Gamma_i'\backslash X^+$ is the quotient by $\mathfrak{G}_i$, which acts freely. So h_K is finite.*

(b) Each element of $\mathfrak{G}_i$ is an automorphism of S_i that extends to the Baily–Borel compactification and preserves $\mathcal{L}_K$. If S_i is defined over $F\supset E(G,X)$, all these automorphisms are defined over one extension $F_{\mathfrak{G}}$ of F of degree at most $h_K!$. Each $\bar\gamma\in\Gamma_i'\cap\lambda(G(\mathbb{Q}))$ acts as right translation by an element of $Z(G)(\mathbb{A}_f)$, which is defined over $E(G,X)$; so $F_{\mathfrak{G}}=F$ if $\Gamma_i'\subset\lambda(G(\mathbb{Q}))$ for every i.

(c) $f^*\mathcal{L}_{K^{\mathrm{ad}}}=\mathcal{L}_K$ on the compactification of S_i. Hence, for every closed irreducible $Y\subset S_i$, we have $\deg_{\mathcal{L}_{K^{\mathrm{ad}}}}f(Y)\le\deg Y$ and $\deg Y\le h_K\deg_{\mathcal{L}_{K^{\mathrm{ad}}}}f(Y)$.

(d) Let $\mathrm{Sel}_{\mathcal{S}}(Z(G))$ be the set of $c\in H^1(\mathbb{Q},Z(G))$ with $c_\infty=0$, $c_p=0$ for $p\in\mathcal{S}$, and c_p in the image of $H^1(\mathbb{Z}_p,\mathcal{Z}_p)$ for $p\notin\mathcal{S}$. Then $|\mathfrak{G}_i|\le|\mathrm{Cl}(K)|\cdot|\mathrm{Sel}_{\mathcal{S}}(Z(G))|$. If $\mathrm{Sel}_{\mathcal{S}}(Z(G))=0$, which holds if $H^1(\mathbb{Q},Z(G))=0$ or if $Z(G)$ is split, then $\Gamma_i'\subset\lambda(G(\mathbb{Q}))$ and $|\mathfrak{G}_i|\le|\mathrm{Cl}(K)|$.

Proof. Here $G(\mathbb{Q})_+=G(\mathbb{Q})\cap\lambda^{-1}(G^{\mathrm{ad}}(\mathbb{R})^+)$, as in Section 2.1.

(a) $\lambda(\Gamma_i)\subset\Gamma_i'$ by Lemma 9.2(i). By Lemma 2.1 with $U=K^{\mathrm{ad}}$, $\lambda(\Gamma_i)$ and Γ_i' are arithmetic and $[\Gamma_i':\lambda(\Gamma_i)]$ is finite. They are neat by Lemma 9.2(ii) and [68, 0.5–0.6]. Let $\bar\gamma\in\Gamma_i'$, $\bar\gamma=\lambda(g_i)\kappa\lambda(g_i)^{-1}$ with $\kappa\in K^{\mathrm{ad}}$. On $G(\mathbb{A}_f)$, $\mathrm{Int}(\bar\gamma)$ preserves $g_iKg_i^{-1}$ by Lemma 9.2(i); it preserves $G(\mathbb{Q})$, and $G(\mathbb{Q})_+$ because $\lambda(\mathrm{Int}(\bar\gamma)\gamma')=\bar\gamma\lambda(\gamma')\bar\gamma^{-1}$. So $\mathrm{Int}(\bar\gamma)\Gamma_i=\Gamma_i$, and $\lambda(\Gamma_i)$ is normal in Γ_i'. As Γ_i' is neat, it acts freely on X^+, and $f|_{S_i}$ is the Galois covering with group $\mathfrak{G}_i$.

(b) Each $\bar\gamma\in\Gamma_i'$ normalizes $\lambda(\Gamma_i)$, so it induces an automorphism of S_i, which extends to the compactification and preserves the extension of the canonical bundle. $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ acts on the finite group $\mathfrak{G}_i=\mathrm{Aut}(S_i/f(S_i))$ by conjugation, and the kernel of this action

has index at most $|\mathrm{Aut}\,\mathfrak{G}_i| \le h_K!$. If $\bar\gamma = \lambda(\gamma)$ with $\gamma \in G(\mathbb{Q})$, then $\gamma \in G(\mathbb{Q})_+$, and $\gamma \in g_i\lambda^{-1}(K^{\mathrm{ad}})g_i^{-1} = Z(G)(\mathbb{A}_f)g_iKg_i^{-1}$ by Lemma 9.2(iii); writing $\gamma = zk$ we get $[\bar\gamma^{-1}x, g_i] = [x, \gamma g_i] = [x, g_izk']$ with $k' \in K$, as z is central. So $\bar\gamma$ acts as right translation by z^{-1}, a Hecke translation, defined over $E(G,X)$.

(c) $f|_{S_i}$ is étale between quotients of X^+ by neat groups, so it pulls top-degree forms back to top-degree forms; the two extensions agree in codimension one on the normal compactification, as logarithmic forms pull back to logarithmic forms, and so are isomorphic (compare [50, Prop. 5.3.2]). The projection formula and $\deg(f|_Y) \le |\mathfrak{G}_i|$ give the inequalities.

(d) Let $\partial\colon G^{\mathrm{ad}}(\mathbb{Q}) \to H^1(\mathbb{Q}, Z(G))$ be the connecting map of Lemma 9.3, a homomorphism with kernel $\lambda(G(\mathbb{Q}))$. For each place v, $\partial_v(\bar\gamma) = \partial_v(\kappa_v)$, since ∂_v kills $\lambda(G(\mathbb{Q}_v)) \ni \lambda(g_{i,v})$. So $\partial_p(\bar\gamma) = 0$ for $p \in \mathcal{S}$, as $\kappa_p \in \lambda(K_p)$; $\partial_p(\bar\gamma)$ comes from $H^1(\mathbb{Z}_p, \mathcal{Z}_p)$ for $p \notin \mathcal{S}$, as $\kappa_p \in \mathcal{G}_p^{\mathrm{ad}}(\mathbb{Z}_p)$; and $\partial_\infty(\bar\gamma) = 0$, as $G(\mathbb{R})^+ \to G^{\mathrm{ad}}(\mathbb{R})^+$ is onto. Hence $\Gamma_i'/(\Gamma_i' \cap \lambda(G(\mathbb{Q})))$ embeds in $\mathrm{Sel}_{\mathcal{S}}(Z(G))$. Next, $\bar\gamma = \lambda(zk) \mapsto [z]$, with z, k as in (b), is a well-defined homomorphism $(\Gamma_i' \cap \lambda(G(\mathbb{Q})))/\lambda(\Gamma_i) \to \mathrm{Cl}(K)$: γ is determined up to $Z(G)(\mathbb{Q})$, and then z up to $Z(G)(\mathbb{A}_f) \cap K$. It is injective: if $z = rw$ with $r \in Z(G)(\mathbb{Q})$ and $w \in Z(G)(\mathbb{A}_f) \cap K$, then $r^{-1}\gamma = wk \in \Gamma_i$. If $Z(G)$ is split, a product of copies of $\mathbb{G}_{\mathrm{m}}$ and of groups μ_m, then $H^1(\mathbb{Q}, \mathbb{G}_{\mathrm{m}}) = 0$ and $H^1(\mathbb{Q}, \mu_m) = \mathbb{Q}^\times/\mathbb{Q}^{\times m}$, and a class a in $\mathrm{Sel}_{\mathcal{S}}$ has $v_p(a) \equiv 0 \bmod m$ for $p \notin \mathcal{S}$ and is an m-th power in $\mathbb{Q}_p$ for $p \in \mathcal{S}$, so $a \in \pm\mathbb{Q}^{\times m}$; for m odd -1 is an m-th power, and for m even $c_\infty = 0$ excludes -1. Thus $\mathrm{Sel}_{\mathcal{S}}(Z(G)) = 0$. □

For example, for $\mathrm{SO}(W)$ with W of signature $(2,m)$, $m \ge 2$ even, $Z(G) = \mu_2$ is split, and $h_K \le 2^{|\mathcal{S}|-1}$, since $-1 \notin K_3$.

9.3. **A closed bound for h_K.** The factor h_K enters both bounds of Theorem 9.12; here we bound it in closed form (Corollary 9.10). Proposition 9.4(d) bounds h_K by $|\mathrm{Cl}(K)| \cdot |\mathrm{Sel}_{\mathcal{S}}(Z(G))|$. We bound both factors in terms of the center. Let $\Xi := X^*(Z(G))$, a finitely generated abelian group with a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, r_Z its rank, $Z(G)^\circ$ the maximal subtorus of $Z(G)$, with character group Ξ/Ξ_{tors}, and $\Pi_Z := Z(G)/Z(G)^\circ$, finite of multiplicative type with character group Ξ_{tors}. Write $\Xi_{\mathrm{tors}} \cong \bigoplus_{i=1}^q \mathbb{Z}/m_i\mathbb{Z}$ with all $m_i \ge 2$, and $m_Z := |\Xi_{\mathrm{tors}}| = \prod_i m_i$. Let L_0 and L_1 be the fields fixed by the kernels of the Galois actions on Ξ_{tors} and on Ξ/Ξ_{tors}, Galois over $\mathbb{Q}$, with $n_{L_0} := [L_0 : \mathbb{Q}]$ and $n_{L_1} := [L_1 : \mathbb{Q}]$. For a number field L let $r_1(L)$, $r_2(L)$ be its numbers of real and complex places and h_L its class number; for a finite group $\mathfrak{H}$ let $\mathrm{gen}(\mathfrak{H}) \le \log_2|\mathfrak{H}|$ be its least number of generators; and for $m \ge 1$ put $L(\emptyset, m) := \{a \in L^\times/L^{\times m} : v_w(a) \equiv 0 \bmod m \text{ for every finite place } w\}$. For a torus T over $\mathbb{Q}$ let $T(\mathbb{Q}_p)^c$ be the maximal compact subgroup of $T(\mathbb{Q}_p)$ and $\ker^1_f(\mathbb{Q}, T) := \ker\big(H^1(\mathbb{Q}, T) \to \prod_p H^1(\mathbb{Q}_p, T)\big)$. As in the rest of this section, K satisfies (Lv), and $\mathcal{S}$, $\mathcal{G}_p$, $\mathcal{Z}_p$ are as defined before Lemma 9.2; for $p \notin \mathcal{S}$ let $\mathcal{Z}_p^\circ$ be the maximal subtorus of $\mathcal{Z}_p$ and $\mathcal{P}_p := \mathcal{Z}_p/\mathcal{Z}_p^\circ$.

We use the following facts for $p \notin \mathcal{S}$.

(M1) $\mathcal{Z}_p^\circ$ has generic fiber $Z(G)^\circ$, and $1 \to \mathcal{Z}_p^\circ \to \mathcal{Z}_p \to \mathcal{P}_p \to 1$ is exact as fppf sheaves, with $\mathcal{P}_p$ finite flat of multiplicative type with generic fiber Π_Z: after a finite unramified extension $\mathcal{Z}_p$ becomes diagonalizable [18, Prop. B.3.4], where this is clear, and everything descends, the maximal subtorus being unique and compatible with base change [18, App. B].

(M2) The Galois modules Ξ, Ξ_{tors} and Ξ/Ξ_{tors} are unramified at p [18, Prop. B.3.4]; so p is unramified in L_0 and L_1, and $\mathcal{P}_p \otimes \mathcal{O}_w \cong \prod_i \mu_{m_i}$ for every place $w \mid p$ of L_0.

(M3) $H^1_{\mathrm{fppf}}(\mathbb{Z}_p, \mathcal{T}) = 0$ for every torus $\mathcal{T}$ over $\mathbb{Z}_p$, by Lang's theorem and Hensel's lemma.

(M4) For a finite $\mathbb{Z}_p$-scheme $\mathcal{Q}$, $\mathcal{Q}(\mathbb{Z}_p) = \mathcal{Q}(\mathbb{Q}_p)$; hence $\Pi_Z(\mathbb{Q}_p) = \mathcal{P}_p(\mathbb{Z}_p)$, and $H^1_{\mathrm{fppf}}(\mathbb{Z}_p, \mathcal{P}_p) \to H^1(\mathbb{Q}_p, \Pi_Z)$ is injective, since a $\mathcal{P}_p$-torsor is affine [80, Tag 0245] and finite over $\mathbb{Z}_p$, so a $\mathbb{Q}_p$-point of it is a $\mathbb{Z}_p$-point.

(M5) For a discrete valuation ring $\mathcal{O}$ with fraction field L, $H^1_{\mathrm{fppf}}(\mathcal{O}, \mu_m) = \mathcal{O}^\times/\mathcal{O}^{\times m}$, with image $\mathcal{O}^\times L^{\times m}/L^{\times m}$ in $H^1(L, \mu_m) = L^\times/L^{\times m}$ (Kummer theory and $\mathrm{Pic}(\mathcal{O}) = 0$).

(M6) $Z(G)(\mathbb{Q}_p) \cap K_p = \mathcal{Z}_p(\mathbb{Z}_p)$ and $Z(G)^\circ(\mathbb{Q}_p) \cap K_p = \mathcal{Z}_p^\circ(\mathbb{Z}_p) = Z(G)^\circ(\mathbb{Q}_p)^c$: a point of $\mathcal{G}_p(\mathbb{Z}_p)$ whose generic fiber lies in $\mathcal{Z}_p$ factors through this schematic closure, and for a torus split over an unramified extension the $\mathbb{Z}_p$-points are the points on which all characters are units, as in the proof of Lemma 9.2(iii).

Let $\partial_v \colon \Pi_Z(\mathbb{Q}_v) \to H^1(\mathbb{Q}_v, Z(G)^\circ)$ be the connecting maps, and $\mathrm{Sel}_{\mathcal{S}}(\Pi_Z)$ the set of $a \in H^1(\mathbb{Q}, \Pi_Z)$ with $a_v = 0$ for $v \in \mathcal{S} \cup \{\infty\}$ and a_p in the image of $H^1_{\mathrm{fppf}}(\mathbb{Z}_p, \mathcal{P}_p)$ for $p \notin \mathcal{S}$.

Proposition 9.5. $|\mathrm{Sel}_{\mathcal{S}}(Z(G))| \le |\mathrm{Sel}_{\mathcal{S}}(\Pi_Z)| \cdot |\ker^1_f(\mathbb{Q}, Z(G)^\circ)| \cdot \prod_{p\in\mathcal{S}} |\partial_p(\Pi_Z(\mathbb{Q}_p))|$, *and the last factor is at most* $m_Z^{|\mathcal{S}|}$.

Proof. Let $\phi \colon H^1(\mathbb{Q}, Z(G)) \to H^1(\mathbb{Q}, \Pi_Z)$ and $\iota \colon H^1(\mathbb{Q}, Z(G)^\circ) \to H^1(\mathbb{Q}, Z(G))$ be the maps of $1 \to Z(G)^\circ \to Z(G) \to \Pi_Z \to 1$. First, $\phi(\mathrm{Sel}_{\mathcal{S}}(Z(G))) \subset \mathrm{Sel}_{\mathcal{S}}(\Pi_Z)$: at $p \notin \mathcal{S}$, c_p is the restriction of some $\gamma \in H^1(\mathbb{Z}_p, \mathcal{Z}_p)$, and $\phi_p(c_p)$ is the restriction of the image $\bar\gamma$ of γ in $H^1(\mathbb{Z}_p, \mathcal{P}_p)$. Let $c \in \mathrm{Sel}_{\mathcal{S}}(Z(G))$ with $\phi(c) = 0$. For $p \notin \mathcal{S}$, $\bar\gamma = 0$ by (M4), so $\gamma = 0$ by (M1) and (M3), and $c_p = 0$; so $c_v = 0$ at every place. Write $c = \iota(b)$. For every v, $\iota_v(b_v) = 0$, so $b_v \in \partial_v(\Pi_Z(\mathbb{Q}_v))$; for $p \notin \mathcal{S}$ this is 0, since by (M4) every $x \in \Pi_Z(\mathbb{Q}_p)$ lies in $\mathcal{P}_p(\mathbb{Z}_p)$ and $\partial_p(x)$ is the restriction of a class in $H^1(\mathbb{Z}_p, \mathcal{Z}_p^\circ) = 0$ (Lemma 9.3). The map $b \mapsto (b_p)_{p\in\mathcal{S}} \in \prod_{p\in\mathcal{S}} \partial_p(\Pi_Z(\mathbb{Q}_p))$ has kernel in $\ker^1_f(\mathbb{Q}, Z(G)^\circ)$ on the set of such b. Counting gives the bound, and $|\Pi_Z(\mathbb{Q}_p)| \le m_Z$. □

Proposition 9.6.

$$|\mathrm{Sel}_{\mathcal{S}}(\Pi_Z)| \le |H^1(\mathrm{Gal}(L_0/\mathbb{Q}), \Pi_Z(L_0))| \cdot \prod_{i=1}^{q} |L_0(\emptyset, m_i)| \le m_Z^{\mathrm{gen}(\mathrm{Gal}(L_0/\mathbb{Q}))+r_1(L_0)+r_2(L_0)}\, h_{L_0}^q.$$

Proof. Inflation and restriction [59, II.1.34] give $0 \to H^1(\mathrm{Gal}(L_0/\mathbb{Q}), \Pi_Z(L_0)) \to H^1(\mathbb{Q}, \Pi_Z) \to H^1(L_0, \Pi_Z)$, and $H^1(L_0, \Pi_Z) = \prod_i L_0^\times/L_0^{\times m_i}$, as $\Pi_Z \cong \prod_i \mu_{m_i}$ over L_0 and $H^1(L_0, \mathbb{G}_{\mathrm{m}}) = 0$ [59, II.1.22]. Let $a \in \mathrm{Sel}_{\mathcal{S}}(\Pi_Z)$ with restriction (a_i). At a place w of L_0 above $p \in \mathcal{S}$, $a_p = 0$, so $v_w(a_i) \equiv 0 \bmod m_i$. Above $p \notin \mathcal{S}$, a_p comes from $H^1(\mathbb{Z}_p, \mathcal{P}_p)$, and its restriction to $L_{0,w}$ factors through $\prod_i H^1(\mathcal{O}_w, \mu_{m_i})$ by (M2), whose image consists of classes of units by (M5); so again $v_w(a_i) \equiv 0 \bmod m_i$. This gives the first inequality. For the second, $1 \to \mathcal{O}_L^\times/\mathcal{O}_L^{\times m} \to L(\emptyset, m) \to \mathrm{Cl}(L)[m] \to 0$, where a with $(a) = \mathfrak{a}^m$ maps to the class of $\mathfrak{a}$; by Dirichlet's unit theorem $|\mathcal{O}_L^\times/\mathcal{O}_L^{\times m}| \le m^{r_1(L)+r_2(L)}$; and $|H^1(\mathfrak{H}, A)| \le |A|^{\mathrm{gen}(\mathfrak{H})}$ for a finite group $\mathfrak{H}$ and a finite module A, as a cocycle is determined by its values on generators, with $|\Pi_Z(L_0)| \le m_Z$. □

For a torus T over $\mathbb{Q}$ split by a Galois field L, put $T(\mathcal{O}_L) := \mathrm{Hom}(X^*(T), \mathcal{O}_L^\times)$, $T(\mathbb{A}_f)^c := \prod_p T(\mathbb{Q}_p)^c$ and $h(T) := |T(\mathbb{Q})\backslash T(\mathbb{A}_f)/T(\mathbb{A}_f)^c|$.

Proposition 9.7. *Let T be a torus over $\mathbb{Q}$ of dimension r, split by a Galois field L with group $\mathfrak{H}$ of order n_L. Then $|\ker^1_f(\mathbb{Q},T)|$ and $h(T)$ are both at most $|H^1(\mathfrak{H},T(\mathcal{O}_L))|\cdot|(X_*(T)\otimes \mathrm{Cl}(L))^{\mathfrak{H}}| \le n_L^{r(\log_2 n_L + r_1(L)+r_2(L))} h_L^r$.*

Proof. As $\mathfrak{H}$-modules $T(L) = X_*\otimes L^\times$ with $X_* := X_*(T)\cong\mathbb{Z}^r$, and tensoring $1\to\mathcal{O}_L^\times\to L^\times\to \mathrm{Id}_L\to \mathrm{Cl}(L)\to 0$ (Id_L the fractional ideals) with X_* gives an exact sequence $0\to T(\mathcal{O}_L)\to T(L)\to I\to C\to 0$ with $I := X_*\otimes \mathrm{Id}_L$ and $C := X_*\otimes \mathrm{Cl}(L)$; let $\Pi\subset I$ be the image of $T(L)$.

$\ker^1_f$: by Hilbert's Theorem 90 and inflation–restriction, $H^1(\mathbb{Q},T) = H^1(\mathfrak{H},T(L))$ and $H^1(\mathbb{Q}_p,T) = H^1(\mathfrak{H}_w,T(L_w))$ for $w\mid p$, $\mathfrak{H}_w$ the decomposition group. As $\mathfrak{H}$-modules $I=\bigoplus_p I_p$ with $I_p\cong \mathrm{Ind}_{\mathfrak{H}_w}^{\mathfrak{H}} X_*$, and by Shapiro's lemma [59, II.1.11] $H^1(\mathfrak{H},I_p)\cong H^1(\mathfrak{H}_w,X_*)$, through restriction to $\mathfrak{H}_w$ and the w-coordinate, which on $T(L)$ is the valuation $T(L_w)\to X_*$. If $c\in\ker^1_f(\mathbb{Q},T)$, its restriction to $\mathfrak{H}_w$ is a coboundary in $T(L_w)$, so the image of c in $H^1(\mathfrak{H},I)$ is 0. So the image of c in $H^1(\mathfrak{H},\Pi)$ lies in $\ker(H^1(\mathfrak{H},\Pi)\to H^1(\mathfrak{H},I))$, the image of $C^{\mathfrak{H}}$; and the fibers of $H^1(\mathfrak{H},T(L))\to H^1(\mathfrak{H},\Pi)$ are cosets of the image of $H^1(\mathfrak{H},T(\mathcal{O}_L))$.

$h(T)$: for each p the valuations embed $T(\mathbb{Q}_p)/T(\mathbb{Q}_p)^c$ in $\bigoplus_{w\mid p} X_*$, since an element whose image lies in the compact group $\prod_w X_*\otimes\mathcal{O}_w^\times$ generates a bounded subgroup. So $I_T := T(\mathbb{A}_f)/T(\mathbb{A}_f)^c$ embeds in $I^{\mathfrak{H}}$, compatibly with $T(\mathbb{Q})\subset T(L)$, and $h(T) = |I_T/T(\mathbb{Q})|$. The induced map $I_T/T(\mathbb{Q})\to C^{\mathfrak{H}}$ has kernel embedding in $H^1(\mathfrak{H},T(\mathcal{O}_L))$: for a in the kernel choose $\varrho\in T(L)$ with image a; $\sigma\mapsto\sigma(\varrho)/\varrho$ is a cocycle with values in $T(\mathcal{O}_L)$, well defined up to coboundaries, and if it is $\sigma(\varepsilon)/\varepsilon$, then $\varrho/\varepsilon\in T(\mathbb{Q})$ has image a.

Sizes: with $E := T(\mathcal{O}_L)\cong(\mathcal{O}_L^\times)^r$, n_L kills $H^1(\mathfrak{H},E)$, so $|H^1(\mathfrak{H},E)|\le |H^1(\mathfrak{H},E[n_L])|\cdot |E/n_LE| \le n_L^{r\,\mathrm{gen}(\mathfrak{H})} n_L^{r(r_1(L)+r_2(L))}$, and $|C^{\mathfrak{H}}|\le h_L^r$. □

Theorem 9.8. *For every Shimura datum (G,X) and neat product level, with $\mathcal{S}$ as defined before Lemma 9.2,*

$$|\mathrm{Sel}_{\mathcal{S}}(Z(G))| \le m_Z^{\log_2 n_{L_0}+r_1(L_0)+r_2(L_0)+|\mathcal{S}|}\, h_{L_0}^q\, n_{L_1}^{r_Z(\log_2 n_{L_1}+r_1(L_1)+r_2(L_1))}\, h_{L_1}^{r_Z}.$$

Proof. Propositions 9.5 and 9.6, and Proposition 9.7 for $T = Z(G)^\circ$, split by L_1. □

Lemma 9.9. *Let $\iota_K := \prod_{p\in\mathcal{S}}[Z(G)^\circ(\mathbb{Q}_p)^c : Z(G)^\circ(\mathbb{Q}_p)\cap K_p]$; it is finite. Then*

$$|\mathrm{Cl}(K)| \le m_Z^{|\mathcal{S}|}\,\iota_K\, h(Z(G)^\circ) \le m_Z^{|\mathcal{S}|}\,\iota_K\, n_{L_1}^{r_Z(\log_2 n_{L_1}+r_1(L_1)+r_2(L_1))}\, h_{L_1}^{r_Z}.$$

Proof. The number ι_K is finite, since $Z(G)^\circ(\mathbb{Q}_p)\cap K_p$ is a compact open subgroup of $Z(G)^\circ(\mathbb{Q}_p)$, so it lies in the maximal compact subgroup $Z(G)^\circ(\mathbb{Q}_p)^c$ with finite index. Let $U := Z(G)(\mathbb{A}_f)\cap K$ and $U^\circ := Z(G)^\circ(\mathbb{A}_f)\cap K$. The map $\mathrm{Cl}(K)\to \Pi_Z(\mathbb{A}_f)/(\mathrm{im}\, Z(G)(\mathbb{Q})\cdot \mathrm{im}\, U)$ has kernel the image of $Z(G)^\circ(\mathbb{A}_f)$, which is a quotient of $Z(G)^\circ(\mathbb{Q})\backslash Z(G)^\circ(\mathbb{A}_f)/U^\circ$. Its image lies in $\mathrm{im}\, Z(G)(\mathbb{A}_f)/\mathrm{im}\, U$, and $\mathrm{im}\, U\supset\prod_{p\notin\mathcal{S}}\Pi_Z(\mathbb{Q}_p)$: for $p\notin\mathcal{S}$, $U_p = \mathcal{Z}_p(\mathbb{Z}_p)$ by (M6), $\mathcal{Z}_p(\mathbb{Z}_p)\to\mathcal{P}_p(\mathbb{Z}_p)$ is onto as its cokernel lies in $H^1(\mathbb{Z}_p,\mathcal{Z}_p^\circ)=0$, and $\mathcal{P}_p(\mathbb{Z}_p) = \Pi_Z(\mathbb{Q}_p)$ by (M4). So the image has at most $\prod_{p\in\mathcal{S}}|\Pi_Z(\mathbb{Q}_p)|\le m_Z^{|\mathcal{S}|}$ elements. Next, $U^\circ\subset Z(G)^\circ(\mathbb{A}_f)^c$ with index ι_K, by (M6) at $p\notin\mathcal{S}$, so $|Z(G)^\circ(\mathbb{Q})\backslash Z(G)^\circ(\mathbb{A}_f)/U^\circ|\le \iota_K\, h(Z(G)^\circ)$, and Proposition 9.7 bounds $h(Z(G)^\circ)$. □

Corollary 9.10. *With K as in this section,*

$$h_K \le m_Z^{\log_2 n_{L_0}+n_{L_0}+2|\mathcal{S}|}\, h_{L_0}^{q}\, \iota_K\, \big(n_{L_1}^{r_Z(\log_2 n_{L_1}+n_{L_1})} h_{L_1}^{r_Z}\big)^2.$$

If K is the level made by Lemma 9.1 from a neat product level K_0, then $\iota_K \le |\mathrm{GL}_{n_\rho}(\mathbb{F}_3)|\,\iota_{K_0}$, with ι_{K_0} defined with the same set $\mathcal{S}$. For a number field L of degree n_L, $h_L \le \max(1, B_L(1+\log B_L)^{n_L-1})$ with $B_L := (n_L!/n_L^{n_L})(4/\pi)^{r_2(L)} d_L^{1/2}$.

Proof. Proposition 9.4(d), Theorem 9.8 and Lemma 9.9, with $r_1(L)+r_2(L) \le [L:\mathbb{Q}]$. The bound for ι_K holds since $Z(G)^\circ(\mathbb{Q}_3)\cap K_3$ is the kernel of the reduction map from $Z(G)^\circ(\mathbb{Q}_3)\cap K_{0,3}$ to $\mathrm{GL}_{n_\rho}(\mathbb{F}_3)$ and $K_p = K_{0,p}$ for $p \ne 3$. Every ideal class contains an integral ideal of norm at most B_L [58, Thm. 4.3], and the number of integral ideals of norm at most x is at most $\sum_{m\le x}\tau_{n_L}(m) \le x(1+\log x)^{n_L-1}$, with τ_{n_L} the n_L-fold divisor function. □

All the quantities in Corollary 9.10 are read off the character group of $Z(G)$, the representation ρ and the level. If $Z(G)$ is connected, $m_Z = 1$ and $q = 0$; if $Z(G)$ is split, $\mathrm{Sel}_{\mathcal{S}}(Z(G)) = 0$ by Proposition 9.4(d); if $Z(G)$ is finite, the torus factors are 1.

9.4. **Where maximality is lost.**

Lemma 9.11. *Let $Z \subset S_i$ be closed and irreducible and $V \subset Z$ special and maximal in Z. If $f(V)$ is not maximal in $f(Z)$, there is $\theta \in \mathfrak{G}_i$ with $\theta Z \ne Z$ and $V \subset Z \cap \theta Z$.*

Proof. Let $V^\sharp$ be special with $f(V) \subsetneq V^\sharp \subset f(Z)$, and $\widetilde{V}$ a component of $f^{-1}(V^\sharp)$ containing V. It is special [50, p. 14], of dimension $\dim V^\sharp > \dim V$ as f is finite, and lies in S_i, being connected. By Proposition 9.4(a), $\widetilde{V} \subset f^{-1}(f(Z)) \cap S_i = \bigcup_{\theta\in\mathfrak{G}_i} \theta Z$, so $\widetilde{V} \subset \theta Z$ for some θ. If $\theta Z = Z$, $\widetilde{V}$ would contradict the maximality of V. □

9.5. **The theorem.** For a special subvariety V of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$, $f(V)$ is special with torus T_V, the connected center of $\lambda(H_V)$, as in the introduction. We put

$$\alpha_V := \alpha_{f(V)} \text{ in } \mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \text{ (Definition 2.2)}, \qquad \Delta(V) := \max\big(\deg V,\ d_{L_V},\ \alpha_V\big),$$

with $\deg V$ taken for $\mathcal{L}_K$ at the original level. For V at the original level, where $\lambda(K)$ need not lie in K^{ad}, $\alpha_V := \alpha_{f(V')}$ for a component V' of the preimage of V in $\mathrm{Sh}_{K'}(G,X)$; as K' is normal in K, these components are the $V'k$, $k\in K$, and $f(V'k) = f(V')\lambda(k)$ with $\lambda(k)$ normalizing K^{ad}, so $\alpha_{f(V')}$ does not depend on V'. If G is adjoint and K satisfies (Lv), then $K^{\mathrm{ad}} = K$ and these are the invariants of Section 2 for the representation ρ^{ad} and the lattice Λ; they are those for ρ formed with the same constant B, as the groups $K_{T_V,p}$ and $K^m_{T_V,p}$ do not depend on the representation; this B is admissible for ρ as well if a also bounds the indices of [85, pp. 21–22] for ρ, as $a = |\mathrm{GL}_{\dim G}(\mathbb{F}_3)|$ does when $n_\rho \le \dim G$. At any compact open product level containing K^{ad}, $\alpha_{f(V)}$ is at most this α_V.

Let $\bar g_1,\dots,\bar g_m$ represent $G^{\mathrm{ad}}(\mathbb{Q})_+\backslash G^{\mathrm{ad}}(\mathbb{A}_f)/K^{\mathrm{ad}}$, a finite set [56, Lemma 5.12], and let $\bar c^{\mathrm{ad}}(\delta,z,D,d)$ and $\bar c^{\mathrm{ad}}_{\deg}(z,D)$ be the maxima over t of the constants c^{ad} and $c^{\mathrm{ad}}_{\deg}$ of Sections 7 and 8 for $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ at the level $\bar g_t K^{\mathrm{ad}} \bar g_t^{-1}$, with ρ^{ad} and the lattice $\mathrm{Ad}(\bar g_t)(\Lambda\otimes\widehat{\mathbb{Z}})$ (Section 2), and with $n := \dim \mathrm{Sh}_K(G,X)$; put $e_0 := e^*(n)$. The constant B of Table 1 is the same at all these levels (we take $a := |\mathrm{GL}_{\dim G^{\mathrm{der}}}(\mathbb{F}_3)|$, which bounds the indices of [85, pp. 21–22] for every lattice $\mathrm{Ad}(\bar g_t)(\Lambda\otimes\widehat{\mathbb{Z}})$). Put $h_{\mathfrak{G}} := 1$ if $\Gamma_i' \subset \lambda(G(\mathbb{Q}))$ for every i, and $h_{\mathfrak{G}} := h_K!$ otherwise, and define

$$D_0 := g_3 D, \quad d_0 := g_3 d\, h_{\mathfrak{G}}, \qquad D_{j+1} := e_0^{2n} D_j^2, \quad d_{j+1} := e_0^{2n} D_j^2\, d_j.$$

Theorem 9.12. *Let (G,X) be a Shimura datum, K a neat product level and ρ as above, $F \supset E(G,X)$ a number field with $[F:\mathbb{Q}] = d$, $Z \subset \mathrm{Sh}_K(G,X)$ closed and geometrically irreducible, defined over F, of degree D and dimension z, and V a special subvariety maximal in Z. Let K' be the level of Lemma 9.1, and K^{ad}, h_K, $h_{\mathfrak{G}}$ as above, built from K'; Corollary 9.10 bounds h_K in closed form, and $h_{\mathfrak{G}} \le h_K!$.*

(a) $\deg V \le h_K\, \bar{c}^{\mathrm{ad}}_{\deg}(z, D_z)$, *and if* $\mathrm{QRH}(\delta; L_V)$ *holds, then* $\Delta(V) \le h_K\, \bar{c}^{\mathrm{ad}}(\delta, z, D_z, d_z)$.

(b) If $\lambda(K)$ is open in $G^{\mathrm{ad}}(\mathbb{A}_f)$, for instance if $Z(G)$ is connected, the same holds with $K^{\mathrm{ad}} := \lambda(K)$ and h_K, $h_{\mathfrak{G}}$ computed at the level K itself, with $D_0 := D$, $d_0 := d\,h_{\mathfrak{G}}$, and with $\bar{c}^{\mathrm{ad}}$, $\bar{c}^{\mathrm{ad}}_{\deg}$ the maxima for $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ at the level that Lemma 9.1 makes from $\lambda(K)$ for $(\rho^{\mathrm{ad}}, \Lambda)$, evaluated at $(g_3^{\mathrm{ad}} D_z, g_3^{\mathrm{ad}} d_z)$, where $g_3^{\mathrm{ad}} := |\mathrm{GL}_{\dim G^{\mathrm{der}}}(\mathbb{F}_3)|$. If $\lambda(K)$ satisfies (Lv′) for $(\rho^{\mathrm{ad}}, \Lambda)$, the constants are those at the level $\lambda(K)$ itself, evaluated at (D_z, d_z) (Proposition 8.9).

If $Z(G)$ is connected, $K^{\mathrm{ad}} = \lambda(K') \subset \lambda(K)$ (Lemma 9.2(iv)), so the bound for α_V holds with α_V computed at $\lambda(K)$; for $\mathcal{A}_g$ at $K(N)$, $N \ge 3$, this is the level of Theorem 1.3(i). If G is adjoint, $K^{\mathrm{ad}} = K' \subset K$, so α_V may be computed at K.

Proof. (a) Lemma 9.1 gives $Z' \subset \mathrm{Sh}_{K'}(G,X)$, geometrically irreducible over a field F' of degree at most $g_3 d$, with $\deg_{\mathcal{L}_{K'}} Z' \le g_3 D$, and V' maximal in Z' with $\pi(V') = V$, $\deg V \le \deg_{\mathcal{L}_{K'}} V'$ and $T_{V'} = T_V$. Let S_i be the component containing Z'. Put $Z^{(0)} := Z'$, defined over $F_0 := F'F_{\mathfrak{G}}$, of degree at most d_0 over $\mathbb{Q}$.

Given $Z^{(j)} \supset V'$ closed and irreducible, defined over F_j with $[F_j:\mathbb{Q}] \le d_j$, and of degree at most D_j: V' is maximal in $Z^{(j)} \subset Z'$. If $f(V')$ is maximal in $f(Z^{(j)})$, stop. Otherwise Lemma 9.11 gives $\theta \in \mathfrak{G}_i$ with $\theta Z^{(j)} \ne Z^{(j)}$ and $V' \subset Z^{(j)} \cap \theta Z^{(j)}$, and we let $Z^{(j+1)}$ be a component of this intersection containing V'; $\dim Z^{(j+1)} < \dim Z^{(j)}$. Put $\mathcal{L} := \mathcal{L}_{K'}|_{\overline{S}_i}$. As θ preserves $\mathcal{L}$ (Proposition 9.4(b)), $\theta Z^{(j)}$ has $\mathcal{L}$-degree at most D_j, and by Lemma 5.2(b) at the neat level K', with $e = e_0$, the components of $Z^{(j)} \cap \theta Z^{(j)}$ have total $\mathcal{L}$-degree at most $e_0^{2n} D_j^2$. So $\deg_{\mathcal{L}} Z^{(j+1)} \le D_{j+1}$, and $Z^{(j)} \cap \theta Z^{(j)}$ has at most $e_0^{2n} D_j^2$ components. It is defined over F_j, since θ is defined over $F_{\mathfrak{G}} \subset F_j$, so $Z^{(j+1)}$ is defined over an extension F_{j+1} of F_j of degree at most $e_0^{2n} D_j^2$. The recursion stops after $j^* \le z - \dim V \le z$ steps, since $Z^{(j)} = V'$ once $\dim Z^{(j)} = \dim V$.

Put $W := f(Z^{(j^*)})$; then $f(V')$ is maximal in W. Let $\bar{g}_t$ be the representative for which W lies in the image of $X^+ \times \{\bar{g}_t\}$ in $\mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$. The level K^{ad} satisfies (Lv) for $(\rho^{\mathrm{ad}}, \Lambda)$ (Lemma 9.2(ii)), so $\bar{g}_t K^{\mathrm{ad}} \bar{g}_t^{-1}$ satisfies it for the lattice $\mathrm{Ad}(\bar{g}_t)(\Lambda \otimes \widehat{\mathbb{Z}})$, and we apply Theorem 8.1 at that level to the translates of W and $f(V')$ by $\bar{g}_t^{-1}$ (Section 2); the translation changes no degree, field of definition, T_V, L_V or α_V. The variety W is geometrically irreducible, defined over $F_{j^*} \supset E(G,X) \supset E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ [25, 2.2.1], of dimension at most z and degree at most D_{j^*} by Proposition 9.4(c). This bounds $\deg f(V')$, d_{L_V} and α_V, and $\deg V \le \deg_{\mathcal{L}_{K'}} V' \le h_K \deg f(V')$ by Proposition 9.4(c). The constants are non-decreasing in z, D and d, and D_j, d_j increase with j, so the values at $j = z$ bound all cases.

(b) When $\lambda(K)$ is open, the argument runs at the level K with $K^{\mathrm{ad}} := \lambda(K)$: Proposition 9.4(a)–(d) and Lemma 9.11 hold with the same proofs, $\lambda(K)$ being neat [68, 0.6] and normalizing K, and $\lambda^{-1}(\lambda(K)) = Z(G)(\mathbb{A}_f)K$. Theorem 8.1 is applied at the level $\lambda(K)$ through Lemma 9.1 for $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, \lambda(K), \rho^{\mathrm{ad}})$, which costs the factor g_3^{ad} and can only

increase α. Here too the constants at a level K'' are the maxima over the levels $\bar g_t K'' \bar g_t^{-1}$, with the lattices $\mathrm{Ad}(\bar g_t)(\Lambda\otimes\widehat{\mathbb{Z}})$, for $\bar g_t$ representing $G^{\mathrm{ad}}(\mathbb{Q})_+\backslash G^{\mathrm{ad}}(\mathbb{A}_f)/K''$. □

Proof of Theorem 1.1. If $G^{\mathrm{ad}}=1$, then G is a torus, $\mathrm{Sh}_K(G,X)$ is finite and $V=Z$ is a point with $T_V=1$, so $\deg V = d_{L_V}=\alpha_V=1$, and we put $c_{\deg}(D)=c(\delta,D,d):=1$. Otherwise a special point x has $\lambda(\mathrm{MT}(x))\neq 1$, so $n_T\geq 2$ (Section 2.2) and $s_\rho,t_\rho\geq 1$ for $(G^{\mathrm{ad}},X^{\mathrm{ad}},\rho^{\mathrm{ad}})$, and Theorem 9.12(a) with $z\leq n$ and the monotonicity of the constants give the bounds of Theorem 1.1 with $h_K\,\bar c^{\mathrm{ad}}_{\deg}(n,D_n)$ and $h_K\,\bar c^{\mathrm{ad}}(\delta,n,D_n,d_n)$. Let H_K be the integer part of the right side of Corollary 9.10 for the level K' of Lemma 9.1, and let d_n^* be d_n computed from $d_0:=g_3 d\,H_K!$. As $h_K\leq H_K$, $h_{\mathfrak{G}}\leq h_K!\leq H_K!$ and the constants are non-decreasing in d, Theorem 1.1 holds with

$$c_{\deg}(D):=H_K\,\bar c^{\mathrm{ad}}_{\deg}(n,D_n),\qquad c(\delta,D,d):=H_K\,\bar c^{\mathrm{ad}}(\delta,n,D_n,d_n^*),$$

and these are effectively computable, H_K being read off the character group of $Z(G)$, ρ and the level. Smaller bounds for h_K and $h_{\mathfrak{G}}$, such as those in Theorem 9.13, may be used in place of H_K and $H_K!$. □

Proof of Corollary 1.2. By [19, Theorem 1.1], $\mathrm{QRH}(1/32;L)$ holds for every number field L; apply Theorem 1.1 with $\delta=1/32$. □

9.6. **Siegel modular varieties.** In this subsection $\nu\colon \mathrm{GSp}_{2g}\to\mathbb{G}_{\mathrm{m}}$ is the similitude character.

Theorem 9.13. *Let $g\geq 1$, $(G,X)=(\mathrm{GSp}_{2g},\mathcal{H}_g^{\pm})$ with its standard representation, $n=g(g+1)/2$, $e_0=e^*(n)$, and $s_g:=|\mathrm{Sp}_{2g}(\mathbb{F}_3)|=3^{g^2}\prod_{i=1}^g(3^{2i}-1)$ [79, p. 70]. Let $c^{\mathrm{ad}}(\delta,z,D,d)$ and $c^{\mathrm{ad}}_{\deg}(z,D)$ be the constants of Section 8 for $(\mathrm{PGSp}_{2g},X^{\mathrm{ad}})$ with $(\mathrm{Ad},\mathfrak{sp}_{2g}(\widehat{\mathbb{Z}}))$ at the level named below. Let F, d, Z, D, z and V be as in Theorem 9.12, and define D_j, d_j by $D_{j+1}:=e_0^{2n}D_j^2$, $d_{j+1}:=e_0^{2n}D_j^2d_j$ from the initial values given below.*

(a) Let $N\geq 3$, $K=K(N)$ the principal congruence subgroup of level N, $D_0:=D$ and $d_0:=d$. Then $h_K\leq \#\{u\in(\mathbb{Z}/N)^\times : u^2=1\}/2$, $h_{\mathfrak{G}}=1$, and

$$\deg V\leq h_K\,c^{\mathrm{ad}}_{\deg}(z,D_z),\qquad \Delta(V)\leq h_K\,c^{\mathrm{ad}}(\delta,z,D_z,d_z)\ \text{ if } \mathrm{QRH}(\delta;L_V)\text{ holds,}$$

with the constants, and α_V, at the level $\lambda(K(N))$.

(b) Let $K(1):=\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$, so that $\mathrm{Sh}_{K(1)}(G,X)_{\mathbb{C}}=\mathcal{A}_{g,1}$, take degrees on $\mathcal{A}_{g,1}$ as in Lemma 9.14, and put $D_0:=(s_g/2)D$ and $d_0:=s_g d$. Then

$$\deg V\leq c^{\mathrm{ad}}_{\deg}(z,D_z),\qquad \max(\deg V,d_{L_V},\alpha_V)\leq c^{\mathrm{ad}}(\delta,z,D_z,d_z)\ \text{ if } \mathrm{QRH}(\delta;L_V)\text{ holds,}$$

with the constants at the level $\lambda(K(3))$ and α_V computed at the level $\lambda(K(1))$.

In both cases d_{L_V} is also the discriminant of the splitting field of the connected center of H_V.

For $\mathcal{A}_{g,1}$ we need a degree. Let $\pi\colon \mathrm{Sh}_{K(3)}(G,X)\to\mathrm{Sh}_{K(1)}(G,X)$ be the natural map. The group $K(1)/K(3)=\mathrm{GSp}_{2g}(\mathbb{F}_3)$ acts on $\mathrm{Sh}_{K(3)}(G,X)$ by right translations, which preserve $\mathcal{L}_{K(3)}$, through $\overline{G}:=\mathrm{GSp}_{2g}(\mathbb{F}_3)/\{\pm1\}$, the image of $Z(G)(\mathbb{Q})\cap K(1)=\{\pm1\}$ acting trivially; π is the quotient by $\overline{G}$, and $\deg\pi=|\overline{G}|=s_g$, since a general point of $\mathcal{H}_g$ has stabilizer $\{\pm1\}$ in $\mathrm{Sp}_{2g}(\mathbb{Z})$. The two geometric components of $\mathrm{Sh}_{K(3)}(G,X)$ correspond to $\mathbb{F}_3^\times$, and k moves them by $\nu(k)$, so the stabilizer of a component has order $s_g/2$.

Lemma 9.14. *For $Y\subset\mathrm{Sh}_{K(1)}(G,X)_{\mathbb{C}}$ closed and irreducible and C an irreducible component of $\pi^{-1}(Y)\subset\mathrm{Sh}_{K(3)}(G,X)_{\mathbb{C}}$, both over $\mathbb{C}$, put $\deg Y:=\deg_{\mathcal{L}_{K(3)}}C/[\kappa(C):\kappa(Y)]$.*

(a) *This does not depend on C.*
(b) *Let $G_C \subset \overline{G}$ be the stabilizer of C and $I_C \subset G_C$ the subgroup fixing C pointwise. Then $[\kappa(C):\kappa(Y)] = |G_C/I_C| \le s_g/2$, and $\deg_{\mathcal{L}_{K(3)}} \pi^{-1}(Y)/\deg\pi = \deg Y/|I_C|$, with $1 \le |I_C| \le s_g/2$.*
(c) *If $\mathcal{M}$ is a $\mathbb{Q}$-line bundle on the Baily–Borel compactification of $\mathcal{A}_{g,1}$ with $\pi^*\mathcal{M} \cong \mathcal{L}_{K(3)}$, then $\deg Y = \deg_{\mathcal{M}} \overline{Y}$.*
(d) *Replacing $K(3)$ by any neat K'' normal in $K(1)$ gives the same number.*

Proof. (a) $\overline{G}$ permutes the components of $\pi^{-1}(Y)$ transitively; each $\sigma \in \overline{G}$ preserves $\mathcal{L}_{K(3)}$, so $\deg \sigma C = \deg C$, and $\sigma\colon C \to \sigma C$ is an isomorphism over Y. (b) There are $|\overline{G}|/|G_C|$ components, all of degree $\deg_{\mathcal{L}_{K(3)}} C$. Take $x \in C$ off the proper closed subsets $C \cap \sigma C$ ($\sigma \notin G_C$) and C^σ ($\sigma \in G_C \smallsetminus I_C$); the fiber of $C \to Y$ through x is $G_C \cdot x$, with $|G_C/I_C|$ points, and a finite dominant morphism of complex varieties has $[\kappa(C):\kappa(Y)]$ points in its general fiber. So $\deg_{\mathcal{L}_{K(3)}} \pi^{-1}(Y) = (|\overline{G}|/|G_C|)\,|G_C/I_C|\deg Y$, and we divide by $\deg\pi = |\overline{G}|$. Finally G_C stabilizes a geometric component, so $|G_C| \le s_g/2$. (c) The projection formula for the finite surjective map $\overline{C} \to \overline{Y}$, applied to a multiple of $\mathcal{M}$ that is a line bundle, gives $\deg_{\pi^*\mathcal{M}} \overline{C} = [\kappa(C):\kappa(Y)]\deg_{\mathcal{M}} \overline{Y}$. (d) For neat $K''' \subset K''$, both normal in $K(1)$, the map $\mathrm{Sh}_{K'''} \to \mathrm{Sh}_{K''}$ is finite étale and pulls $\mathcal{L}_{K''}$ back to $\mathcal{L}_{K'''}$ [50, Prop. 5.3.2]; for a component C''' over C, the projection formula gives $\deg C''' = [\kappa(C'''):\kappa(C)]\deg C$, and the field degrees multiply. Two such subgroups compare through their intersection. □

Proof of Theorem 9.13. (a) $K(N)$ is a neat product level for $N \ge 3$. The center $Z(G) = \mathbb{G}_{\mathrm{m}}$ is connected and $H^1(\mathbb{Q}, \mathbb{G}_{\mathrm{m}}) = 0$, so $\lambda(K)$ is open and Theorem 9.12(b) applies; $\Gamma_i' \subset \lambda(G(\mathbb{Q}))$, so $h_{\mathfrak{G}} = 1$ and $d_0 = d$; and Proposition 9.4(d) gives $h_K \le |\mathbb{Q}^\times\backslash\mathbb{A}_f^\times / \ker(\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N)^\times)| = |\widehat{\mathbb{Z}}^\times/\{\pm1\}\ker(\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N)^\times)| = \varphi(N)/2$. More precisely, the class of $\bar\gamma = \lambda(zk)$ in $\mathrm{Cl}(K)$ is that of z, and $\nu(zk) = z^2\nu(k) \in \mathbb{Q}_{>0}$ with $\nu(k) \in \ker(\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N)^\times)$, so writing $z = qu$ with $q \in \mathbb{Q}^\times$, $u \in \widehat{\mathbb{Z}}^\times$, we get $u^2 \in \ker(\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N)^\times)$. Hence $h_K \le \#\{u \in (\mathbb{Z}/N)^\times : u^2 = 1\}/2$. Let p_0 be an odd prime dividing N, or $p_0 = 2$ if N is a power of 2, and $v := v_{p_0}(N)$, so $v \ge 2$ if $p_0 = 2$. Every eigenvalue ξ of an element of $K(N)_{p_0} \subset 1 + p_0^v M_{2g}(\mathbb{Z}_{p_0})$ has $v_{p_0}(\xi - 1) \ge v$; the elements with this property form a group; and a root of unity $\zeta \ne 1$ has $v_{p_0}(\zeta - 1) \le 1/(p_0 - 1) < v$. So $K(N)_{p_0}$ is neat, hence so is $\lambda(K(N))_{p_0}$ [68, 0.6]; and $\mathrm{Ad}(\lambda(K(N)))$ preserves $\mathfrak{sp}_{2g}(\widehat{\mathbb{Z}})$, as in the proof of Lemma 9.2(ii). Thus $\lambda(K(N))$ satisfies (Lv′), and Proposition 8.9 applies at the level $\lambda(K(N))$ with no factor. For PGSp_{2g} the representatives $\bar g_t$ can be taken in $\lambda(\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}))$. Indeed $\lambda\colon \mathrm{GSp}_{2g}(\mathbb{A}_f) \to \mathrm{PGSp}_{2g}(\mathbb{A}_f)$ is onto, as $H^1(\mathbb{Q}_p, \mathbb{G}_{\mathrm{m}}) = H^1(\mathbb{Z}_p, \mathbb{G}_{\mathrm{m}}) = 0$, and $\mathrm{GSp}_{2g}(\mathbb{A}_f) = \mathrm{GSp}_{2g}(\mathbb{Q})_+\,\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$: for $x \in \mathrm{GSp}_{2g}(\mathbb{A}_f)$ write $\nu(x) = qu$ with $q \in \mathbb{Q}_{>0}$ and $u \in \widehat{\mathbb{Z}}^\times$, and take $\gamma \in \mathrm{GSp}_{2g}(\mathbb{Q})$ with $\nu(\gamma) = q$ and $k \in \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$ with $\nu(k) = u$; then $\gamma^{-1}xk^{-1} \in \mathrm{Sp}_{2g}(\mathbb{A}_f) = \mathrm{Sp}_{2g}(\mathbb{Q})\,\mathrm{Sp}_{2g}(\widehat{\mathbb{Z}})$ by strong approximation for Sp_{2g}, and $\nu > 0$ on $\gamma\,\mathrm{Sp}_{2g}(\mathbb{Q})$. The group $\lambda(\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}))$ normalizes $\lambda(K(M))$ for every M, and the level that Lemma 9.1 makes from it, and preserves $\mathfrak{sp}_{2g}(\widehat{\mathbb{Z}})$; so the maxima of Theorem 9.12 are the constants at the level named.

(b) Choose a component V' of $\pi^{-1}(V)$ and a component Z' of $\pi^{-1}(Z)$ containing it. As in Lemma 9.1, V' is special and maximal in Z', with $T_{V'} = T_V$, and Z' is defined over a field of degree at most $s_g d$, since $\pi^{-1}(Z)$ has at most $|\overline{G}| = s_g$ components. By Lemma 9.14(b), $\deg_{\mathcal{L}_{K(3)}} Z' = [\kappa(Z'):\kappa(Z)]\,D \le (s_g/2)D$. Part (a) at $N = 3$, where $h_{K(3)} = 1$, bounds $\deg_{\mathcal{L}_{K(3)}} V'$ and the torus data at $\lambda(K(3))$. Finally $\deg V = \deg_{\mathcal{L}_{K(3)}} V'/[\kappa(V'):\kappa(V)] \le$

$\deg_{\mathcal{L}_{K(3)}} V'$, and since $\lambda(K(3)) \subset \lambda(K(1))$, each index $|K^m_{T_V,p}/K_{T_V,p}|$ of Definition 2.2, and hence α_V (the constant B does not depend on the level), is at least as large at $\lambda(K(3))$ as at $\lambda(K(1))$.

In both cases the connected center of H_V contains the central $\mathbb{G}_{\mathrm{m}}$ (the weight) and is isogenous to $\mathbb{G}_{\mathrm{m}} \times T_V$, so the two splitting fields coincide. □

The bound for h_K in Theorem 9.13(a) is 1 unless $(\mathbb{Z}/N)^\times$ has a square root of 1 other than ± 1, for instance for $N = 8$, 12 or 15; only then can f fail to be injective on a component, and only then is Lemma 9.11 needed.

10. Listing for abelian type

The bounds of Theorem 9.13 leave a finite set of candidates for the maximal special subvarieties of Z, but the set has to be found. The bounds for d_{L_V} and α_V bound the index of the order of endomorphisms attached to V (Section 10.1), which leaves finitely many PEL data; in Mumford's model (Section 10.2) the locus of each datum is computable (Section 10.3), and V is rigid in the normalization of one of its components (Section 10.4), with degree there bounded in terms of $\deg V$ (Sections 10.5 and 10.6). Urbanik's algorithm [87, Cor. 6.3] then computes a finite set of weakly special subvarieties of bounded degree that contains the rigid ones, a test on the algebraic monodromy selects the rigid ones, which are special (Section 10.7), Section 10.8 passes to Hodge and abelian type, and Section 10.9 states the algorithm and proves Theorem 10.21.

In this section $G = \mathrm{GSp}(W, \psi)$ with $W = \mathbb{Q}^{2g}$, $W_{\mathbb{Z}} = \mathbb{Z}^{2g}$ and ψ perfect on $W_{\mathbb{Z}}$, $X = \mathcal{H}_g^{\pm}$, and $\lambda\colon G \to G^{\mathrm{ad}}$. For a special subvariety V we write $T := T_V^G$ for the connected center of H_V itself, which contains the scalars, and $\bar{T} := \lambda(T)$ for the torus of Section 9.

10.1. **Bounded torus data give bounded orders.** Let $K = \prod_p K_p$ be compact open with $K(N) \subset K \subset G(\widehat{\mathbb{Z}})$ for some $N \geq 1$, and let $V \subset \mathrm{Sh}_K(G, X)_{\mathbb{C}}$ be special, defined by (H, X_H), the image of $X_H^+ \times \{\gamma\}$ with $\gamma \in G(\mathbb{A}_f)$. Put $\mathcal{E} := \mathbb{Q}[T]$, the $\mathbb{Q}$-span of $T(\mathbb{Q})$ in $\mathrm{End}(W)$, $n_{\mathcal{E}} := \dim_{\mathbb{Q}} \mathcal{E}$, $\Lambda_V \subset W$ the lattice with $\Lambda_{V,p} = \gamma_p W_{\mathbb{Z}_p}$, and $R := \mathcal{E} \cap \mathrm{End}(\Lambda_V)$. For a prime p let $\iota_p := [K^m_{T,p} : T(\mathbb{Q}_p) \cap \gamma_p K_p \gamma_p^{-1}]$, and $\bar{\iota}_p$ the same index for $\bar{T}$ and $\lambda(\gamma_p K_p \gamma_p^{-1})$; so $\alpha_V = \prod_{\bar{\iota}_p > 1} \max(1, \bar{B}\bar{\iota}_p)$ with $\bar{B}$ the constant B of the adjoint datum. Let m_V be the least common multiple of the dimensions of the irreducible summands of $W \otimes \overline{\mathbb{Q}}$ under H_V; so $m_V = 1$ if V is a point, and $m_V \leq \mathrm{lcm}(1, \dots, 2g)$.

Lemma 10.1. *Suppose $d_{L_V} \leq D_1$ and $\alpha_V \leq A_1$ with $A_1 \geq 1$. Put $A_1' := A_1 \max(1, 1/\bar{B})$ and $P_1 := 1 + m_V A_1'$.*

(a) $\mathcal{E} = \prod_i F_i$ with each $F_i \subset L_V$ a totally real or CM field, $d_{F_i} \leq D_1$ and $n_{\mathcal{E}} \leq 2g$. R is an order of $\mathcal{E}$ and $R = \mathcal{E} \cap \mathrm{End}(A_x)$ for the abelian variety A_x of the point $[x, \gamma]$, for every $x \in X_H^+$ (so $H_1(A_x, \mathbb{Z}) = \Lambda_V$); if V is a point, R is the center of $\mathrm{End}(A_x)$.

(b) For every prime $p \geq m_V + 2$ unramified in L_V, either $R_p = \mathcal{O}_{\mathcal{E},p}$ or $\iota_p \geq (p-1)/m_V$.

(c) $R_p = \mathcal{O}_{\mathcal{E},p}$ for every prime $p > P_1$ with $p \nmid N d_{L_V}$.

(d) Both as written and with $n_{\mathcal{E}}$ replaced by $2g$,

$$[\mathcal{O}_{\mathcal{E}} : R] \leq \left(2 \cdot 4^{P_1} N^2 D_1 A_1'^{P_1 + \log_2(N D_1)}\right)^{n_{\mathcal{E}}(n_{\mathcal{E}}-1)/2} \left(2 m_V n_{\mathcal{E}} (n_{\mathcal{E}} - 1)(n_{\mathcal{E}}!)^2 D_1^{1/2}\right)^{n_{\mathcal{E}}(n_{\mathcal{E}}-1)/2}.$$

Proof. For a special point, (d) is a form, with explicit constants, of [82, Lemma 7.2] for $\mathcal{A}_g$ and of [23, Thm. 4.1]; (b) is an explicit form of [23, Prop. 4.11] combined with [85, Prop. 3.15], and the proof of (d) follows [82, Lemmas 7.4 and 7.5], [23, Props. 4.8 and 4.12].

Weights and the involution. Let Φ be the set of weights of T on $W \otimes \overline{\mathbb{Q}}$. As $T(\mathbb{Q})$ is Zariski dense in T, $\mathcal{E} \otimes \overline{\mathbb{Q}}$ is the algebra of operators that are scalar on each weight space; so $\mathcal{E}$ is étale of dimension $|\Phi|$, with embeddings φ_χ, $\chi \in \Phi$, taking values in L_V. The ψ-adjoint of $t \in T(\mathbb{Q})$ is $\nu(t)t^{-1} \in T(\mathbb{Q})$, with ν the similitude character, so the adjoint preserves $\mathcal{E}$; on $\mathcal{E} \subset \mathrm{End}^0(A_x)$ it is the Rosati involution, which is positive, so it preserves each factor of $\mathcal{E}$ and is the identity or complex conjugation on it. As $d_{F_i}^{[L_V:F_i]}$ divides d_{L_V}, $d_{F_i} \le D_1$; and $n_\mathcal{E} \le \dim W$. For a point, $\mathrm{End}^0(A_x) = \mathrm{End}_\mathcal{E}(W)$ has center $\mathcal{E}$. This gives (a).

Let $\mu_c \in X_*(T) \otimes \mathbb{Q}$ be the preimage, under the isogeny $T \to H^{\mathrm{ab}}$, of the Hodge cocharacter of H^{ab}, which is the same for all $x \in X_H$. If T acts on an irreducible $H_{\overline{\mathbb{Q}}}$-summand W_j by χ, then $\det_{W_j}$ is trivial on H^{der} and equals $\dim(W_j)\chi$ on T, so $\langle \chi, \mu_c \rangle = a_j / \dim W_j \in [0,1] \cap m_V^{-1}\mathbb{Z}$, with a_j the multiplicity of the weight 1 of a Hodge generic cocharacter on W_j. Let Y be the Galois orbit of $m_V\mu_c$. Then $\langle \chi, y \rangle \in \{0, \dots, m_V\}$ for $y \in Y$ and $\chi \in \Phi$, so $Y \subset X_*(T)$; and Y spans $X_*(T) \otimes \mathbb{Q}$, since otherwise the Hodge cocharacter would factor through a proper subtorus and h_x through a proper $\mathbb{Q}$-subgroup of H. For $\chi \ne \chi'$ in Φ some $y \in Y$ has $\langle \chi - \chi', y \rangle \ne 0$, and $|\langle \chi - \chi', y \rangle| \le m_V$ for all $y \in Y$.

(b) Let $p \ge m_V + 2$ be unramified in $L := L_V$, $\mathfrak{P} \mid p$, f the degree and σ the Frobenius of $L_\mathfrak{P}/\mathbb{Q}_p$, $q := p^f$, and ζ a generator of the roots of unity of order $q-1$ in $L_\mathfrak{P}$. For $y \in Y$ the element $t := \sum_{k<f} \sigma^k y \otimes \zeta^{p^k}$ of $X_*(T) \otimes L_\mathfrak{P}^\times$ is σ-invariant, so $t \in T(\mathbb{Q}_p)$, and $t \in K_{T,p}^m$ as it has finite order. For $\chi \ne \chi'$ choose y with $b_0 := \langle \chi - \chi', y \rangle \ne 0$; then $(\chi/\chi')(t) = \zeta^a$ with $a = \sum_{k<f} p^k \langle \chi - \chi', \sigma^k y \rangle \equiv b_0 \not\equiv 0 \bmod p$ and $|a| \le m_V(q-1)/(p-1) < q-1$, so ζ^a has order at least $(p-1)/m_V$. If $R_p \ne \mathcal{O}_{\mathcal{E},p}$, the image of R_p in the étale algebra $\mathcal{O}_{\mathcal{E},p}/p$ is a proper subalgebra by Nakayama's lemma, so two embeddings $\varphi_\chi \ne \varphi_{\chi'}$ agree modulo the maximal ideal on R_p. As $T(\mathbb{Q}_p) \cap \gamma_p K_p \gamma_p^{-1} \subset R_p^\times$, every element s of $\langle t \rangle$ in this group has $\varphi_\chi(s) \equiv \varphi_{\chi'}(s)$, and both are roots of unity of order prime to p, so they are equal. Hence $\iota_p \ge [\langle t \rangle : \ker(\chi/\chi')] \ge (p-1)/m_V$.

(c) For every p, $\iota_p \le [\mathbb{Z}_p^\times : \mathbb{Z}_p^\times \cap K_p]\, \bar{\iota}_p \le p^{v_p(N)} \bar{\iota}_p$: the map $T(\mathbb{Q}_p) \to \bar{T}(\mathbb{Q}_p)$ is onto by Hilbert's Theorem 90, it maps the level subgroup of T onto that of $\bar{T}$ and $K_{T,p}^m$ into $K_{\bar{T},p}^m$, not necessarily onto, which is all the inequality needs, and its kernel on $K_{T,p}^m$ is $\mathbb{Z}_p^\times$. If $p > P_1$, $p \nmid N d_{L_V}$ and $R_p \ne \mathcal{O}_{\mathcal{E},p}$, then $p \ge m_V + 2$ and (b) gives $\bar{\iota}_p \ge \iota_p \ge (p-1)/m_V > A_1' \ge 1$, so $\alpha_V \ge \bar{B}\bar{\iota}_p > A_1$, a contradiction.

(d) Let $\mathfrak{t} := \mathrm{Lie}\, T \subset \mathcal{E}$, $k_p := 1$ for odd p and $k_p := 2$ for $p = 2$. For $y_0 \in \mathfrak{t} \cap \mathcal{O}_\mathcal{E}$ with $\Delta_0 := \mathrm{disc}(y_0) \ne 0$, the element $u := \exp(p^{k_p} y_0)^{\iota_p}$ lies in $(K_{T,p}^m)^{\iota_p} \subset R_p^\times$, since exp converges on $p^{k_p}\mathcal{O}_{\mathcal{E},p}$ and maps $p^{k_p}\mathcal{O}_{\mathcal{E},p} \cap (\mathfrak{t} \otimes \mathbb{Q}_p)$ into $T(\mathbb{Q}_p) \cap \mathcal{O}_{\mathcal{E},p}^\times = K_{T,p}^m$. As $v_p(e^z - 1) = v_p(z)$ for $v_p(z) > 1/(p-1)$, $v_p(\mathrm{disc}\, u) = n_\mathcal{E}(n_\mathcal{E}-1)(v_p(\iota_p) + k_p) + v_p(\Delta_0)$, so $\mathbb{Z}_p[u] \subset R_p$ gives $[\mathcal{O}_{\mathcal{E},p} : R_p] \le p^{(n_\mathcal{E}(n_\mathcal{E}-1)(v_p(\iota_p)+k_p)+v_p(\Delta_0))/2}$. To choose y_0, identify $\mathcal{E} \otimes L = L^\Phi$, let $w := (\langle \chi, m_V\mu_c \rangle)_\chi$ and $y_\ell := (\mathrm{id} \otimes \mathrm{Tr}_{L/\mathbb{Q}})(w\ell) \in \mathfrak{t} \cap \mathcal{O}_\mathcal{E}$ for $\ell \in \mathcal{O}_L$. The $\mathbb{Q}$-linear map $L \to \mathfrak{t}$, $\ell \mapsto y_\ell$, is onto: in $\mathfrak{t} \otimes L = X_*(T) \otimes L$ we have $y_\ell = \sum_\sigma \sigma(\ell)\, \sigma(m_V\mu_c)$, so by the independence of the embeddings of L the L-span of its image is that of Y, which is $\mathfrak{t} \otimes L$; so the image, a $\mathbb{Q}$-subspace, is $\mathfrak{t}$, and disc is a nonzero polynomial on $\mathfrak{t}$, of degree $n_\mathcal{E}(n_\mathcal{E}-1)$. With $v_1, \dots, v_{[L:\mathbb{Q}]} \in \mathcal{O}_L$ attaining the successive minima of the house, which are at most $D_1^{1/2}$ by Minkowski's second theorem, some $\ell = \sum c_k v_k$ with $0 \le c_k \le n_\mathcal{E}(n_\mathcal{E}-1)$ has $\Delta_0 := \mathrm{disc}(y_\ell) \ne 0$, and $|\Delta_0| \le (2m_V n_\mathcal{E}(n_\mathcal{E}-1)[L:\mathbb{Q}]^2 D_1^{1/2})^{n_\mathcal{E}(n_\mathcal{E}-1)}$, with $[L:\mathbb{Q}] \le n_\mathcal{E}!$ since $\mathrm{Gal}(L/\mathbb{Q})$ acts faithfully on Φ. By (c) the bad primes number at most $P_1 + \log_2(ND_1)$, and for them $p^{v_p(\iota_p)} \le p^{v_p(N)} A_1'$ and $\prod p^{k_p} \le 2 \cdot 4^{P_1} N D_1$. Multiplying gives (d); the bound increases with $n_\mathcal{E}$.

Parts (c) and (d) hold, with the same proofs, when $\bar{B}$ is replaced in A_1' by any $\bar{B}_0 \in (0, \bar{B}]$: $\bar{B}$ enters only through $A_1' \geq A_1/\bar{B}$. □

The estimate in the proof of Lemma 10.1(b) is attained, for instance at $p = 5$: take $\mathcal{E} = \mathbb{Q}(i) \times \mathbb{Q}$ and $T = \{(t, s) : N(t) = s^2\}$, the torus of $\{x_0\} \times Y(1) \subset \mathcal{A}_2$ for the point x_0 of $Y(1)$ with complex multiplication by $\mathbb{Q}(i)$, with $m_V = 2$.

10.2. **Mumford's model and its level.** To list special subvarieties we need $\mathcal{A}_g$ and its universal family by explicit equations. Let d_Υ be a positive multiple of 8, $\Upsilon := (d_\Upsilon, \dots, d_\Upsilon)$ with g entries, $K(\Upsilon) := (\mathbb{Z}/d_\Upsilon\mathbb{Z})^g$, and V_Υ the $\mathbb{Q}$-vector space of functions on $K(\Upsilon)$, so that $\mathbb{P}(V_\Upsilon)$ has coordinates $Q(a)$, $a \in K(\Upsilon)$. Mumford considers the functor $\mathcal{M}_\Upsilon$ of abelian schemes with an Υ-marking: an abelian scheme $\mathcal{X}$ over S, a relatively ample, totally symmetric, normalized invertible sheaf L on $\mathcal{X}$ of type Υ, and a symmetric ϑ-structure for L [61, pp. 78–79]. An Υ-marking embeds $\mathcal{X}$ in $\mathbb{P}(V_\Upsilon) \times S$, and the theta-null map $t\colon \mathcal{M}_\Upsilon \to \mathbb{P}(V_\Upsilon)$ takes the image of the zero section [61, pp. 82–83]. Mumford writes down a closed subscheme $\overline{M}_\Upsilon \subset \mathbb{P}(V_\Upsilon)$, cut out by Riemann's quartic theta relations and the relations $Q(a) = Q(-a)$, and a closed subscheme $\overline{\mathcal{A}}_\Upsilon \subset \mathbb{P}(V_\Upsilon) \times \overline{M}_\Upsilon$, cut out by relations of bidegree $(2, 2)$ in the coordinates of the two factors [61, p. 83], and he proves the following [61, Theorem, p. 83, and Further Theorem, p. 84]: there is an open subscheme $M_\Upsilon \subset \overline{M}_\Upsilon$ such that t is an isomorphism of $\mathcal{M}_\Upsilon$ with the functor of points of M_Υ, and $\mathcal{A}_\Upsilon := \overline{\mathcal{A}}_\Upsilon \times_{\overline{M}_\Upsilon} M_\Upsilon$ is the universal abelian scheme over M_Υ, with its zero section, embedded in $\mathbb{P}(V_\Upsilon) \times M_\Upsilon$; in particular the origin of the fiber over $s \in M_\Upsilon$ is the point s itself. We use these schemes over $\mathbb{Q}$. The open set M_Υ is not given explicitly; the algorithm computes $M_\Upsilon \otimes \overline{\mathbb{Q}}$ from Mumford's equations and those of Z (Lemma 10.4).

Over $\mathbb{C}$, $M_\Upsilon \otimes \mathbb{C}$ is a Shimura variety $\mathrm{Sh}_{K_\Upsilon}(G, X)$: a point is a principally polarized abelian variety A with a similitude $\eta\colon W_{\widehat{\mathbb{Z}}} \to H_1(A, \widehat{\mathbb{Z}})$, and η, η' give the same point if and only if $\eta' = \alpha\eta k$ with $\alpha \in \mathrm{Aut}(A, \varphi_0)$ and k in the subgroup $K_\Upsilon \subset G(\widehat{\mathbb{Z}})$ of elements for which η and ηk induce the same Υ-marking on A. Here φ_0 is the principal polarization, L is the totally symmetric sheaf with $\varphi_L = d_\Upsilon\varphi_0$, which has type Υ and is unique [60, p. 307], and the marking induced by η is the symmetric ϑ-structure for L that [60, §2, Remark 3, p. 319] attaches to the symplectic isomorphism $H(L^{\otimes 2}) = A[2d_\Upsilon] \cong H(2\Upsilon)$ defined by η as in the proof of Lemma 10.2(i); every Υ-marking on A arises in this way [60, §2, Remark 4]. Let $e_1, \dots, e_g, f_1, \dots, f_g$ be a symplectic basis of $W_\mathbb{Z}$, with $\psi(e_i, e_j) = \psi(f_i, f_j) = 0$, $\psi(e_i, f_i) = 1$ and $\psi(e_i, f_j) = 0$ for $i \neq j$, let ν be the similitude character, and for $m \geq 1$ put

$$K'(m) := \{k \in G(\widehat{\mathbb{Z}}) : k \equiv \mathrm{diag}(1_g, \nu(k)1_g) \bmod m\} \supset K(m),$$

and $\Gamma(m) := \mathrm{Sp}_{2g}(\mathbb{Z}) \cap K(m)$.

Lemma 10.2. *(i) $K'(2d_\Upsilon) \subset K_\Upsilon \subset K'(d_\Upsilon)$. In particular $K(2d_\Upsilon) \subset K_\Upsilon$ and $K_\Upsilon \cap \mathrm{Sp}(W)(\mathbb{A}_f) \subset K(d_\Upsilon)$.*

(ii) $\nu(K_\Upsilon) = \widehat{\mathbb{Z}}^\times$. Hence $S := M_\Upsilon \otimes \mathbb{C}$ is connected, $S = \Gamma_\Upsilon\backslash\mathcal{H}_g$ with $\Gamma_\Upsilon := \mathrm{Sp}_{2g}(\mathbb{Z}) \cap K_\Upsilon$, and $\Gamma(2d_\Upsilon) \subset \Gamma_\Upsilon \subset \Gamma(d_\Upsilon)$.

(iii) K_Υ is not neat. But if $(H', X_{H'})$ is a Shimura subdatum of (G, X), $X_{H'}^+$ a connected component of $X_{H'}$ and $h \in G(\mathbb{A}_f)$, then every subgroup of $H'(\mathbb{Q}) \cap hK_\Upsilon h^{-1}$ that preserves $X_{H'}^+$ is neat. In particular $G(\mathbb{Q})_+ \cap hK_\Upsilon h^{-1}$ is neat for every h.

Proof. (i) Mumford defines $H(\Upsilon) := K(\Upsilon) \oplus \widehat{K}(\Upsilon)$ with $\widehat{K}(\Upsilon) := \mathrm{Hom}(K(\Upsilon), \mathbb{C}^*)$ [60, p. 294], so the commutator pairing of the Heisenberg group $\mathcal{G}(\Upsilon)$ takes values in μ_{d_Υ} with

no choice of a root of unity. For L of type Υ the group $H(L)$ is $A[d_\Upsilon]$, and an Υ-marking induces a symplectic isomorphism $H(L) \cong H(\Upsilon)$. On $A[d_\Upsilon]$ the Weil pairing $\langle\ ,\ \rangle$ satisfies $\langle \eta v, \eta w\rangle = \zeta^{\nu(\eta)\psi(v,w)}$ with $\zeta := \exp(2\pi i/d_\Upsilon)$; the other sign convention replaces ζ by ζ^{-1} and changes nothing below. The isomorphism attached to η matches $a \in K(\Upsilon)$ with $\sum_i a_i \eta e_i$, and the character $a \mapsto \zeta^{\sum_i a_i b_i}$ of $K(\Upsilon)$ with the unique $P \in \sum_i (\mathbb{Z}/d_\Upsilon\mathbb{Z})\,\eta f_i$ such that $\langle \eta e_i, P\rangle = \zeta^{b_i}$ for all i, namely $P = \nu(\eta)^{-1}\sum_i b_i \eta f_i$. So η and ηk give the same isomorphism if and only if $ke_i \equiv e_i$ and $\nu(k)^{-1}kf_i \equiv f_i$ modulo d_Υ for all i, that is, $k \in K'(d_\Upsilon)$. Since the Υ-marking determines this isomorphism, $K_\Upsilon \subset K'(d_\Upsilon)$.

The same computation for $L^{\otimes 2}$, which has type 2Υ and $H(L^{\otimes 2}) = A[2d_\Upsilon]$, shows that η and ηk induce the same symplectic isomorphism $H(L^{\otimes 2}) \cong H(2\Upsilon)$ if and only if $k \in K'(2d_\Upsilon)$; and by [60, §2, Remark 3, p. 319] such an isomorphism determines the symmetric ϑ-structure for L. So $K'(2d_\Upsilon) \subset K_\Upsilon$. For $k \in K'(d_\Upsilon)$ write $k = (1 + d_\Upsilon M)\,\mathrm{diag}(1_g, \nu(k)1_g)$. By the formula for P at level $2d_\Upsilon$, the isomorphisms $H(L^{\otimes 2}) \cong H(2\Upsilon)$ attached to ηk and η differ by $\mathrm{diag}(1_g, \nu(\eta)1_g)(1 + d_\Upsilon M)\,\mathrm{diag}(1_g, \nu(\eta)^{-1}1_g)$, which is $1 + N$ with $N := d_\Upsilon M$ modulo $2d_\Upsilon$, as $\nu(\eta)$ is odd and the entries of N are 0 or d_Υ; so it depends only on k, and $N(H(2\Upsilon)) \subset H(2\Upsilon)[2]$. In blocks for $K(2\Upsilon) \oplus \widehat{K}(2\Upsilon)$, $1 + N = (1 + N_\mathrm{d})(1 + N_\mathrm{o})$ with N_d block-diagonal and N_o off-diagonal, both factors symplectic. For $1 + N_\mathrm{o}$, the computation after [60, §2, Remark 3, p. 319] shows that the marking is replaced by its composite with the automorphism $F_\sigma\colon (t, y, m) \mapsto (\sigma(y, m)t, y, m)$ of $\mathcal{G}(\Upsilon) = \mathbb{C}^* \times K(\Upsilon) \times \widehat{K}(\Upsilon)$ [60, pp. 294 and 318], where $\Delta_1\colon K(2\Upsilon) \to \widehat{K}(2\Upsilon)$ and $\Delta_2\colon \widehat{K}(2\Upsilon) \to K(2\Upsilon)$ are the two blocks of N_o, $\bar{l}$ is the image of l in $\widehat{K}(\Upsilon)$, and $\sigma((2x, \bar{l})) = \Delta_1 x(x) \cdot l(\Delta_2 l)$ for $x \in K(2\Upsilon)$ and $l \in \widehat{K}(2\Upsilon)$; this depends only on N_o. The factor $1 + N_\mathrm{d}$ acts by $x \mapsto \varepsilon x$ and $l \mapsto l \circ \varepsilon^{-1}$, with ε its block on $K(2\Upsilon)$. Recall Mumford's homomorphisms $D_{-1}\colon \mathcal{G}(2\Upsilon) \to \mathcal{G}(2\Upsilon)$, $(t, x, l) \mapsto (t, -x, -l)$, $H_2\colon \mathcal{G}(2\Upsilon) \to \mathcal{G}(\Upsilon)$, $(t, x, l) \mapsto (t^2, 2x, \bar{l})$, and $E_2\colon \mathcal{G}(\Upsilon) \to \mathcal{G}(2\Upsilon)$, $(t, x, l) \mapsto (t^2, x, 2 * l)$, where $K(\Upsilon) \subset K(2\Upsilon)$ by $a \mapsto 2a$ and $(2 * l)(x) := l(2x)$ [60, p. 316]. The automorphism $U\colon (t, x, l) \mapsto (t, \varepsilon x, l \circ \varepsilon^{-1})$ of $\mathcal{G}(2\Upsilon)$ commutes with D_{-1} and satisfies $H_2 \circ U = H_2$ and $U \circ E_2 = E_2$, as $\varepsilon - 1$ has values of order 2; so U carries a symmetric lift of the first isomorphism to one of the second with the same composite with H_2, and by [60, §2, Remarks 1 and 3] the marking does not change. Hence the markings induced by ηk and η differ by an automorphism of $\mathcal{G}(\Upsilon)$ that depends only on k, so K_Υ does not depend on (A, η); and it is a group, since for $k, k' \in K_\Upsilon$ the pair $(A, \eta k)$ gives that $\eta k k'$ induces the marking of ηk, which is that of η, and likewise ηk^{-1} that of $\eta k^{-1}k = \eta$. Elements of $K'(d_\Upsilon)$ with $\nu = 1$ are $\equiv 1$ modulo d_Υ. Both $K'(d_\Upsilon)$ and $K'(2d_\Upsilon)$ are product levels with the same factor at every odd p, so K_Υ is a product level.

(ii) $\mathrm{diag}(1_g, u1_g) \in K'(2d_\Upsilon)$ for every $u \in \widehat{\mathbb{Z}}^\times$, so $\nu(K_\Upsilon) = \widehat{\mathbb{Z}}^\times$. As Sp_{2g} is simply connected, $\pi_0(\mathrm{Sh}_K(G, X)_\mathbb{C}) = \mathbb{Q}_{>0}\backslash\mathbb{A}_f^\times/\nu(K) = \widehat{\mathbb{Z}}^\times/\nu(K)$ for every compact open $K \subset G(\widehat{\mathbb{Z}})$ [25, 2.1.3], which is trivial for $K = K_\Upsilon$. An element of $G(\mathbb{Q})_+ \cap K_\Upsilon$ has similitude in $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^\times = \{1\}$, so it lies in $\mathrm{Sp}_{2g}(\mathbb{Z}) \cap K_\Upsilon = \Gamma_\Upsilon$, and the inclusions follow from (i).

(iii) The element $\mathrm{diag}(1_g, -1_g)$ of $G(\mathbb{Z})$ has similitude -1 and lies in $K'(2d_\Upsilon) \subset K_\Upsilon$; its eigenvalues at every p are ± 1, so it is not neat [68, 0.6]. Let Γ' be a subgroup of $H'(\mathbb{Q}) \cap hK_\Upsilon h^{-1}$ that preserves $X^+_{H'}$. Then Γ' preserves the component of $\mathcal{H}_g^\pm$ that contains $X^+_{H'}$, so $\nu(\Gamma') \subset \mathbb{Q}_{>0} \cap \nu(K_\Upsilon) = \{1\}$, and $\Gamma' \subset G(\mathbb{Q}) \cap h(K_\Upsilon \cap \mathrm{Sp}(W)(\mathbb{A}_f))h^{-1} \subset G(\mathbb{Q}) \cap hK(d_\Upsilon)h^{-1}$ by (i). As $d_\Upsilon \geq 3$, $K(d_\Upsilon)$ is neat, and neatness passes to conjugates and to subgroups [68, 0.6]. □

So K_Υ is not a neat level in the sense of Section 2.1, but S is the quotient of $\mathcal{H}_g$ by the neat group Γ_Υ, and everything below uses only the neatness of the arithmetic groups in (iii). In particular the bundle of top-degree forms on S extends to the Baily–Borel compactification as a line bundle $\mathcal{L}_{K_\Upsilon}$ [62, Prop. 3.4(b)], whose sections are the modular forms of weight $g+1$ for Γ_Υ, and degrees on S are taken for it. The bounds of Section 9.6 are proved at neat product levels, and reach K_Υ through $K(2d_\Upsilon)$.

Lemma 10.3. *Let $K' := K(2d_\Upsilon)$, $\pi\colon \mathrm{Sh}_{K'}(G,X) \to S$ the natural map, and $c_\Upsilon := [K_\Upsilon : K'] \le |\mathrm{GSp}_{2g}(\mathbb{Z}/2d_\Upsilon\mathbb{Z})|$. Let $Z \subset S$ be closed and geometrically irreducible, defined over a number field F with $[F:\mathbb{Q}] = d$, with $D := \deg_{\mathcal{L}_{K_\Upsilon}} Z$, and let V be special and maximal in Z. Then there are a geometrically irreducible component Z' of $\pi^{-1}(Z)$, defined over a field of degree at most $c_\Upsilon d$, with $\deg_{\mathcal{L}_{K'}} Z' \le c_\Upsilon D$, and a special subvariety V' maximal in Z' with $\pi(V') = V$, $T_{V'} = T_V$, $\deg_{\mathcal{L}_{K_\Upsilon}} V \le \deg_{\mathcal{L}_{K'}} V'$ and $\alpha_V \le \alpha_{V'}$, where α_V is computed at the level $\lambda(K_\Upsilon)$ and $\alpha_{V'}$ at $\lambda(K')$. Since the constants of Theorem 9.13(a) are non-decreasing in (D,d), Theorem 9.13(a) with $N = 2d_\Upsilon$, evaluated at $(c_\Upsilon D, c_\Upsilon d)$, bounds $\deg_{\mathcal{L}_{K_\Upsilon}} V$, and, if $\mathrm{QRH}(\delta; L_V)$ holds, d_{L_V} and α_V.*

Proof. This is the proof of Lemma 9.1 with $K_\Upsilon \supset K'$ in place of $K \supset K'$; that proof uses the neatness of K only through the arithmetic groups, which are neat here by Lemma 10.2(iii). Each component of $\mathrm{Sh}_{K'}(G,X)_{\mathbb{C}}$ is $\Gamma(2d_\Upsilon)\backslash\mathcal{H}_g$, and π restricted to it is the covering $\Gamma(2d_\Upsilon)\backslash\mathcal{H}_g \to \Gamma_\Upsilon\backslash\mathcal{H}_g$ of quotients by neat groups, so π is finite étale of degree at most c_Υ; top-degree forms pull back to top-degree forms, and the two extensions are given by modular forms of weight $g+1$, so $\pi^*\mathcal{L}_{K_\Upsilon} = \mathcal{L}_{K'}$. If V is the image of $X_H^+ \times \{\gamma\}$, let V' be the image of $X_H^+ \times \{\gamma\}$ in $\mathrm{Sh}_{K'}(G,X)$: a component of $\pi^{-1}(V)$ with the same datum (H, X_H), so $T_{V'} = T_V$. Let Z' be a component of $\pi^{-1}(Z)$ containing V'. As π is defined over $\mathbb{Q}$ (it sends $(A, \varphi_0, \eta K')$ to the Υ-marking induced by η, which involves no choice of a root of unity and so commutes with $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$), $\pi^{-1}(Z)$ is defined over F and has at most c_Υ geometric components, Z' is defined over an extension of degree at most c_Υ; by the projection formula $\deg_{\mathcal{L}_{K'}} \pi^{-1}(Z) = \deg\pi \cdot D \le c_\Upsilon D$ and $\deg V \le \deg(\pi|_{V'}) \deg V = \deg_{\mathcal{L}_{K'}} V'$. If $V' \subsetneq V'' \subset Z'$ with V'' special, then $\pi(V'') \subset Z$ is special of dimension larger than $\dim V$, against maximality. Finally, with the same γ, $\bar{T}(\mathbb{Q}_p) \cap \lambda(\gamma_p K'_p \gamma_p^{-1}) \subset \bar{T}(\mathbb{Q}_p) \cap \lambda(\gamma_p K_{\Upsilon,p} \gamma_p^{-1})$, so each index $\bar{\iota}_p$ can only grow when K_Υ is replaced by K'; each factor $\max(1, \bar{B}\bar{\iota}_p)$ is at least 1 and non-decreasing in $\bar{\iota}_p$, the set of p with $\bar{\iota}_p > 1$ can only grow, and $\bar{B}$ does not change. The constants of Theorem 9.13(a) are non-decreasing in D and d because D_z and d_z are, and the constants of Section 8 are non-decreasing in D and d (proof of Theorem 9.12). □

Lemma 10.4. *Let $Z \subset M_\Upsilon \otimes \overline{\mathbb{Q}}$ be closed and irreducible. Put $n' := g(g+1)/2$ and $P_\Upsilon(t) := d_\Upsilon^g t^g$. Let $Y := (\overline{M}_\Upsilon \otimes \overline{\mathbb{Q}})_{\mathrm{red}}$, let $Y^\circ \subset Y$ be the open set of points at which Y is smooth over $\overline{\mathbb{Q}}$ of local dimension n', $\mathcal{X}^\circ := \overline{\mathcal{A}}_\Upsilon \times_{\overline{M}_\Upsilon} Y^\circ$, $H_\Upsilon \subset Y^\circ$ the set of points y whose fiber $\mathcal{X}_y^\circ \subset \mathbb{P}(V_\Upsilon)_{\kappa(y)}$ is smooth of pure dimension g with Hilbert polynomial P_Υ, U the interior of H_Υ in Y°, C_Z the irreducible component of Y containing Z, and $N_0 := U \cap C_Z$.*

(i) $\mathcal{X}^\circ \times_{Y^\circ} U \to U$ *is smooth.*

(ii) $M_\Upsilon \otimes \overline{\mathbb{Q}} \subset U$. *In particular $Z \subset Y^\circ$, and C_Z is well defined.*

(iii) $N_0 = M_\Upsilon \otimes \overline{\mathbb{Q}}$.

(iv) N_0 *can be computed from Mumford's equations and equations for Z, as a quasi-projective variety given by explicit equations and inequations.*

Proof. (i) Each $y \in U$ has an open neighborhood $W \subset H_\Upsilon$. As Y° is smooth over $\overline{\mathbb{Q}}$, each connected component of W is integral, and over it the fibers of $\mathcal{X}^\circ$ have the same Hilbert polynomial P_Υ, so $\mathcal{X}^\circ_W$ is flat over W [44, III.9.9]. A flat morphism of finite presentation with smooth fibers is smooth [41, 17.5.1].

(ii) $M_\Upsilon \otimes \overline{\mathbb{Q}}$ is open in $\overline{M}_\Upsilon \otimes \overline{\mathbb{Q}}$ and smooth of dimension n' (it is a fine moduli space, and $S = \Gamma_\Upsilon \backslash \mathcal{H}_g$ with Γ_Υ neat by Lemma 10.2), so it is an open subscheme of Y contained in Y°. Its fibers are abelian varieties embedded by a sheaf L of type Υ, which are smooth of pure dimension g with Hilbert polynomial $\chi(L^{\otimes t}) = t^g \chi(L) = d_\Upsilon^g t^g$, because $\chi(L)^2 = \deg \varphi_L = |K(L)| = d_\Upsilon^{2g}$ and $\chi(L) > 0$ for ample L [57, Thm. 11.1]. So $M_\Upsilon \otimes \overline{\mathbb{Q}}$ is an open subset of Y° contained in H_Υ, hence in U. A point of Y° lies on exactly one irreducible component of Y.

(iii) Two distinct components of Y of dimension n' meet only at points where Y is not smooth, so the connected components of Y° are the non-empty $C_j \cap Y^\circ$, for the components C_j of Y of dimension n'; as each $U \cap C_j$ is open in the irreducible C_j, the connected components of U are the non-empty $U \cap C_j$. By (ii), $Z \subset U \cap C_Z = N_0$, and N_0 is connected. The family $\overline{\mathcal{A}}_\Upsilon \times_{\overline{M}_\Upsilon} N_0$ is smooth over N_0 by (i), and every geometric point of Z is in the image of $\mathcal{M}_\Upsilon$ under t. Mumford's Step IV [61, p. 94] says, with Υ for his δ: "Suppose that N is a connected subscheme of $\overline{M}_\Upsilon$ such that if $\overline{\mathcal{A}}_\Upsilon$ induces a subscheme B over N, then B is smooth over N. Suppose that some geometric point x of N is in the image of $\mathcal{M}_\Upsilon$ under t. Then the N-valued point 1_N of N is in the image of $\mathcal{M}_\Upsilon$." Its proof [61, pp. 94–95] uses N only as a connected scheme with a morphism to $\overline{M}_\Upsilon$, so it applies to $N_0 \to \overline{M}_\Upsilon$. Hence $N_0 \subset M_\Upsilon \otimes \overline{\mathbb{Q}}$. Now N_0 is open in U, hence open in $M_\Upsilon \otimes \overline{\mathbb{Q}}$, and closed in $U \supset M_\Upsilon \otimes \overline{\mathbb{Q}}$, hence closed in $M_\Upsilon \otimes \overline{\mathbb{Q}}$. By Lemma 10.2(ii), $M_\Upsilon \otimes \overline{\mathbb{Q}}$ is connected, so $N_0 = M_\Upsilon \otimes \overline{\mathbb{Q}}$.

(iv) The components of Y are computed by absolute primary decomposition, Y° by the Jacobian criterion, and C_Z by ideal membership. Urbanik's flattening stratification [87, Prop. 5.1(i)], applied to $\mathcal{X}^\circ \to Y^\circ$, gives locally closed $T_1, \dots, T_k$ covering Y° over which $\mathcal{X}^\circ$ is flat. Over each irreducible component T' of each T_i the fibers have one Hilbert polynomial [44, III.9.9], computed at the generic point by a Gröbner basis. Let H'_Υ be the set of $y \in Y^\circ$ whose fiber has Hilbert polynomial P_Υ and whose Jacobian matrix has rank $\dim \mathbb{P}(V_\Upsilon) - g$ at every point of the fiber. The points of $\mathcal{X}^\circ_{T'}$ at which the Jacobian matrix of the fiber has rank different from $\dim \mathbb{P}(V_\Upsilon) - g$ form a constructible set, and its image in T' is constructible and computable [7, Thm. 1.0]. So H'_Υ is constructible and computable, and so is its interior U' in Y°, the complement of the closure of $Y^\circ \smallsetminus H'_\Upsilon$. Clearly $H_\Upsilon \subset H'_\Upsilon$, so $U \subset U'$. Conversely, let $y \in U' \cap C_Z$ and let $W \subset U' \cap C_Z$ be an irreducible open neighborhood of y in Y°. The open set $M_\Upsilon \otimes \overline{\mathbb{Q}}$ meets C_Z (it contains Z), so W contains the generic point of C_Z, over which the fiber is an abelian variety. As the fibers over W all have Hilbert polynomial P_Υ, $\mathcal{X}^\circ_W$ is flat over the integral W [44, III.9.9], so its associated points lie over the generic point of W; hence $\mathcal{X}^\circ_W$ is integral of dimension $n' + g$, and every fiber over W is of pure dimension g [44, III.9.6]. On a fiber of pure dimension g, rank $\dim \mathbb{P}(V_\Upsilon) - g$ of the Jacobian matrix at every point is smoothness. Hence $W \subset H_\Upsilon$, and $U' \cap C_Z = U \cap C_Z = N_0$. □

10.3. **PEL loci.** *PEL data.* A PEL datum $\mathfrak{d} = (\mathcal{E}, R, \tau)$ consists of a product $\mathcal{E} = \prod_i F_i$ of totally real and CM fields with its positive involution $*$ (the identity on each totally real factor and complex conjugation on each CM factor), an order $R \subset \mathcal{O}_\mathcal{E}$ stable under $*$, given by generators $e_1, \dots, e_m$ of R as a $\mathbb{Z}$-module, and a type τ: multiplicities $m_i \geq 1$ with $\sum_i m_i [F_i : \mathbb{Q}] = 2g$, together with, for each CM factor F_i and each place v of its maximal totally real subfield F_i^+, an integer $0 \leq a_v \leq m_i$, the multiplicity on the tangent space of a fixed embedding φ_v of F_i above v (its conjugate then has multiplicity $m_i - a_v$); a totally real F_i carries $m_i/2$ copies of each of its embeddings, m_i even. Then τ gives, for

each $r \in R$, a characteristic polynomial $\tau(r)$ of degree g, and we let W' be the $\mathcal{E}$-module whose character is $\tau\bar{\tau}$. The *PEL locus* $\mathfrak{E}(\mathfrak{d}) \subset S$ is the set of points s for which there is a ring homomorphism $\iota\colon R \to \mathrm{End}(A_s)$ with $\iota(r)^\dagger = \iota(r^*)$ for the Rosati involution $\dagger$ of the polarization, and $\det(T - \iota(r) \mid \mathrm{Lie}\, A_s) = \tau(r)$ for all $r \in R$; then $H_1(A_s, \mathbb{Q}) \cong W'$ as $\mathcal{E}$-modules, as $H_1(A_s, \mathbb{C}) = \mathrm{Lie}\, A_s \oplus \overline{\mathrm{Lie}\, A_s}$.

For a special V with torus algebra $\mathcal{E} = \mathbb{Q}[T]$ as in Section 10.1 we put $\mathfrak{d}_V := (\mathcal{E}, R, \tau_V)$, with $R = \mathcal{E} \cap \mathrm{End}(\Lambda_V)$ and $\tau_V(r)$ the characteristic polynomial of r on $\mathrm{Lie}\, A_x = W_{\mathbb{C}}/F^0_x$ for $x \in X_H^+$, which does not depend on x, being continuous with values in a countable set.

Endomorphisms by their graphs. Let $s \in N_0$, with abelian variety $A_s \subset \mathbb{P}(V_\Upsilon)$ embedded by L of type Υ. A closed subscheme $\Gamma \subset A_s \times A_s$ such that $\mathrm{pr}_1|_\Gamma$ is an isomorphism and $(0,0) \in \Gamma$ is the graph of the homomorphism $f := \mathrm{pr}_2 \circ (\mathrm{pr}_1|_\Gamma)^{-1}$, since a regular map of abelian varieties is a homomorphism followed by a translation [57, Cor. 1.2]. On $\Gamma_f \cong A_s$ the bundle $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ restricts to $L \otimes f^*L$, and $\varphi_{L \otimes f^*L} = \varphi_L \circ (1 + f^\dagger f)$, so

$$\chi\big((L \otimes f^*L)^{\otimes t}\big) = t^g\, d_\Upsilon^g\, \det_{H_1(A_s,\mathbb{Q})}(1 + f^\dagger f)^{1/2}, \tag{10.1}$$

because $\chi(M)^2 = \deg \varphi_M$ and $\chi(M) > 0$ for ample M, $\chi(M^{\otimes t}) = t^g \chi(M)$, and the degree of an endomorphism is its determinant on $H_1(A_s, \mathbb{Q})$ [57, Thm. 11.1 and Prop. 10.20]. If $f = \iota(e)$ with $\iota(e)^\dagger = \iota(e^*)$, this is the polynomial $P_e(t) := t^g d_\Upsilon^g \det_{W'}(1 + e^* e)^{1/2}$, computed from $\mathfrak{d}$.

Lemma 10.5. *Let B be a quasi-projective variety over a number field, $\mathcal{X} \subset \mathbb{P}^M \times B$ a closed subscheme whose ideal sheaf is generated by explicitly given bihomogeneous $F_1, \dots, F_k$ of degrees at most m in the coordinates x of $\mathbb{P}^M$, and $p(t)$ a Hilbert polynomial with Gotzmann number r_0. Put $r := \max(r_0, m)$, $N(t) := \binom{M+t}{M}$ and $q(t) := N(t) - p(t)$. Then the relative Hilbert scheme $\mathrm{Hilb}^p(\mathcal{X}/B)$ is the closed subscheme of $\mathrm{Hilb}^p(\mathbb{P}^M) \times B \subset \mathrm{Gr}(q(r), N(r)) \times B$ defined by $x^\alpha F_j(\cdot\,; b) \in \Omega$ for all j and all monomials x^α of degree $r - \deg_x F_j$, where $\Omega \subset \overline{\mathbb{Q}}[x]_r$ is the point of the Grassmannian; and its universal family is cut out in $\mathbb{P}^M \times \mathrm{Hilb}^p(\mathcal{X}/B)$ by Ω.*

Proof. For the absolute case, by Gotzmann's regularity theorem the ideal sheaf of a subscheme $\mathcal{Z} \subset \mathbb{P}^M_A$ with Hilbert polynomial p is r-regular, $I(\mathcal{Z})_{\geq r}$ is generated by $I(\mathcal{Z})_r$, and $\mathcal{Z} \mapsto I(\mathcal{Z})_r$ embeds $\mathrm{Hilb}^p(\mathbb{P}^M)$ in $\mathrm{Gr}(q(r), N(r))$ as a closed subscheme with explicit equations [54, Thms. 1.21, 1.24 and 1.25, (1.22), (1.23)]; Lella states these for $r = r_0$, and they hold for every $r \geq r_0$ with the same proofs, since an r_0-regular sheaf is r-regular. For a T-point $T \to B$ and $\mathcal{Z} \subset \mathbb{P}^M_T$ flat with Hilbert polynomial p, let Ω be the module of sections of $I_{\mathcal{Z}}(r)$, locally a direct summand of rank $q(r)$. If F_j vanishes on $\mathcal{Z}$, then $x^\alpha F_j \in \Omega$. Conversely, if $x^\alpha F_j \in \Omega$ for all $|\alpha| = r - \deg F_j$, then on $D_+(x_i)$ the function $F_j / x_i^{\deg F_j} = x_i^{r - \deg F_j} F_j / x_i^r$ is a section of $I_{\mathcal{Z}}$, so F_j vanishes on $\mathcal{Z}$. Membership of a given vector in a subbundle of rank $q(r)$ is the vanishing of exterior products, a closed condition compatible with base change. □

Let $\mathcal{H}$ be a scheme of finite type over a number field k, $\mathcal{A} \to \mathcal{H}$ an abelian scheme of relative dimension g embedded in $\mathbb{P}(V_\Upsilon) \times \mathcal{H}$, and $\Gamma \subset \mathcal{A} \times_{\mathcal{H}} \mathcal{A}$ closed, flat and proper over $\mathcal{H}$ with fibers Γ_t of dimension g; in our application $\mathcal{H}$ is one of the relative Hilbert schemes $\mathcal{H}_{i,n}$ in the proof of Lemma 10.9, $\mathcal{A}$ the pull-back of the universal abelian scheme over N_0, and Γ the universal subscheme. Let $\pi_t := \mathrm{pr}_1|_{\Gamma_t}\colon \Gamma_t \to \mathcal{A}_t$ and $U_{\mathrm{iso}} := \{t \in \mathcal{H} : \pi_t \text{ is an isomorphism}\}$.

Lemma 10.6. *(a) For $t \in \mathcal{H}$ the following are equivalent: π_t is an isomorphism; π_t is a monomorphism; for every $y \in \mathcal{A}_t$ the fiber $\pi_t^{-1}(y)$ is empty or $\kappa(y)$-isomorphic to* $\operatorname{Spec}\kappa(y)$.

(b) U_{iso} is open in $\mathcal{H}$.

(c) Let $t \in \mathcal{H}$, $\kappa := \kappa(t)$, and $J \subset \kappa[x,y]$ a bihomogeneous ideal whose sheaf is the ideal sheaf of Γ_t in $\mathbb{P}_x \times \mathbb{P}_y$, two copies of $\mathbb{P}(V_\Upsilon)$ with coordinates $x_0,\dots,x_M$ and $y_0,\dots,y_M$; let J' be the same ideal in variables (x,y'). Then π_t is an isomorphism if and only if for every (i,j,l) each minor $y_a y'_b - y_b y'_a$, dehomogenized by $x_i = y_j = y'_l = 1$, lies in the dehomogenization of $J + J' \subset \kappa[x,y,y']$.

Proof. (a) Let π_t be a monomorphism. It is proper, so it is a closed immersion [41, 18.12.6]; its image is closed of dimension g in the integral g-dimensional $\mathcal{A}_t$, hence all of $\mathcal{A}_t$, and as $\mathcal{A}_t$ is reduced, π_t is an isomorphism. The third condition holds for all y if and only if π_t is a closed immersion [41, 18.12.6], hence by the same argument if and only if it is an isomorphism. (b) $\mathrm{pr}_1 \colon \Gamma \to \mathcal{A}$ is proper. Let V_0 be the set of $y \in \mathcal{A}$ over which the fiber is empty or $\kappa(y)$-isomorphic to $\operatorname{Spec}\kappa(y)$. By [41, 18.12.7], $y \in V_0$ if and only if pr_1 is a closed immersion over a neighborhood of y, so V_0 is open; the structure map $\mathcal{A} \to \mathcal{H}$ is proper, so U_{iso}, the complement of the image of $\mathcal{A} \smallsetminus V_0$, is open by (a). (c) As $\mathcal{A}_t \to \mathbb{P}_x$ is a closed immersion, $\Gamma_t \times_{\mathcal{A}_t} \Gamma_t = \Gamma_t \times_{\mathbb{P}_x} \Gamma_t$ is the closed subscheme of $\mathbb{P}_x \times \mathbb{P}_y \times \mathbb{P}_{y'}$ with ideal $J + J'$, and the diagonal is cut out in it by the 2×2 minors of the matrix with rows y and y'. So the diagonal is an isomorphism if and only if these minors vanish on $\Gamma_t \times_{\mathcal{A}_t} \Gamma_t$, which on the charts $D_+(x_i y_j y'_l)$ is the stated membership. A morphism is a monomorphism if and only if its diagonal is an isomorphism [80, Tag 01L3]; conclude by (a). □

On an affine chart $C = V(I_C) \cap D(h_C)$ of the Hilbert base $\mathcal{H} \subset \mathrm{Gr}(q(r), N(r)) \times N_0$ with coordinates u, where I_C includes the Hilbert equations with their original scheme structure, the point $\Omega(t)$ of Lemma 10.5 has a basis $w_1(u),\dots,w_{q(r)}(u)$ with coefficients polynomial in u, and pulling back along the Segre map gives generators $w_s(t)(x \otimes y)$ of the ideal sheaf of Γ_t in $\mathbb{P}_x \times \mathbb{P}_y$, admissible in Lemma 10.6(c) at every $t \in C$, including generic points of closed subsets.

Algorithm 10.7 (U_{iso})**.** (1) For each chart C, compute the minimal primes $\mathfrak{p}$ of I_C not containing h_C, and put each pair $(C,\mathfrak{p})$ on a list. (2) Take a pair $(C,\mathfrak{p})$ off the list, let $\kappa := \mathrm{Frac}(k[u]/\mathfrak{p})$, and run the test of Lemma 10.6(c) at the generic point of $V(\mathfrak{p})$ over κ, by Buchberger's algorithm with cofactors on each of the $(M+1)^3$ charts. (3) If the answer is yes, choose $h \in k[u] \smallsetminus \mathfrak{p}$ that is a common denominator of all coefficients of all cofactors; record $V(\mathfrak{p}) \cap D(h) \cap D(h_C)$, and put on the list the pairs $(C,\mathfrak{p}')$ for the minimal primes $\mathfrak{p}'$ of $\mathfrak{p} + (h)$ not containing h_C. (4) If the answer is no, record nothing. (5) Repeat until the list is empty, and output the union of the recorded sets.

Lemma 10.8. *Algorithm 10.7 terminates, and its output is U_{iso}.*

Proof. A recorded set lies in U_{iso}: after multiplying by a power of h the cofactor identities hold in $(k[u]/\mathfrak{p})_h[x,y,y']$, hence at every $t \in V(\mathfrak{p}) \cap D(h)$, and Lemma 10.6(c) applies. A "no" loses nothing: $U_{\mathrm{iso}} \cap V(\mathfrak{p})$ is open in the irreducible $V(\mathfrak{p}) \cap C$ by Lemma 10.6(b) and misses its generic point, so it is empty. Every $t \in U_{\mathrm{iso}}$ is recorded: it lies on a component $V(\mathfrak{p}_0)$ from step (1), which answered yes; either t is in the recorded set, or t lies on a component of $V(\mathfrak{p}_0 + (h_0))$, of smaller dimension, which again answered yes; and a 0-dimensional component $\{t\}$ that answers yes is recorded. Each pair put on the list has smaller dimension than its

parent, so the algorithm terminates. This is the induction of the proof of [87, Prop. 5.1(iii)], with "reduced fiber" replaced by "π_t is an isomorphism". □

Lemma 10.9. *For every PEL datum $\mathfrak{d}$, the set $\mathfrak{E}(\mathfrak{d}) \cap N_0$ is constructible, and it can be computed from Mumford's equations, N_0 and $\mathfrak{d}$ as a finite union of explicit locally closed subsets of N_0.*

Proof. Let $\mathcal{A} := \overline{\mathcal{A}}_\Upsilon \times_{\overline{M}_\Upsilon} N_0$, the universal abelian scheme over N_0 (Lemma 10.4), and embed $\mathcal{A} \times_{N_0} \mathcal{A}$ in $\mathbb{P}(V_\Upsilon \otimes V_\Upsilon) \times N_0$ by the Segre map, so that $\mathcal{O}(1)$ restricts to $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$; its ideal sheaf is generated by the Segre quadrics and by $F(x;b)y^\beta$, $F(y;b)x^\beta$ ($|\beta| = \deg F$) for Mumford's equations F of $\mathcal{A}$. For each generator e_i and each $n = 1, \dots, 2g+1$, let $\mathcal{H}_{i,n}$ be the relative Hilbert scheme of Lemma 10.5 for the polynomial P_{ne_i}, with universal subscheme $\Gamma_{i,n}$, and let $\mathcal{H}$ be their fiber product over N_0. Impose: (1) pr_1 is an isomorphism on each $\Gamma_{i,n}$ (the open set U_{iso}, computed by Algorithm 10.7), and $(0,0) \in \Gamma_{i,n}$, i.e. every element of Ω vanishes at $s \otimes s$, as the origin of A_s is the point s; so $\Gamma_{i,n}$ is the graph of a homomorphism $f_{i,n}$. (2) $df_{i,n} = n\, df_{i,1}$, the $\mathbb{Z}$-linear relations among the e_i, and the expressions of 1 and of each product e_ie_j as $\mathbb{Z}$-linear combinations of the e_k, hold for the differentials $df_{i,1}$ at the origin, and $\det(T - df_{i,1} \mid \mathrm{Lie}) = \tau(e_i)$; these are closed conditions on the tangent spaces of the graphs at $(0,0)$. In characteristic 0, $f \mapsto df$ is injective on $\mathrm{End}(A_s)$, by complex uniformization; so (2) gives $f_{i,n} = nf_{i,1}$, a ring homomorphism $\iota\colon R \to \mathrm{End}(A_s)$ with $\iota(e_i) = f_{i,1}$, and the type τ, as the traces on $\mathrm{Lie}\, A_s$, additive in r, agree on the e_i, hence on $\mathcal{E}$, and fix the multiplicity of each embedding. (3) By Equation (10.1), the Hilbert polynomials give $\det(1 + n^2 f^\dagger f) = \det_{W'}(1 + n^2 e^* e)$ for $f = f_{i,1}$, $e = e_i$ and $2g+1$ values of n^2, so $f^\dagger f$ and $e^* e$ have the same characteristic polynomial. Hence $\mathrm{Tr}(f^\dagger f) = \sum |\xi|^2$ over the eigenvalues ξ of f, which is the case of equality in Schur's inequality for the positive definite Rosati form; so f is normal, and $f^\dagger = \iota(e_i^*)$, as both act by $\bar{\xi}$ on the ξ-eigenspace of f. Conversely every $s \in \mathfrak{E}(\mathfrak{d}) \cap N_0$ with its ι gives such graphs. So $\mathfrak{E}(\mathfrak{d}) \cap N_0$ is the image in N_0 of an explicit finite union of locally closed subsets of $\mathcal{H}$, and the image is constructible and computable [7, Thm. 1.0], on finitely many affine charts. □

10.4. **Containers.** We use Urbanik's algorithm [87], with his definitions. Let $\widetilde{S}$ be a smooth quasi-projective variety carrying a polarizable integral variation of Hodge structure $\mathbb{V}$. The algebraic monodromy group H_Y of an irreducible subvariety Y (not the Mumford–Tate group of Section 2.2) is the identity component of the Zariski closure of the monodromy of $\mathbb{V}$ on the normalization of Y; Y is *weakly special* if it is maximal among irreducible subvarieties with its algebraic monodromy group [87, Defs. 1.3–1.4]; and Y is *rigid* if there is no weakly special Y' with $Y \subsetneq Y' \subset \widetilde{S}$ such that H_Y is normal in $H_{Y'}$ [87, Def. 4.28]. Special subvarieties are weakly special. For a projective compactification of $\widetilde{S}$, a very ample line bundle on it and an integer d', [87, Cor. 6.3] (with the remark below) gives an algorithm that computes a finite set of weakly special subvarieties of degree at most d' containing the rigid ones, from equations for $\widetilde{S}$ and the data $(\mathcal{H}, F^\bullet, \nabla)$ of the variation [87, Thm. 2.4].

Remark. The stopping test in the proof of [87, Prop. 5.14] needs one more condition. There, for a component $\mathcal{T}$ of $\mathrm{Var}(\widetilde{S})$ and the distinct types $\mathcal{C}(V_{i_1}), \dots, \mathcal{C}(V_{i_p})$ of dimension ℓ among $\mathcal{C}(V_0), \dots, \mathcal{C}(V_k)$ (repeats removed), the loci $W_{r,i_j} \subset \mathcal{T}$ cut out by the constraints of order at most r are computed until (i) they are closed and (ii) of their components of generic type dimension ℓ for different j, no one contains another. The general linear group is transitive on the flag variety $\check{L}$, so $W_{0,i} = \mathcal{T}$ for every i, and (i) and (ii) hold at $r = 0$, with nothing returned, whenever $\dim \mathcal{G}(\mathcal{T}) \neq \ell$; but $\mathcal{T}$ can have optimal components of

smaller type dimension (for $H^1 \oplus H^1$ on $Y(3) \times Y(3)$, the diagonal among the $(1,1)$-curves). Add (iii): every component of every W_{r,i_j} has generic type of dimension at most ℓ. This is decidable [87, Lemma 5.8] and holds, with (i) and (ii), once $W_{r,i_j} = \mathcal{T} \cap W(\mathcal{C}(V_{i_j}))$ for all j. Then a component A of W_{r,i_j} with $\dim \mathcal{G}(A) = \ell$ lies in the closed locus $W(\mathcal{G}(A))$ [87, Prop. 4.31], so in a component A' of $W_{r,i_{j'}}$ with $\mathcal{C}(V_{i_{j'}}) = \mathcal{G}(A)$; $\dim \mathcal{G}(A') \geq \dim \mathcal{G}(A) = \ell$, from $A \subset A' \subset W(\mathcal{G}(A'))$, and $\dim \mathcal{G}(A') \leq \ell$ by (iii), so $j' = j$ by (ii), and A is an optimal component of $\mathcal{T} \cap W(\mathcal{C}(V_{i_j}))$. Conversely, an optimal component lies in a component of W_{r,i_j} whose generic type has dimension ℓ by (iii), which is therefore optimal and equal to it. So [87, Prop. 5.14, Thm. 5.15] hold as stated. Let $\mathcal{W} \subset \mathrm{Var}(\widetilde{S})$ be the locus of weakly special subvarieties [87, §4.8], and $\mathrm{Var}(\widetilde{S})_{d'} \subset \mathrm{Var}(\widetilde{S})$ the subscheme parametrizing subvarieties whose closures have degree at most d' [87, Lemma 5.10]. The proof of [87, Cor. 6.3] takes the isolated points of $\mathcal{W} \cap \mathrm{Var}(\widetilde{S})_{d'}$ by [87, Thm. 6.1(ii)]; the proof of that theorem shows that a rigid weakly special Y is an isolated point of $W(\mathcal{C}(Y))$, and isolation in $\mathcal{W}$ itself can fail on a general base. Taking instead, for each type, the isolated points of the locus of weakly special subvarieties of that type and of degree at most d', which the algorithm of [87, Thm. 5.15] computes type by type, gives a finite set with the properties stated in [87, Cor. 6.3]. (On the bases $\Gamma\backslash X^+$ with Γ neat used below, every weakly special subvariety has the dimension of its type, and isolation in $\mathcal{W}$ holds.) We use only these statements.

A maximal special subvariety of Z need not be rigid in S: a special curve $\{x\} \times Y(1)$ in $\mathcal{A}_2$, with x a CM point, moves in the family $\{x'\} \times Y(1)$. We make it rigid by passing to the normalization of a component of the PEL locus defined by the connected center of its generic Mumford–Tate group (Lemma 10.10).

Lemma 10.10. *Let $V \subset S$ be special, the image of $X_H^+ \times \{\gamma\}$ for a Shimura subdatum (H, X_H) with H the generic Mumford–Tate group, let T be the connected center of H, $\mathcal{E} := \mathbb{Q}[T]$, and $M := Z_G(T) = Z_G(\mathcal{E})$.*

(a) For $x \in X_H$, $h_x(\mathbb{S}) \subset M_{\mathbb{R}}$ and $T \subset Z(M)$. Let X_M^+ be the component of $M(\mathbb{R})h_x$ containing X_H^+, and S_V the image of $X_M^+ \times \{\gamma\}$. Then S_V is special and $V \subset S_V$. If H' is the generic Mumford–Tate group on X_M^+, then $T \subset Z(H')$, $X_{H'}^+ = X_M^+$, and S_V is the image of $X_{H'}^+ \times \{\gamma\}$.

(b) For every PEL datum $\mathfrak{d}$, $\mathfrak{E}(\mathfrak{d})$ is closed in S, and each of its irreducible components is a special subvariety of a dimension $\dim \mathfrak{d}$ determined by $\mathfrak{d}$. S_V is an irreducible component of $\mathfrak{E}(\mathfrak{d}_V)$.

(c) Let $K_{H'} := \gamma K_\Upsilon \gamma^{-1} \cap H'(\mathbb{A}_f)$, $\widetilde{S}_V$ the image of $X_{H'}^+ \times \{1\}$ in $\mathrm{Sh}_{K_{H'}}(H', X_{H'})_{\mathbb{C}}$, and $\nu\colon \widetilde{S}_V \to S_V$ the map induced by $[x,h] \mapsto [x, h\gamma]$. Then $\widetilde{S}_V$ is smooth and ν is the normalization. The image W_V of $X_H^+ \times \{1\}$ in $\widetilde{S}_V$ is special, $\nu(W_V) = V$, and $\nu|_{W_V}$ is birational onto V. The first two sentences hold for every special $S' \subset S$ in place of S_V, with its generic Mumford–Tate group and its γ' in place of H' and γ; this covers every irreducible component of every $\mathfrak{E}(\mathfrak{d})$.

(d) W_V is rigid in $\widetilde{S}_V$, for the variation of Hodge structure $\nu^*\mathbb{V}$ given by W.

(e) For every very ample line bundle $\mathcal{L}'$ on a projective compactification of $\widetilde{S}_V$ and every integer $d' \geq \deg_{\mathcal{L}'} \overline{W}_V$, the point $[W_V]$ lies in the finite set that [87, Cor. 6.3] computes for $(\widetilde{S}_V, \nu^*\mathbb{V}, d')$. If $\mathcal{L}' = \nu^*\mathcal{O}(1)$ for a projective embedding of $\overline{S}_V$, then $\deg_{\mathcal{L}'} \overline{W}_V = \deg_{\mathcal{O}(1)} \overline{V}$.

We call S_V the *container* of V.

Proof. (a) $h_x(\mathbb{S}) \subset H_{\mathbb{R}}$ and T is central in H, so $h_x(\mathbb{S}) \subset M_{\mathbb{R}}$; M is reductive, being the centralizer of a torus, and $T \subset Z(M)$. $X_H^+ \subset M(\mathbb{R})h_x$ is connected, so it lies in one component X_M^+, and $V \subset S_V$. For $y \in X_H$, $(M, M(\mathbb{R})h_y)$ satisfies (2.1.1.1) and (2.1.1.2) of [25], as $\operatorname{Lie} M$ is a sub-Hodge structure of $\operatorname{Lie} G$ and $\psi(\cdot, h_y(i)\cdot)$ is definite, but (2.1.1.3) can fail (for $A_y \cong A_0^2$ with A_0 a CM elliptic curve, M^{ad} is compact); so h_y projects trivially to the $\mathbb{Q}$-simple factors of M^{ad} with compact real points [25, 2.1.1] and factors through $M_1 := Z(M)^\circ M_{\mathrm{nc}}$, with M_{nc} the product of the other $\mathbb{Q}$-simple factors of M^{der}. As $Z(M)^\circ$ and the compact factors centralize h_y, $X_M^+ = M_1(\mathbb{R})^+ h_y$ for $y \in X_H^+$, a component of the Shimura subdatum $(M_1, M_1(\mathbb{R})h_y)$, which satisfies (2.1.1.3) because a $\mathbb{Q}$-factor of M_1^{ad} on which h_y projects trivially has compact real points by (2.1.1.2); so S_V is special. By [85, Lemma 2.1], S_V is the image of $X_{H'}^+ \times \{\gamma\}$, and $X_{H'}^+ = X_M^+$, both being connected complex submanifolds of the same dimension, the first closed in the second. As $\mathrm{MT}(y) \subset H'$ for $y \in X_H$, $H \subset H'$ and $T \subset Z(M) \cap H' \subset Z(H')$.

(b) Let $s = [x', \gamma'] \in \mathfrak{E}(\mathfrak{d})$ with ι as in the definition. Then $\iota \otimes \mathbb{Q}$ embeds $\mathcal{E}$ in $\mathrm{End}(W)$, compatibly with ψ-adjoints, and commutes with $h_{x'}$. Let $M_\iota := Z_G(\iota(\mathcal{E}))$, the group of $\iota(\mathcal{E})$-linear symplectic similitudes, which is connected and reductive ([56, Prop. 8.7], on each factor of $\mathcal{E}$) and contains $h_{x'}(\mathbb{S})$. The image $S_{\iota,s}$ of the component of $M_\iota(\mathbb{R})h_{x'}$ through x', times $\{\gamma'\}$, is special as in (a), contains s, and lies in $\mathfrak{E}(\mathfrak{d})$, since the lattice and $\iota(R)$ do not change along it and τ is locally constant. Its dimension is that of the symmetric space of $M_{\iota,\mathbb{R}}$, which is determined by $\mathfrak{d}$: the ψ-orthogonal decomposition $W_{\mathbb{R}} = \bigoplus_v W_v$ over the places of the factors of $\mathcal{E}$ gives a Siegel space of dimension $r(r+1)/2$ for a real place of a totally real factor, with $2r = \dim_{\mathbb{R}} W_v$, and a unitary symmetric space of dimension $a_v(m_i - a_v)$ for a place v of F_i^+ with F_i CM, as $\dim_{\mathbb{C}} W_v = m_i$; call the sum $\dim \mathfrak{d}$. Now $\mathfrak{E}(\mathfrak{d})$ is constructible (Lemma 10.9; here $N_0 = M_\Upsilon \otimes \overline{\mathbb{Q}}$ by Lemma 10.2(ii) and Lemma 10.4(iii)) and is the union of the countably many $S_{\iota,s}$. Let C be an irreducible component of its Zariski closure. Then $\mathfrak{E}(\mathfrak{d}) \cap C$ contains a dense open subset of C, which is a countable union of the closed sets $S_{\iota,s} \cap C$; by Baire's theorem one of them has interior, so $C \subset S_{\iota,s} \subset \mathfrak{E}(\mathfrak{d})$ and $C = S_{\iota,s}$. So $\mathfrak{E}(\mathfrak{d})$ is closed and its components have dimension $\dim \mathfrak{d}$. Finally $S_V = S_{\iota_V,x}$ for the inclusion ι_V of R_V, so S_V is irreducible of dimension $\dim \mathfrak{d}_V$ in $\mathfrak{E}(\mathfrak{d}_V)$, and the component containing it is S_V.

(c) $\widetilde{S}_V = \Gamma_{H'} \backslash X_{H'}^+$ with $\Gamma_{H'} := \{q \in H'(\mathbb{Q}) : qX_{H'}^+ = X_{H'}^+\} \cap \gamma K_\Upsilon \gamma^{-1}$, which is neat by Lemma 10.2(iii); so $\widetilde{S}_V$ is smooth. The map ν is finite, and it is injective on the Hodge generic locus by [85, Lemma 2.2]. That lemma assumes $Z(G)(\mathbb{R})$ compact; its proof uses this only to show that $N_G(H')/H'$ has compact real points, from the compactness of the stabilizer of a point of X in $G(\mathbb{R})$. For $G = \mathrm{GSp}$ this stabilizer is compact modulo $Z(G)(\mathbb{R}) = \mathbb{R}^\times$, and $Z(G) \subset H'$, so $N_G(H')/H'$ still has compact real points. Ullmo and Yafaev assume K neat. Here we use instead that two Hodge generic points of $X_{H'}^+$ with the same image in S differ by some $q \in N_G(H')(\mathbb{Q}) \cap G(\mathbb{Q})_+ \cap \gamma K_\Upsilon \gamma^{-1}$. As the similitude of q is in $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^\times = \{1\}$, q lies in $N_G(H')(\mathbb{A}_f) \cap \gamma K(d_\Upsilon)\gamma^{-1}$ (Lemma 10.2(i)), a compact neat group, whose image K'' in $\overline{N}(\mathbb{A}_f)$, $\overline{N} := N_G(H')/H'$, is compact and neat [68, 0.6]. So the image of q lies in $\overline{N}(\mathbb{Q}) \cap K''$, which is neat [68, 0.6] and discrete in the compact $\overline{N}(\mathbb{R})$, hence finite and trivial, so $q \in H'(\mathbb{Q}) \cap \gamma K_\Upsilon \gamma^{-1}$ preserves $X_{H'}^+$ and the two points agree in $\widetilde{S}_V$. A finite morphism from a normal variety that has degree 1 onto its image is the normalization. The same argument for $(H, X_H) \subset (G, X)$ shows that the image of $X_H^+ \times \{\gamma\}$ in S is reached generically injectively; as that map factors through $W_V \to V$, $\nu|_{W_V}$ is birational. W_V is special in $\widetilde{S}_V$, defined by $(H, X_H) \subset (H', X_{H'})$. For a special $S' \subset S$ in place of S_V the first

two sentences of (c) have the same proof, and by (b) every irreducible component of every $\mathfrak{E}(\mathfrak{d})$ is such an S'.

(d) Suppose that Y is weakly special in $\widetilde{S}_V$ with $W_V \subsetneq Y$ and H_{W_V} normal in H_Y. For a special subvariety defined by (H, X_H) of a quotient $\Gamma\backslash X$ with Γ neat and the variation of a faithful representation, the algebraic monodromy group is H^{der} (proof of [87, Cor. 6.2]); so $H_{W_V} = H^{\mathrm{der}}$. Algebraic monodromy groups are semisimple (proof of [87, Lemma 4.29]), so $H_Y = H^{\mathrm{der}}H''$ with H'' the product of the other $\mathbb{Q}$-simple factors, which commutes with H^{der}. As H_Y is normal in $\mathrm{MT}(Y) \subset H'$, $H'' \subset Z_{H'}(H^{\mathrm{der}})$, which also centralizes $T \subset Z(H')$, hence centralizes $H = T \cdot H^{\mathrm{der}}$ and $h_x(\mathbb{S})$ for $x \in X_H$. Being semisimple, $H'' \subset \mathrm{Sp}(W, \psi)$; it fixes $h_x(i)$, so $H''(\mathbb{R})$ preserves the positive definite form $\psi(\cdot, h_x(i)\cdot)$ and is compact. By [87, Lemma 4.11] an algebraic monodromy group has no $\mathbb{Q}$-simple factor with compact real points, so $H'' = 1$ and $H_Y = H_{W_V}$. Then $Y \supsetneq W_V$ has the algebraic monodromy group of the weakly special W_V, which contradicts the maximality in the definition. So W_V is rigid.

(e) $\widetilde{S}_V$ is smooth and $\nu^*\mathbb{V}$ is a polarizable integral variation, so [87, Cor. 6.3] applies, and $[W_V]$ is in the output by (d). The projection formula and (c) give $\deg_{\nu^*\mathcal{O}(1)} \overline{W}_V = \deg_{\mathcal{O}(1)} \overline{V}$. □

A special point Q has $S_Q = \{Q\}$: h_x factors through $T \subset Z(M)$, so $M(\mathbb{R})h_x = \{h_x\}$. For the curve $\{x\} \times Y(1) \subset \mathcal{A}_2$ above, with x a CM point with CM by k, $\mathcal{E} = k \times \mathbb{Q}$ and $M = H$, so $S_V = V$; the curve is its own container and is rigid there.

10.5. **Degrees in Mumford's coordinates.** Urbanik's degrees are taken for $\mathcal{O}(1)$ on the closure in $\mathbb{P}(V_\Upsilon)$, ours for the Baily–Borel bundle. Theta constants have weight $1/2$, so the first is the smaller.

Lemma 10.11. *Let $g \geq 2$, $\varphi\colon S \to \mathbb{P}(V_\Upsilon)$ the theta-null immersion and $\mathcal{L} := \mathcal{L}_{K_\Upsilon}$. For every closed irreducible $Y \subset S$ of dimension r, with closure $\overline{Y}$ in $\mathbb{P}(V_\Upsilon)$,*

$$\deg_{\mathcal{O}(1)} \overline{Y} \leq \deg_{\mathcal{L}} Y/(2(g+1))^r \leq \deg_{\mathcal{L}} Y.$$

Proof. Let ω be the determinant of the Hodge bundle on S and $N_\Upsilon := d^g_\Upsilon$, the rank of the push-forward p_*L of the marking sheaf along the universal family p. Mumford shows that $p_*L \cong V_\Upsilon \otimes \mathcal{K}$ for the invertible sheaf $\mathcal{K} := \varphi^*\mathcal{O}(-1)$ [61, pp. 80–82], so $\det p_*L \cong \mathcal{K}^{\otimes N_\Upsilon}$. For a relatively ample symmetric line bundle trivialized along the zero section, with p_* of rank N_Υ, Faltings and Chai prove $8N_\Upsilon^3 \det(p_*L) = -4N_\Upsilon^4\,\omega$ in $\mathrm{Pic}(S)$ [29, Thm. I.5.1], as stated in [71, Introduction]; Mumford's L is totally symmetric, relatively ample and normalized. Hence $\varphi^*\mathcal{O}(1)^{\otimes 8N_\Upsilon^4} \cong \omega^{\otimes 4N_\Upsilon^4}$. Also $\mathcal{L}|_S = \omega^{\otimes(g+1)}$ [47, §II.1], and as $g \geq 2$ the boundary of the Baily–Borel compactification S^{BB} has codimension at least 2, S^{BB} is normal and $\mathcal{L}$ is a line bundle on it (Lemma 10.2(iii), [62, Prop. 3.4(b)]), so sections of its powers over S extend to S^{BB}.

Put $e := 4N_\Upsilon^4$. Monomials of degree $2e(g+1)$ in the coordinates of $\mathbb{P}(V_\Upsilon)$ are therefore sections of $\mathcal{L}^{\otimes e}$ over S^{BB}, and $v \circ \varphi$, with v the Veronese map of degree $2e(g+1)$, is the rational map of S^{BB} given by a linear subsystem of $|\mathcal{L}^{\otimes e}|$. Let $\overline{Y}^{\mathrm{BB}}$ be the closure of Y in S^{BB} and $\varpi\colon \widetilde{Y} \to \overline{Y}^{\mathrm{BB}}$ a resolution of the base locus, with $\varpi^*\mathcal{L}^{\otimes e} = \mathcal{M}' + F'$, $\mathcal{M}'$ base-point free and F' effective. As $\varpi^*\mathcal{L}^{\otimes e}$ is nef, $(\varpi^*\mathcal{L}^{\otimes e})^r - \mathcal{M}'^r = \sum_{j<r} F' \cdot (\varpi^*\mathcal{L}^{\otimes e})^j \cdot \mathcal{M}'^{r-1-j} \geq 0$. As φ is injective, $\widetilde{Y} \to v(\overline{Y})$ is birational, and $(2e(g+1))^r \deg_{\mathcal{O}(1)} \overline{Y} = \deg v(\overline{Y}) = \mathcal{M}'^r \leq e^r \deg_{\mathcal{L}} Y$. □

10.6. **Normalization.** Let k be a number field, $\mathcal{N} \subset \mathbb{P}^{M'}_k$ an integral closed subscheme with homogeneous prime ideal $I \subset R' := k[x_0, \dots, x_{M'}]$, $A := R'/I$, and $\bar{A}$ the integral closure of A in its fraction field. The algorithm of Greuel, Laplagne and Seelisch [39, Algorithm 3 and Remark 4.6] (in its notation, with the stopping test $U_{i+1} = pU_i$) returns $d_A \in R' \smallsetminus I$ and $u_1, \dots, u_s \in R'$ with $\bar{A} = d_A^{-1}(u_1, \dots, u_s)$ (images in A); it does not say that these are homogeneous. The following lemma makes the graded and Veronese steps explicit.

Lemma 10.12. *Fix $x \in A_1 \smallsetminus \{0\}$ and let $\widetilde{B}$ be the integral closure of $A_{(x)} := (A_x)_0$.*

(a) $\bar{A} \subset \widetilde{B}[x, x^{-1}]$, and $\bar{A} = \bigoplus_{n \geq 0} \bar{A}_n$ is a graded subring, with x of degree 1 and $\widetilde{B}$ of degree 0.

(b) $\operatorname{Proj} \bar{A} \to \mathcal{N}$ is finite, and $\operatorname{Proj} \bar{A}$ is the normalization $\widetilde{\mathcal{N}}$ of $\mathcal{N}$.

(c) Let c be a homogeneous component of d_A with $c \notin I$, and $a_j \in R'$ with $d_A a_j \equiv c u_j \bmod I$. Then the homogeneous components $a_{j,m}$ of the a_j give elements $a_{j,m}/c \in \bar{A}$ of degree $m - \deg c$, which with 1 generate $\bar{A}$ as an A-module.

(d) Let $b_1, \dots, b_\mu$ be the elements of positive degree among $x_0, \dots, x_{M'}$ and the generators of (c), of degrees $d_1, \dots, d_\mu$, and $h := \mu \operatorname{lcm}(d_1, \dots, d_\mu)$. Then $\bar{A}_{nh} = (\bar{A}_h)^n$ for all $n \geq 1$.

(e) With a basis $T_1, \dots, T_{M''}$ of $\bar{A}_h$ and $\mathfrak{a}$ the kernel of $k[T] \to \bar{A}$, $\widetilde{\mathcal{N}} \cong \operatorname{Proj}(k[T]/\mathfrak{a}) \subset \mathbb{P}^{M''-1}_k$, with explicit equations.

(f) This embedding of $\widetilde{\mathcal{N}}$ is given by $\nu^*\mathcal{O}_{\mathcal{N}}(h)$, $\nu \colon \widetilde{\mathcal{N}} \to \mathcal{N}$ the normalization.

Proof. (a) $A_x = A_{(x)}[x, x^{-1}]$, and integral closure commutes with localization [80, Tag 0307], so $\bar{A}_x$ is the integral closure of $A_{(x)}[x, x^{-1}]$, which is $\widetilde{B}[x, x^{-1}]$, as this ring is integral over it and normal [80, Tags 030A, 00GY]. For $\lambda \in k^\times$ the automorphism $x \mapsto \lambda x$ of $\widetilde{B}[x, x^{-1}]$ preserves A, hence $\bar{A}$; applying it for enough distinct λ and inverting a Vandermonde matrix shows that each homogeneous component of an element of $\bar{A}$ lies in $\bar{A}$. A homogeneous $b \in \bar{A}$ of negative degree $-e'$ satisfies $b^n + \sum_i (a_i)_{-ie'} b^{n-i} = 0$, the degree $-ne'$ part of an integral equation, with $(a_i)_{-ie'} \in A_{-ie'} = 0$; so $b = 0$.

(b) If a homogeneous prime $\mathfrak{q}$ of $\bar{A}$ contains A_1, then it contains every homogeneous element of positive degree, by the degree ne' part of an integral equation; so the $D_+(x)$, $x \in A_1 \smallsetminus \{0\}$, cover $\operatorname{Proj} \bar{A}$. On $D_+(x)$ the ring of sections is $(\bar{A}_x)_0 = \widetilde{B}$, the normalization of the chart $\operatorname{Spec} A_{(x)}$; these identifications glue, and $\widetilde{B}$ is finite over $A_{(x)}$ as $\bar{A}$ is finite over A.

(c) The conductor $\{a \in A : a\bar{A} \subset A\}$ contains d_A and is homogeneous, as A is a graded subring of $\widetilde{B}[x, x^{-1}]$; so it contains c, which gives a_j, and $c\bar{A}$ is a homogeneous ideal of A containing a_j, hence the $a_{j,m}$. As $u_j/d_A = a_j/c = \sum_m a_{j,m}/c$, these elements generate.

(d) $\bar{A}_{nh}$ is spanned by products $\beta \prod_i b_i^{\alpha_i}$ with $\beta \in \bar{A}_0$ and $\sum_i \alpha_i d_i = nh$. Put $\ell := \operatorname{lcm}(d_i)$ and $\alpha_i = q_i \ell/d_i + r_i$ with $0 \leq r_i < \ell/d_i$. The blocks b_i^{ℓ/d_i} have degree ℓ, and $\prod_i b_i^{r_i}$ has degree $j\ell$ with $0 \leq j \leq \mu - 1$; there are $n\mu - j$ blocks. Grouping $\beta \prod_i b_i^{r_i}$ with $\mu - j$ blocks, and the rest in $n - 1$ groups of μ blocks, writes the product as a product of n elements of $\bar{A}_h$.

(e) $\bar{A}_h$ is finite-dimensional, and by (d) the subalgebra $k[\bar{A}_h]$ agrees with $\bar{A}^{(h)}$ in positive degrees, so $\operatorname{Proj} k[\bar{A}_h] \cong \operatorname{Proj} \bar{A}^{(h)} \cong \operatorname{Proj} \bar{A}$ [40, Prop. 2.4.7], and $k[\bar{A}_h] = k[T]/\mathfrak{a}$ is generated in degree 1.

(f) On $D_+(x)$ both $\mathcal{O}_{\operatorname{Proj}\bar{A}}(1)$ and $\nu^*\mathcal{O}_{\mathcal{N}}(1)$ are free with generator x and transition functions x/x', so they agree; and $\mathcal{O}(1)$ on $\operatorname{Proj} k[\bar{A}_h]$ is generated by x^h on $D_+(x^h)$, with transition functions $(x/x')^h$. □

10.7. **Rigidity and monodromy.** In this subsection $\widetilde{S} = \Gamma\backslash X^+$ is a connected Shimura variety with Γ neat, $\mathbb{V}$ the variation of a faithful representation, and $G_{\widetilde{S}}$ the generic Mumford–Tate group of $\widetilde{S}$. For $Y \subset \widetilde{S}$ weakly special and $x \in Y(\overline{\mathbb{Q}})$, let $\mathfrak{h}_Y$ be the Lie algebra of H_Y in $\mathrm{End}(\mathbb{V}_x) \otimes \overline{\mathbb{Q}}$, $F^\bullet$ the Hodge filtration of $\mathbb{V}_x$, $F^j\mathrm{End}(\mathbb{V}_x) := \{\phi : \phi(F^p) \subset F^{p+j} \text{ for all } p\}$, and for a subspace $\mathfrak{n}$ put $t(\mathfrak{n}) := \dim \mathfrak{n} - \dim(\mathfrak{n} \cap F^0\mathrm{End}(\mathbb{V}_x))$.

Lemma 10.13. *Let $Y \subset \widetilde{S}$ be weakly special, x a smooth point of Y, and $\mathfrak{z}$ the centralizer of $\mathfrak{h}_Y$ in $\mathfrak{h}_{\widetilde{S}}$. Then Y is rigid if and only if $\mathfrak{z} \cap F^1\mathrm{End}(\mathbb{V}_x) = 0$. If Y is rigid, it is special.*

Proof. $\mathrm{MT}(x) \subset \mathrm{MT}(Y)$ normalizes H_Y and $H_{\widetilde{S}}$, which are normal in $\mathrm{MT}(Y)$ and $G_{\widetilde{S}}$ [4, Thm. 1 and its proof], so $\mathfrak{z}$ is a sub-Hodge structure of $\mathfrak{h}_{\widetilde{S}}$ of types $(-1,1)$, $(0,0)$, $(1,-1)$, and $\mathfrak{z} \cap F^1 = \mathfrak{z}^{1,-1}$ is the complex conjugate of $\mathfrak{z}^{-1,1}$. If $\mathfrak{z}^{-1,1} = 0$ and $Y' \supsetneq Y$ is weakly special with H_Y normal in $H_{Y'}$, then $H_{Y'} = H_Y H''$ with $\mathrm{Lie}\, H'' \subset \mathfrak{z}$, both summands stable under $\mathrm{Ad}\, h_x$, and Y' is the image of $H_{Y'}(\mathbb{R})^+ \cdot \tilde{x}$ [87, Prop. 4.8, Lemma 4.10(ii)], of dimension $\dim \mathfrak{h}_{Y'}^{-1,1} = \dim \mathfrak{h}_Y^{-1,1} = \dim Y$, a contradiction. If $\mathfrak{z}^{-1,1} \neq 0$, then $N' := Z_{G_{\widetilde{S}}}(H_Y)^\circ$ is reductive and normalized by $\mathrm{MT}(Y)$, so h_x factors through the reductive group $\mathrm{MT}(Y)N'$, which defines a special $Y' \supset Y$ whose tangent space contains $\mathfrak{h}_Y^{-1,1} \oplus \mathfrak{z}^{-1,1}$; so $Y' \supsetneq Y$, and H_Y is normal in $(\mathrm{MT}(Y)N')^{\mathrm{der}} \supset H_{Y'}$, so Y is not rigid. Finally, if Y is rigid, let Y^{sp} be the smallest special subvariety containing Y, the image of $\mathrm{MT}(Y)(\mathbb{R})^+ \cdot \tilde{x}$ for x Hodge generic in Y; H_Y is normal in $\mathrm{MT}(Y)^{\mathrm{der}} \supset H_{Y^{\mathrm{sp}}}$, so $Y^{\mathrm{sp}} = Y$. □

The test is linear algebra over $\overline{\mathbb{Q}}$ once $\mathfrak{h}_Y$, $\mathfrak{h}_{\widetilde{S}}$ and $F^\bullet$ are known in one frame at one point; F^1 is algebraic, unlike the Hodge decomposition. The group H_Y is computed on a curve, and the curve is certified by the following test.

Lemma 10.14. *Let $Y \subset \widetilde{S}$ be weakly special of dimension r, $C \subset Y^{\mathrm{sm}}$ an irreducible curve and $x \in C$. Then*

(a) $\mathfrak{h}_C \subset \mathfrak{h}_Y$, both are stable under $\mathrm{Ad}\, h_x(\mathbb{S})$, and $t(\mathfrak{h}_C) = \dim \mathfrak{h}_C^{-1,1}$, $t(\mathfrak{h}_Y) = \dim \mathfrak{h}_Y^{-1,1}$;
(b) $t(\mathfrak{h}_Y) = r$;
(c) $t(\mathfrak{h}_C) = r$ if and only if $\mathfrak{h}_C = \mathfrak{h}_Y$.

Proof. (a) Loops in C are loops in Y^{sm}, and algebraic monodromy does not change when Y is replaced by an open subvariety [87, proof of Lemma 4.11], so $H_C \subset H_Y$. A tensor that is Hodge at a point outside the Hodge locus of a subvariety is Hodge everywhere on it, so $\mathrm{MT}(x) \subset \mathrm{MT}(C) \subset \mathrm{MT}(Y)$; H_C and H_Y are normal in $\mathrm{MT}(C)$ and $\mathrm{MT}(Y)$ [4, Thm. 1 and its proof], so $h_x(\mathbb{S})$ normalizes both, and $\mathfrak{h}_C$, $\mathfrak{h}_Y$ are sub-Hodge structures of $\mathrm{End}(\mathbb{V}_x)$ of types $(-1,1)$, $(0,0)$, $(1,-1)$. For such $\mathfrak{n}$, $\mathfrak{n}_{\mathbb{C}} \cap F^0 = \mathfrak{n}^{0,0} \oplus \mathfrak{n}^{1,-1}$. (b) Y is the image of $H_Y(\mathbb{R})^+ \cdot \tilde{x}$ [87, Prop. 4.8, Lemma 4.10(ii)], whose tangent space at $\tilde{x}$ is $\mathfrak{h}_Y^{-1,1}$ [87, proof of Lemma 4.9]. (c) If $t(\mathfrak{h}_C) = r$, then $\mathfrak{h}_C^{-1,1} = \mathfrak{h}_Y^{-1,1}$ by (a) and (b), and $\mathfrak{h}_C^{1,-1} = \mathfrak{h}_Y^{1,-1}$ by conjugation. Let $\mathfrak{s}$ be a non-compact $\mathbb{R}$-simple ideal of $\mathfrak{h}_{Y,\mathbb{R}}$. The connected group $h_x(\mathbb{S})$ fixes each simple ideal, so $\mathfrak{s}_{\mathbb{C}} = \mathfrak{s}^{-1,1} \oplus \mathfrak{s}^{0,0} \oplus \mathfrak{s}^{1,-1}$; put $\mathfrak{p} := \mathfrak{s}^{-1,1} \oplus \mathfrak{s}^{1,-1}$. If $\mathfrak{p} = 0$, $\mathfrak{s}$ commutes with $h_x(i)$ and, as $H_Y \subset \mathrm{Sp}(W, \psi)$, preserves a positive definite form, so it is compact; hence $\mathfrak{p} \neq 0$. As $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{s}^{0,0}$ and $[\mathfrak{s}^{0,0}, \mathfrak{p}] \subset \mathfrak{p}$, $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ is a non-zero ideal of $\mathfrak{s}_{\mathbb{C}}$ stable under conjugation, so it is $\mathfrak{s}_{\mathbb{C}}$; and $\mathfrak{p} \subset \mathfrak{h}_{C,\mathbb{C}}$, so $\mathfrak{s}_{\mathbb{C}} \subset \mathfrak{h}_{C,\mathbb{C}}$. Each $\mathbb{Q}$-simple ideal $\mathfrak{h}_i$ of the semisimple $\mathfrak{h}_Y$ has a non-compact $\mathbb{R}$-simple ideal [87, Lemma 4.11], so $\mathfrak{h}_C \otimes \overline{\mathbb{Q}}$ contains a $\overline{\mathbb{Q}}$-simple ideal of $\mathfrak{h}_i \otimes \overline{\mathbb{Q}}$, hence all of them, as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ permutes them transitively and preserves $\mathfrak{h}_C \otimes \overline{\mathbb{Q}}$. So $\mathfrak{h}_Y \subset \mathfrak{h}_C$. □

Proposition 10.15. *Let $\widetilde{S}$ be the normalization of an irreducible component S' of some $\mathfrak{E}(\mathfrak{d})$, as in Lemma 10.10(c) (for instance $\widetilde{S}_V$), embedded by Lemma 10.12, and $Y \subset \widetilde{S}$ weakly special of dimension $r \geq 1$ and defined over $\overline{\mathbb{Q}}$. The following procedure terminates and returns an irreducible curve $C \subset Y^{\mathrm{sm}}$, a point $x \in C(\overline{\mathbb{Q}})$, and $\mathfrak{h}_Y$, $\mathfrak{h}_{\widetilde{S}}$ and $F^\bullet$ as subspaces of $\mathrm{End}(H^1_{\mathrm{dR}}(A_x)) \otimes \overline{\mathbb{Q}}$ in one frame. If $r = 1$, put $C := Y^{\mathrm{sm}}$. If $r \geq 2$, enumerate the $(r-1)$-tuples of hyperplanes defined over $\mathbb{Q}(i)$ by increasing height; for each, let C be the intersection of Y^{sm} with the hyperplanes, and pass to the next tuple unless C is a non-empty smooth geometrically irreducible curve; choose $x \in C(\overline{\mathbb{Q}})$ outside the finitely many points excluded below, compute $\mathfrak{h}_C$ at x, and accept C if and only if $t(\mathfrak{h}_C) = r$. On acceptance $\mathfrak{h}_Y := \mathfrak{h}_C$.*

Proof. Computing $\mathfrak{h}_C$. The Gauss–Manin connection on H^1_{dR} of the pull-back of the universal family to C has regular singular points and quasi-unipotent local monodromy [49, Thm. 14.1]. Let $u\colon C \to \mathbb{P}^1$ be the morphism given by a general linear projection (a non-constant rational function on the smooth curve C), and $\ell^0 \subset \mathbb{A}^1$ the complement of a finite set of points, chosen so that u is étale over ℓ^0 and including the images of the points of the closure of C that are not in C, so that u is finite over ℓ^0. Its push-forward is a system $\mathrm{Y}' = A\mathrm{Y}$ over $\overline{\mathbb{Q}}(t)$ of rank $2g \deg u$, for the coordinates of horizontal sections in the frame fixed below, whose local system on ℓ^0 is induced from that of $C^0 := u^{-1}(\ell^0)$. The image of $\pi_1(C^0)$, of finite index, preserves the summand V_x of the fiber $\bigoplus_{y \in u^{-1}(c)} V_y$, and the kernel of the permutation action on $u^{-1}(c)$, normal and of finite index, preserves every summand. The identity component of the Zariski closure lies in the closure of the image of this kernel. So the projection of the Lie algebra of the identity component of the Zariski closure of the monodromy to $\mathrm{End}(V_x)$ is $\mathfrak{h}_C$. The system is regular singular in Singer's sense, as a local coordinate z on the smooth completion of C at a point over c satisfies $t - c = z^e \cdot (\text{unit})$, and $1/t = z^e \cdot (\text{unit})$ over $c = \infty$, so the differential Galois group is the Zariski closure of the monodromy [77, Prop. 1.3.27], both taken for one fundamental matrix. Feng's algorithm [30, Algorithm 4.1] computes the Galois group for a fundamental matrix $F_c \cdot h'$, where F_c is normalized at a point c chosen in step (g) of the algorithm and $h' \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ is computed [30, §§4.2, 4.4]; we take $c := u(x)$, which excludes finitely many x. The Galois group over $\mathbb{C}(t)$ is the base change of the one over $\overline{\mathbb{Q}}(t)$: $\mathbb{C} \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}((t-c)) \to \mathbb{C}((t-c))$ is injective, so the ideal of algebraic relations of F_c over $\mathbb{C}(t)$ is generated by the one over $\overline{\mathbb{Q}}(t)$, and the stabilizer condition has coefficients in $\overline{\mathbb{Q}}$. Thus $\mathfrak{h}_C$ is computed in $\mathrm{End}(H^1_{\mathrm{dR}}(A_x)) \otimes \overline{\mathbb{Q}}$, where $F^\bullet$ is algebraic.

Computing $\mathfrak{h}_{\widetilde{S}}$. Write $S' = S_{\iota,s}$ as in the proof of Lemma 10.10(b), $M := M_\iota$, and X^+_M for the component of $M_\iota(\mathbb{R})h_{x'}$ through x'; $X^+_M = X^+_{H'}$ for the generic Mumford–Tate group H' on X^+_M, as in the proof of Lemma 10.10(a). $\widetilde{S}$ is the normalization of a special subvariety with generic Mumford–Tate group H', and its algebraic monodromy group is H'^{der} (proof of [87, Cor. 6.2]), which is the product of the $\mathbb{Q}$-simple factors of M^{der} that are non-compact at $\mathbb{R}$: $\mathrm{Ad}\, h(i)$ is a Cartan involution of $M^{\mathrm{ad}}_{\mathbb{R}}$, trivial on the compact factors, so h projects trivially to them; and the group generated by the $h'(\mathbb{S})$, $h' \in X^+_M$, is normal in $M(\mathbb{R})^+$ and non-trivial on each non-compact $\mathbb{R}$-simple factor.

At x, $M \otimes \overline{\mathbb{Q}}$ is the centralizer in $\mathrm{GSp}(H^1_{\mathrm{dR}}(A_x))$ of $\iota_x(\mathcal{E})$, and ι_x is the fiber at x of an action of $\mathcal{E}$ on the family over all of $\widetilde{S}$ that defines S'; we find it as follows.

The generic point. Let $\mathcal{C}$ be an irreducible component, dominating S', of the locally closed subset of $\mathcal{H}$ in the proof of Lemma 10.9, $\overline{\mathcal{C}}$ its closure in $\mathcal{H} \times_{N_0} S'$, η the generic point of S', and $A_{\bar{\eta}}$ the geometric generic fiber of the universal family over S'. The geometric points of $\mathcal{C}$

over η are tuples of graphs of endomorphisms f of $A_{\bar\eta}$ with $\mathrm{Tr}(f^\dagger f)$ fixed by condition (3) in the proof of Lemma 10.9, so there are finitely many; hence $\dim \mathcal{C} = \dim S'$, and the generic point of $\mathcal{C}$ is the only point of $\overline{\mathcal{C}}$ over η.

Rationality. Every endomorphism ϕ of $A_{\bar\eta}$ is defined over $\overline{\mathbb{Q}}(S')$. Indeed, ϕ is defined over a finite extension of $\overline{\mathbb{Q}}(S')$, so on a finite étale cover of a dense open subset of $\widetilde{S}$ it is a flat section of End of the local system H_1, which near each point is an element of $\mathrm{End}(W)$ commuting with h_y for all y in a non-empty open subset of $X^+_{H'}$; such a subset contains Hodge generic points, so ϕ commutes with H' and is fixed by $\Gamma_{H'} \subset H'(\mathbb{Q})$, which is $\pi_1(\widetilde{S}(\mathbb{C}))$, as $\Gamma_{H'}$ is torsion-free and lies in $\mathrm{Sp}(W)(\mathbb{Q})$, so acts freely on the contractible $X^+_{H'}$, and acts on W as the monodromy of $\widetilde{S}$ (Lemma 10.10(c)). As $\widetilde{S} \to S'$ is birational (Lemma 10.10(c)), $\overline{\mathbb{Q}}(S') = \overline{\mathbb{Q}}(\widetilde{S})$; the family is an abelian scheme over all of $\widetilde{S}$, so $\mathrm{Gal}(\overline{\overline{\mathbb{Q}}(S')}/\overline{\mathbb{Q}}(S'))$ acts on the ℓ-adic Tate module of $A_{\bar\eta}$ through $\pi_1^{\mathrm{et}}(\widetilde{S})$, which, as the base field $\overline{\mathbb{Q}}$ is algebraically closed of characteristic 0, is the profinite completion of $\pi_1(\widetilde{S}(\mathbb{C})) = \Gamma_{H'}$ [42, Exp. XII, Cor. 5.2, and Exp. XIII, Prop. 4.6], acting through the closure of $\Gamma_{H'}$ in $\mathrm{GL}(W \otimes \mathbb{Q}_\ell)$; so it fixes ϕ.

Extension. Therefore the point of $\mathcal{H}$ at the generic point of $\mathcal{C}$ is rational over $\overline{\mathbb{Q}}(S')$, $\overline{\mathcal{C}} \to S'$ is birational, and $\overline{\mathcal{C}} \times_{S'} \widetilde{S}$ has exactly one irreducible component $\mathcal{C}'$ dominating $\widetilde{S}$, with function field $\overline{\mathbb{Q}}(\widetilde{S})$. The endomorphisms $f_{i,1}$ at the generic point of $\mathcal{C}'$ are therefore endomorphisms of the generic fiber of the pull-back of the universal family to the smooth $\widetilde{S}$, and they extend to it [29, Prop. I.2.7]; by Equation (10.1) the graphs of the $nf_{i,1}$ define a section of $\mathcal{H} \times_{N_0} \widetilde{S} \to \widetilde{S}$, whose image is closed, irreducible, of dimension $\dim \widetilde{S}$ and contains the generic point of $\mathcal{C}'$, so equals $\mathcal{C}'$. Hence $\mathcal{C}'$ has one point over x, the graphs of ι_x.

The datum. The extended $f_{i,1}$ act on the local system H_1 by flat sections, so on $X^+_{H'}$ they give a constant embedding ι' of $\mathcal{E}$ in $\mathrm{End}(W)$, with the datum $\mathfrak{d}$, which commutes with every h_y, $y \in X^+_{H'}$, hence with H'; so $X^+_{H'} = H'(\mathbb{R})^+ h_{x'}$ lies in the component of $M_{\iota'}(\mathbb{R})h_{x'}$ through x', $S' \subset S_{\iota',s}$, and as both are irreducible of dimension $\dim \mathfrak{d}$ (Lemma 10.10(b)), $S' = S_{\iota',s}$. So we may take $\iota = \iota'$ above; at x it is ι_x.

The action of ι_x on $H^1_{\mathrm{dR}}(A_x)$ and the polarization pairing, up to a scalar, are computed in the frame at x in which $\mathfrak{h}_C$ (conjugated back by Feng's matrix h') and $F^\bullet$, from [87, Thm. 2.4], are written. That frame is a basis of the de Rham bundle of [87, Thm. 2.4] over an affine open set containing x, made free by [87, Prop. 2.1(iii)]; its elements are classes of Čech–de Rham cocycles for the affine cover of the family by the complements $U_0, \dots, U_g$ of relative hyperplane sections [87, Lemmas 2.5 and 2.6], and by base change their restrictions to A_x form a basis of $H^1_{\mathrm{dR}}(A_x)$. On the Čech–de Rham complex of A_x, by the standard formulas, we compute f^* on the cover by the sets $U_j \cap f^{-1}(U_k)$, which are affine, being isomorphic to $\Gamma_f \cap (U_j \times U_k)$, with f given on charts by its graph, and we compute the pairing of a and b as the ratio of $a \cup b \cup c_1(\mathcal{O}(1))^{g-1}$ to $c_1(\mathcal{O}(1))^g$ in $H^{2g}_{\mathrm{dR}}(A_x)$, a non-zero multiple of the pairing dual to the Riemann form of $\mathcal{O}(1)|_{A_x}$ (if $c_1 = \sum_i d_i\, dx_i \wedge dy_i$, it pairs dx_i with dy_i to $1/(gd_i)$ and every other pair of these basis vectors to 0). Each class is written in the frame by a search over cochains of increasing degree, which stops because a solution exists. Compactness at each real place is read off the integers a_v of the datum. So $\mathfrak{h}_{\widetilde{S}}$ is linear algebra over $\overline{\mathbb{Q}}$.

Correctness and termination. An accepted C has $\mathfrak{h}_C = \mathfrak{h}_Y$ by Lemma 10.14(c). For $r \geq 2$, by the Lefschetz theorem of Hamm and Lê [43, Thm. 1.1.3] there is an open dense set of hyperplanes L_1 for which $L_1 \cap Y^{\mathrm{sm}}$ is connected and smooth, with $\pi_1(L_1 \cap Y^{\mathrm{sm}}) \to \pi_1(Y^{\mathrm{sm}})$ onto, and so on for the following hyperplanes; hyperplanes defined over $\mathbb{Q}(i)$ are dense in the complex topology, so some tuple over $\mathbb{Q}(i)$ gives a smooth connected curve C with

$\pi_1(C) \to \pi_1(Y^{\mathrm{sm}})$ onto, hence $\mathfrak{h}_C = \mathfrak{h}_Y$ and $t(\mathfrak{h}_C) = r$. The enumeration reaches it after finitely many tuples, each processed by terminating procedures, so the search stops, and whatever C it stops at is correct. □

10.8. **Hodge-type covers.** In this subsection $S = \Gamma_\Upsilon\backslash\mathcal{H}_g$ as in Lemma 10.2(ii), and for a Shimura subdatum $(G_1, X_1) \subset (G, X)$ and compact open subgroups $K^1 \subset G_1(\mathbb{A}_f)$ and $K^2 \subset G(\mathbb{A}_f)$ with $K^1 \subset K^2$ we write $u(K^1, K^2)\colon \mathrm{Sh}_{K^1}(G_1, X_1)_{\mathbb{C}} \to \mathrm{Sh}_{K^2}(G, X)_{\mathbb{C}}$, $[x, a] \mapsto [x, a]$, for the morphism of [24, 1.14]. We show that the image in S of a Shimura variety of Hodge type is closed and that the special subvarieties of S inside one of its components come from its own subdata (Lemmas 10.17 and 10.18), and that every connected component of a Shimura variety of abelian type at a neat level receives a finite étale map from such a component, for d_Υ divisible enough (Proposition 10.19).

Lemma 10.16. *Let $(G', X') \subset (G, X)$ be a Shimura subdatum. For $m \geq 1$ let $T_m := W^{\otimes m} \otimes (W^\vee)^{\otimes m}$, with lattice $T_m(\mathbb{Z}) := W_{\mathbb{Z}}^{\otimes m} \otimes (W_{\mathbb{Z}}^\vee)^{\otimes m}$, and let ψ_m be the bilinear form on T_m that is ψ on each factor W and $\psi^\vee$ on each factor $W^\vee$, where $\psi^\vee(\psi(v,\cdot), \psi(w,\cdot)) := \psi(v, w)$. Let $s_i \in T_{m_i}(\mathbb{Z})$, $1 \leq i \leq r$, be tensors whose common stabilizer in $\mathrm{GL}(W)$ is G', and $c_s := \max_i \psi_{m_i}(s_i, s_i)$. If $N^2 > 4c_s$, then every $\sigma \in \Gamma(N)$ with $\sigma X' \cap X' \neq \emptyset$ lies in $G'(\mathbb{Q})$. In particular, for every subgroup $\Gamma \subset \Gamma(N)$ and every connected component X'^+ of X', the map $(\Gamma \cap G'(\mathbb{Q})_+)\backslash X'^+ \to \Gamma\backslash X$ is injective.*

Such tensors exist. By Chevalley's theorem, as in the proof of [56, Prop. 2.1], G' is the stabilizer in $\mathrm{GL}(W)$ of finitely many tensors in spaces $W^{\otimes a} \otimes (W^\vee)^{\otimes b}$. As G' contains the scalars, which act there by λ^{a-b}, only $a = b$ occurs, and we scale the tensors to be integral.

Proof. Let $\mathrm{GL}(W)$ act on $W^\vee$ by $\phi \mapsto \phi \circ g^{-1}$. As ψ is perfect on $W_{\mathbb{Z}}$, so is $\psi^\vee$ on $W_{\mathbb{Z}}^\vee$, and ψ_m is integral on $T_m(\mathbb{Z})$; it is symmetric, as a product of $2m$ alternating forms. If $\psi(gv, gw) = \nu(g)\psi(v, w)$, then $g \cdot \psi(v,\cdot) = \nu(g)^{-1}\psi(gv,\cdot)$, so $\psi^\vee(g\phi, g\chi) = \nu(g)^{-1}\psi^\vee(\phi, \chi)$ and ψ_m is G-invariant. For $h \in X$ let $J := h(i)$. The form $b_J(v, w) := \psi(v, Jw)$ is symmetric and definite on $W_{\mathbb{R}}$, and since $J \cdot \psi(w,\cdot) = \psi(Jw,\cdot)$, the form $(\phi, \chi) \mapsto \psi^\vee(\phi, J\chi)$ is b_J transported to $W_{\mathbb{R}}^\vee$, with the same sign. So $B_J(a, b) := \psi_m(a, Jb)$ is a tensor product of $2m$ definite forms of one sign, hence positive definite on $T_m \otimes \mathbb{R}$. If $s \in T_m$ is fixed by $h(\mathbb{S})$, then $Js = s$ and $B_J(s, s) = \psi_m(s, s)$.

Let $\sigma \in \Gamma(N)$ and $x, y \in X'$ with $h_y = \sigma h_x \sigma^{-1}$. Each s_i is fixed by $h_x(\mathbb{S})$ and by $h_y(\mathbb{S})$, which lie in $G'_{\mathbb{R}}$, so $\sigma^{-1}s_i$ is fixed by $h_x(\mathbb{S})$. As $\sigma^{\pm 1} \equiv 1$ modulo N on $W_{\mathbb{Z}}$ and on $W_{\mathbb{Z}}^\vee$, $t := \sigma^{-1}s_i - s_i = Nu$ with $u \in T_{m_i}(\mathbb{Z})$, and u is fixed by $J := h_x(i)$. So $\psi_{m_i}(u, u) = B_J(u, u)$ is a non-negative integer, zero only if $u = 0$, and by the triangle inequality for B_J and the invariance of ψ_{m_i},

$$N^2\psi_{m_i}(u, u) = B_J(t, t) \leq \left(B_J(\sigma^{-1}s_i, \sigma^{-1}s_i)^{1/2} + B_J(s_i, s_i)^{1/2}\right)^2 = 4\psi_{m_i}(s_i, s_i) < N^2.$$

Hence $u = 0$ for every i, and σ fixes every s_i: $\sigma \in G'(\mathbb{Q})$. If moreover $\sigma \in \Gamma$ and $x, \sigma x \in X'^+$, then σ maps X'^+ onto the component containing σx, so $\sigma \in G'(\mathbb{Q})_+$ [56, Prop. 5.7(b)]. □

For $g = 2$, write $W = W_1 \oplus W_2$ with $W_k := \langle e_k, f_k\rangle$ and let GL_2 act on each W_k through the basis e_k, f_k. The four block units of $\mathrm{End}(W) = T_1$ for this decomposition (with W_1 and W_2 identified by $e_1 \mapsto e_2$, $f_1 \mapsto f_2$) cut out this $G' = \mathrm{GL}_2$ and have $\psi_1 = 2$, so $N \geq 3$ suffices; and $\mathrm{diag}(-\mathrm{id}_{W_1}, \mathrm{id}_{W_2}) \in \Gamma(2)$ fixes X' pointwise without lying in G', so the bound is sharp there.

Lemma 10.17. *Let $(G_1, X_1) \subset (G, X)$ be a Shimura subdatum.*

(i) Let $X_1^+ \subset \mathcal{H}_g$ be a connected component of X_1 and $\Gamma_1 := G_1(\mathbb{Q})_+ \cap K_\Upsilon$. The map $\iota_1 \colon \Gamma_1 \backslash X_1^+ \to S$ induced by $X_1^+ \subset \mathcal{H}_g$ is finite and unramified, and its image is closed and irreducible of dimension $\dim X_1^+$.

(ii) Let X_1^+ be any connected component of X_1 and $\gamma \in G(\mathbb{A}_f)$. The image C_γ of $X_1^+ \times \{\gamma\}$ in S is closed and irreducible of dimension $\dim X_1^+$. The image of $\mathrm{Sh}_{K_\Upsilon \cap G_1(\mathbb{A}_f)}(G_1, X_1)$ in S is the union of finitely many C_γ with $\gamma \in G_1(\mathbb{A}_f)$, and each C_γ with $\gamma \in G_1(\mathbb{A}_f)$ is one of its irreducible components.

Proof. (i) Put $K^2 := K_\Upsilon \cap K(d_\Upsilon)$ and $K^1 := K^2 \cap G_1(\mathbb{A}_f)$. They are neat, as $K(d_\Upsilon)$ is [68, 0.6]. An element of $G_1(\mathbb{Q})_+$ preserves X_1^+, hence $\mathcal{H}_g$, so it lies in $G(\mathbb{Q})_+$ [56, Prop. 5.7(b)]; and $G(\mathbb{Q})_+ \cap K_\Upsilon = \Gamma_\Upsilon \subset \Gamma(d_\Upsilon)$ (Lemma 10.2(ii) and its proof). So $G(\mathbb{Q})_+ \cap K^2 = \Gamma_\Upsilon$ and $G_1(\mathbb{Q})_+ \cap K^1 = \Gamma_1$, and by [25, 2.1.2] the connected components of $\mathrm{Sh}_{K^2}(G, X)_\mathbb{C}$ and $\mathrm{Sh}_{K^1}(G_1, X_1)_\mathbb{C}$ that contain the images of $\mathcal{H}_g \times \{1\}$ and $X_1^+ \times \{1\}$ are $\Gamma_\Upsilon \backslash \mathcal{H}_g$ and $\Gamma_1 \backslash X_1^+$. The natural morphism from the first of them to S is the identity of $\Gamma_\Upsilon \backslash \mathcal{H}_g$; as Γ_Υ is torsion free, the algebraic structure on $\Gamma_\Upsilon \backslash \mathcal{H}_g$ is unique and analytic morphisms from reduced schemes into it are algebraic [25, 2.1.2], so this is an isomorphism. Through it, ι_1 is the restriction of $u(K^1, K^2)$ to a connected component, with values in a connected component. As K^1 and K^2 are neat, the arithmetic groups of all connected components of $\mathrm{Sh}_{K^1}(G_1, X_1)_\mathbb{C}$ and $\mathrm{Sh}_{K^2}(G, X)_\mathbb{C}$ are torsion free and act freely, so the maps from $X_1 \times G_1(\mathbb{A}_f)/K^1$ and $X \times G(\mathbb{A}_f)/K^2$ to them are étale, and $u(K^1, K^2)$ is finite and unramified [24, proof of Prop. 1.15, p. 132], hence so is ι_1. A finite morphism is closed with finite fibers, and $\Gamma_1 \backslash X_1^+$ is irreducible of dimension $\dim X_1^+$.

(ii) Let $x_0 \in X_1^+$. As S is the image of $\mathcal{H}_g \times \{1\}$ (Lemma 10.2(ii)), $[x_0, \gamma] = [y_0, 1]$ with $y_0 \in \mathcal{H}_g$: there are $q \in G(\mathbb{Q})$ and $k \in K_\Upsilon$ with $y_0 = qx_0$ and $q\gamma k = 1$. Then $[x, \gamma] = [x, q^{-1}] = [qx, 1]$ for $x \in X_1^+$. The connected set qX_1^+ contains y_0, so it lies in $\mathcal{H}_g$, and it is a connected component of the Shimura subdatum (qG_1q^{-1}, qX_1); so (i) applies to it, and C_γ is closed and irreducible of dimension $\dim X_1^+$. By [25, 2.1.2], $\mathrm{Sh}_{K_\Upsilon \cap G_1(\mathbb{A}_f)}(G_1, X_1)_\mathbb{C}$ is the union of the images of $X_1^+ \times \{\gamma\}$ with γ in a finite set of representatives of $G_1(\mathbb{Q})_+ \backslash G_1(\mathbb{A}_f)/(K_\Upsilon \cap G_1(\mathbb{A}_f))$. So its image is the union of finitely many closed irreducible sets C_γ, all of dimension $\dim X_1^+$; none contains another properly, so each is an irreducible component. □

Lemma 10.18. *Let $(G', X') \subset (G, X)$ be a Shimura subdatum, X'^+ a connected component of X', $\gamma \in G(\mathbb{A}_f)$, and $S_1^0 \subset S$ the image of $X'^+ \times \{\gamma\}$. If a special subvariety V of S lies in S_1^0, then V is the image of $X_M^+ \times \{\gamma\}$ for a Shimura subdatum (M, X_M) of (G', X') and a connected component $X_M^+ \subset X'^+$ of X_M.*

This is [85, Lemma 2.1] with the base point taken in $X'^+ \times \{\gamma\}$; what Theorem 10.21(c) uses is that V is the image of $X_M^+ \times \{\gamma\}$ with the same γ and with X_M^+ inside the given component, which the statement of [85, Lemma 2.1] does not give.

Proof. Let V be the image of $X_H^+ \times \{g\}$ (every special subvariety of S is of this form, by Lemma 10.17(ii)), with $(H, X_H) \subset (G, X)$ a Shimura subdatum and $g \in G(\mathbb{A}_f)$. For $y \in X_H^+$, $[y, g] \in S_1^0$ means that $y = qx'$ and $g \in q\gamma K_\Upsilon$ for some $x' \in X'^+$ and $q \in G(\mathbb{Q})$. So X_H^+ is the union of the sets $A_q := X_H^+ \cap qX'^+$ over the countable set of $q \in G(\mathbb{Q})$ with $g \in q\gamma K_\Upsilon$, and by Baire's theorem the closure in X_H^+ of one of them, A_q, contains a non-empty open subset of X_H^+. The set of $y \in X_H^+$ for which h_y factors through $(qG'q^{-1})_\mathbb{R}$ is the common zero set of real-analytic functions on X_H^+ (polynomials defining $(qG'q^{-1})(\mathbb{R})$, evaluated at

$h_y(z)$, $z \in \mathbb{C}^\times$); it contains A_q, hence a non-empty open set, so it is all of X_H^+, which is connected.

Put $Y := q^{-1}X_H^+$, a connected component of the Shimura subdatum $(H_1, X_{H_1}) := (q^{-1}Hq, q^{-1}X_H)$. Then h_y factors through $G'_{\mathbb{R}}$ for every $y \in Y$, and Y contains a point y_0 of X'^+, as $A_q \neq \emptyset$. Let M be the smallest algebraic $\mathbb{Q}$-subgroup of $\mathrm{GL}(W)$ through which h_y factors for every $y \in Y$. Then $M \subset G' \cap H_1$, and M is connected, as $\mathbb{S}$ is. For $\sigma \in H_1(\mathbb{Q}) \cap H_1(\mathbb{R})^+$ we have $\sigma Y = Y$ and $h_{\sigma y} = \sigma h_y \sigma^{-1}$, so $\sigma M \sigma^{-1} = M$; as $H_1(\mathbb{Q}) \cap H_1(\mathbb{R})^+$ is dense in $H_1(\mathbb{R})^+$ [56, Thm. 5.4], the normalizer of M in H_1 has dimension $\dim H_1$, so M is normal in H_1, hence reductive. The projection of every h_y to the $\mathbb{Q}$-simple factors of H_1^{ad} outside the image of M is trivial, so there are no such factors [56, Def. 5.5, SV3]: $H_1 = M \cdot Z(H_1)$, so $H_1^{\mathrm{der}} \subset M$, $Z(M) \subset Z(H_1)$ and $M^{\mathrm{ad}} = H_1^{\mathrm{ad}}$. As $\mathrm{Lie}\, M \subset \mathrm{Lie}\, H_1$ and $M^{\mathrm{ad}} = H_1^{\mathrm{ad}}$, the axioms for (H_1, X_{H_1}) give that (M, X_M) with $X_M := M(\mathbb{R}) \cdot h_{y_0}$ is a Shimura datum, contained in (G', X') as $M \subset G'$ and $y_0 \in X'$. Finally $Y = H_1(\mathbb{R})^+ y_0$ [56, proof of Prop. 5.7(a)], and $H_1(\mathbb{R})^+ = H_1^{\mathrm{der}}(\mathbb{R})^+ Z(H_1)(\mathbb{R})^+$ with $Z(H_1)(\mathbb{R})$ fixing y_0; so $Y = M(\mathbb{R})^+ y_0$ is a connected component X_M^+ of X_M, and $Y \subset G'(\mathbb{R})^+ y_0 = X'^+$. As $g \in q\gamma K_\Upsilon$, V is the set of the points $[x, q\gamma] = [q^{-1}x, \gamma]$, $x \in X_H^+$, which is the image of $X_M^+ \times \{\gamma\}$. □

Proposition 10.19. *Let (G'', X'') be a Shimura datum of abelian type: there are a Shimura datum (G', X') of Hodge type and a central isogeny $f\colon G'^{\mathrm{der}} \to G''^{\mathrm{der}}$ that induces an isomorphism $(G'^{\mathrm{ad}}, X'^{\mathrm{ad}}) \cong (G''^{\mathrm{ad}}, X''^{\mathrm{ad}})$ ([25, 2.3.10] constructs such data when G''^{ad} is $\mathbb{Q}$-simple of type A, B, C, $D^{\mathbb{R}}$ or $D^{\mathbb{H}}$ and G''^{der} is a quotient of the covering of G''^{ad} in 2.3.10(ii): the universal covering, except for type $D^{\mathbb{H}}$, where it is the covering of 2.3.8). Let $K'' \subset G''(\mathbb{A}_f)$ be a neat compact open subgroup, X''^+ a connected component of X'', $g'' \in G''(\mathbb{A}_f)$, and S^0 the image of $X''^+ \times \{g''\}$ in $\mathrm{Sh}_{K''}(G'', X'')_{\mathbb{C}}$, so that $S^0 = \Gamma''\backslash X''^+$ with Γ'' the image of $G''(\mathbb{Q})_+ \cap g''K''g''^{-1}$ in $G''^{\mathrm{ad}}(\mathbb{Q})^+$ [25, 2.1.2]. Then (G', X'), f, an embedding $(G', X') \hookrightarrow (G, X)$ with $g \geq 2$, and an integer $d_0 \geq 3$ can be chosen so that f identifies a connected component $X'^+ \subset \mathcal{H}_g$ of X' with X''^+ and, for every d_Υ divisible by $8d_0$, with $\Gamma'_+ := G'(\mathbb{Q})_+ \cap K_\Upsilon$:*

- *(i) the map $\iota\colon \Gamma'_+\backslash X'^+ \to S$ is a closed immersion, whose image S_1^0 is the image of $X'^+ \times \{1\}$, an irreducible component of the image of $\mathrm{Sh}_{K_\Upsilon \cap G'(\mathbb{A}_f)}(G', X')$ in S;*
- *(ii) the image of Γ'_+ in $G''^{\mathrm{ad}}(\mathbb{Q})^+$ is a subgroup of finite index of Γ''; so the identity of $X'^+ = X''^+$ induces a finite étale morphism $\pi\colon S_1^0 \to S^0$, and $X''^+ \to S^0$ factors as $X'^+ \to S_1^0 \xrightarrow{\pi} S^0$.*

Proof. The datum. Take $(G', X') \hookrightarrow (\mathrm{GSp}(V_1, \psi_1), X(\psi_1))$ and f from the definition; if $\dim V_1 = 2$, replace (V_1, ψ_1) by $(V_1 \oplus V_1, \psi_1 \oplus \psi_1)$ with G' acting diagonally. A symplectic basis identifies (V_1, ψ_1) with (W, ψ), $g \geq 2$, and after conjugating by $\mathrm{diag}(1_g, -1_g)$ a chosen component $X_0'^+$ of X' lies in $\mathcal{H}_g$. Identify $(G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$ with $(G''^{\mathrm{ad}}, X''^{\mathrm{ad}})$ through f; X' and X'' map injectively onto unions of components of X''^{ad} [56, Prop. 5.7(a)]. The elements of $G''^{\mathrm{ad}}(\mathbb{R})$ carrying the component of X''^{ad} that contains $X_0'^+$ onto the one that contains X''^+ form a non-empty union of cosets of $G''^{\mathrm{ad}}(\mathbb{R})^+$, so one of them, q, is rational [56, Thm. 5.4]. Replacing f by $\mathrm{Int}(q) \circ f$, we may take $X'^+ := X_0'^+$ equal to X''^+ in X''^{ad}. Let λ', λ'' be the maps to G''^{ad}; then $\lambda' = \lambda'' \circ f$ on G'^{der}.

The arithmetic groups. Let $K_g'' := g''K''g''^{-1}$. By [25, 2.0.7] there is a compact open subgroup $K_2 = \prod_p K_{2,p}$ of $G''^{\mathrm{der}}(\mathbb{A}_f) \cap K_g''$ with $G''^{\mathrm{der}}(\mathbb{Q}) \cap K_2 \subset G''^{\mathrm{der}}(\mathbb{R})^+$, so $\lambda''(G''^{\mathrm{der}}(\mathbb{Q}) \cap K_2) \subset \Gamma''$. The preimage $K_1 := f^{-1}(K_2)$ is a compact open subgroup of $G'^{\mathrm{der}}(\mathbb{A}_f)$: each f_p

has finite kernel and open image, so $f_p^{-1}(K_{2,p})$ is compact, and for almost all p it is the group of $\mathbb{Z}_p$-points of a smooth model, because f extends to a finite, hence integral, morphism of models. So $\lambda'(G'^{\mathrm{der}}(\mathbb{Q})\cap K_1)\subset\Gamma''$. Let Z' be the neutral component of the center of G', choose a compact open $K_0\subset G'(\mathbb{A}_f)$ with $K_0\cap G'^{\mathrm{der}}(\mathbb{A}_f)\subset K_1$, and put $U:=Z'(\mathbb{Q})\cap K_0$. By [25, 2.0.12] there is a compact open $K_3\subset G'(\mathbb{A}_f)$ with $G'(\mathbb{Q})\cap K_3G'^{\mathrm{der}}(\mathbb{A}_f)\subset G'^{\mathrm{der}}(\mathbb{Q})U$. If $\sigma\in G'(\mathbb{Q})\cap K_0\cap K_3$, then $\sigma=dz$ with $z\in U$ and $d=\sigma z^{-1}\in G'^{\mathrm{der}}(\mathbb{Q})\cap K_1$, so $\lambda'(\sigma)=\lambda'(d)\in\Gamma''$. Choose d_1 with $K(d_1)\cap G'(\mathbb{A}_f)\subset K_0\cap K_3$, tensors as in Lemma 10.16, N_I with $N_I^2>4c_s$ (so $N_I\geq 3$), and put $d_0:=N_Id_1$.

Let $8d_0\mid d_\Upsilon$. An element of Γ'_+ preserves $X'^+\subset\mathcal{H}_g$, so its multiplier lies in $\mathbb{Q}_{>0}\cap\widehat{\mathbb{Z}}^\times=\{1\}$, and by Lemma 10.2(i),(ii) it lies in $\mathrm{Sp}_{2g}(\mathbb{Z})\cap K_\Upsilon=\Gamma_\Upsilon\subset\Gamma(d_\Upsilon)$. So $\Gamma'_+=\Gamma_\Upsilon\cap G'(\mathbb{Q})_+\subset K(d_1)$, and the image of Γ'_+ lies in Γ''; both groups are arithmetic [56, Prop. 3.2], so the index is finite.

(i). As $\Gamma_\Upsilon\subset\Gamma(d_\Upsilon)$ and $d_\Upsilon\geq N_I$, Lemma 10.16 shows that ι is injective on $\mathbb{C}$-points, and by Lemma 10.17(i) it is finite and unramified. Write $Y_1:=\Gamma'_+\backslash X'^+$. The diagonal $Y_1\to Y_1\times_S Y_1$ is an open immersion [41, 17.4.2], and a closed one, as ι is separated; so the complement of its image is open and closed in $Y_1\times_S Y_1$, a scheme of finite type over $\mathbb{C}$. A $\mathbb{C}$-point of that complement would be a pair (x_1,x_2) of $\mathbb{C}$-points of Y_1 with $x_1\neq x_2$ and $\iota(x_1)=\iota(x_2)$; there is none, and a non-empty scheme of finite type over $\mathbb{C}$ has a $\mathbb{C}$-point, so the complement is empty and the diagonal is an isomorphism. Hence ι is a monomorphism [80, Tag 01L3], and, being proper, a closed immersion [41, 18.12.6]. Its image is the image of $X'^+\times\{1\}$, an irreducible component of the image of $\mathrm{Sh}_{K_\Upsilon\cap G'(\mathbb{A}_f)}(G',X')$ by Lemma 10.17(ii).

(ii). The image of Γ'_+ has finite index in Γ'', and both groups are neat (Lemma 10.2(iii) and [68, 0.6]), so $\Gamma'_+\backslash X''^+\to\Gamma''\backslash X''^+$ is a finite étale covering, algebraic as in [25, 2.1.2]. Composed with the inverse of the isomorphism in (i), it is π. □

10.9. **The algorithm.** Let $g\geq 2$ and let $Z\subset M_\Upsilon\otimes\overline{\mathbb{Q}}$ be closed and geometrically irreducible, given by equations over a number field. Let $c_{\deg}$, D_1, A_1 be numbers, with $A_1\geq 1$, such that every special subvariety V maximal in Z has $\deg_{\mathcal{L}_{K_\Upsilon}}V\leq c_{\deg}$, $d_{L_V}\leq D_1$ and $\alpha_V\leq A_1$, with α_V at the level $\lambda(K_\Upsilon)$, and let $\bar{B}_0\in(0,\bar{B}]$, where $\bar{B}$ is the constant B of Table 1 for the adjoint datum with which α_V is formed (Section 10.1). By Lemma 10.3, Theorem 9.13(a) at $N=2d_\Upsilon$ supplies such numbers: $c_{\deg}$ with no hypothesis, and D_1, A_1 under the zero-free half-plane that it assumes for the fields L_V, which holds for every number field by [19] (Corollary 1.2). The same computation gives $\bar{B}_0$: those constants depend on $\bar{B}$ through $B'=\bar{B}c_U$, and one may take $\bar{B}_0:=\bar{B}=(A\,a!)^{-r_U}$, with $a=|\mathrm{GL}_{g(2g+1)}(\mathbb{F}_3)|$ (Section 9), $r_U=2g$ (for a torus T of dimension $t\leq g$, every finite quotient of the maximal compact subgroup of $T(\mathbb{Q}_p)$ has at most $2t$ generators) and A the integer of Table 1 for this datum, taken as the integer the proof of [85, Prop. 2.9] constructs, which it bounds effectively in terms of g [85, Lemmas 2.4–2.8]; upper bounds for A, r_U and a give a valid $\bar{B}_0$.

Algorithm 10.20 (Listing)**.** Input: Mumford's equations for $\overline{M}_\Upsilon$ and $\overline{\mathcal{A}}_\Upsilon$, equations for Z, and $(c_{\deg}, D_1, A_1, \bar{B}_0)$.

(0) *Base.* Compute N_0 (Lemma 10.4(iv)).

(1) *Data.* Enumerate the PEL data $\mathfrak{d} = (\mathcal{E}, R, \tau)$ with $\mathcal{E} = \prod_i F_i$, $\sum_i [F_i : \mathbb{Q}] \le 2g$, $d_{F_i} \le D_1$, and $[\mathcal{O}_\mathcal{E} : R] \le B_\mathcal{O}$, where $B_\mathcal{O}$ is the bound of Lemma 10.1(d) with $n_\mathcal{E} = 2g$, $N = 2d_\Upsilon$, $m_V = \operatorname{lcm}(1, \dots, 2g)$, D_1 and A_1, with $\bar{B}_0$ in place of $\bar{B}$ in A_1'.

(2) *Containers.* For each $\mathfrak{d}$, compute $\mathfrak{E}(\mathfrak{d}) \cap N_0$ (Lemma 10.9), which is closed in N_0 (Lemma 10.10(b)), and its irreducible components S' (primary decomposition).

(3) *Rigid members.* For each S' with $S' \cap Z \neq \emptyset$, compute the normalization $\nu\colon \widetilde{S}' \to \overline{S}'$ of the closure of S' in $\mathbb{P}(V_\Upsilon)$ and its embedding by $\nu^*\mathcal{O}(h)$ (Lemma 10.12), and restrict to $\nu^{-1}(S')$. Pull back the universal family and compute the data of the variation H^1 [87, Thm. 2.4]. Run [87, Cor. 6.3] on $\nu^{-1}(S')$ with degree bound $d' := h^{\dim S'} c_{\deg}$. For each output Y, keep it if $\nu(Y) \subset Z$ (ideal membership) and Y is rigid (Lemma 10.13, with $\mathfrak{h}_Y$ from Proposition 10.15, and $\mathfrak{h}_Y = 0$ if $\dim Y = 0$, with $\mathfrak{h}_{\widetilde{S}}$ and $F^\bullet$ at the point Y computed as in the proof of Proposition 10.15, which does not use $\dim Y \ge 1$ there), and put $\nu(Y)$ on the list.

(4) *Maximality.* Delete every member of the list that is strictly contained in another member.

Theorem 10.21. *(a) Algorithm 10.20 terminates, and its output is exactly the set of special subvarieties of S maximal among the special subvarieties contained in Z.*

(b) Let $(G', X') \hookrightarrow (\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$ be a Shimura datum of Hodge type, $K_{G'} := K_\Upsilon \cap G'(\mathbb{A}_f)$, and suppose Z lies in the image of $\mathrm{Sh}_{K_{G'}}(G', X')$ in S. Then the output of Algorithm 10.20 is exactly the set of special subvarieties of $\mathrm{Sh}_{K_{G'}}(G', X')$ (through their images) maximal in Z.

(c) Let (G'', X'') be a Shimura datum of abelian type, in the sense of Proposition 10.19, and S^0 a connected component of a Shimura variety attached to it. Suppose that $G'^{\mathrm{ad}} = G''^{\mathrm{ad}}$ and $X'^{+} = X''^{+}$ for a connected component X'^{+} of X', and that the quotient map $X''^{+} \to S^0$ factors as $X'^{+} \to S_1^0 \xrightarrow{\pi} S^0$ with π finite, where $S_1^0 \subset S$ is the image of $X'^{+} \times \{\gamma\}$ for some $\gamma \in G'(\mathbb{A}_f)$ and the first map is $x \mapsto [x, \gamma]$. Let $Z \subset S^0$ be closed and irreducible, given by equations for $\widetilde{Z} := \pi^{-1}(Z) \subset S$, and suppose that every special subvariety V maximal in a component of $\widetilde{Z}$ has $\deg_{\mathcal{L}_{K_\Upsilon}} V \le c_{\deg}$, $d_{L_V} \le D_1$ and $\alpha_V \le A_1$, with α_V at the level $\lambda(K_\Upsilon)$. Then running Algorithm 10.20 with $(c_{\deg}, D_1, A_1, \bar{B}_0)$ on each component of $\widetilde{Z}$, and step (4) on the union of the outputs, gives a finite list in S_1^0 whose images in S^0 are exactly the special subvarieties maximal in Z.

Theorem 1.4 is part (a). By Proposition 10.19, the data of (c) exist for every connected component S^0 of a Shimura variety of abelian type at a neat level, with π finite étale. The map π need not be a Galois covering, and (c) does not need it to be. We do not show that (G', X'), S_1^0, π or $\pi^{-1}(Z)$ can be computed from the data defining S^0: unless (G', X') is of PEL type, S_1^0 is not cut out by the moduli problems that Algorithm 10.20 uses, and we have no model of S^0 itself. This is why (c) takes Z through equations for its preimage. A model of S^0 would need the action on Mumford's model of the elements of $G''^{\mathrm{ad}}(\mathbb{Q})^+$ that do not lift to the Hodge-type group; these are not Hecke operators of the Hodge-type datum, and the moduli interpretation does not give them.

Proof. (a) *Every member put on the list is special and lies in Z.* An output Y of step (3) is weakly special and rigid, hence special by Lemma 10.13, which applies to $\nu^{-1}(S')$ by Lemma 10.10(c). The map ν is induced by a morphism of Shimura data, so $\nu(Y)$ is special, and $\nu(Y) \subset Z$ is tested exactly.

Every maximal V is put on the list. By Lemma 10.1(a),(d), applied with $N = 2d_\Upsilon$ (as $K(2d_\Upsilon) \subset K_\Upsilon$, Lemma 10.2) and $m_V \le \operatorname{lcm}(1,\dots,2g)$, with $\bar{B}_0$ in place of $\bar{B}$ in A_1' (as at the end of the proof of Lemma 10.1), the bound being increasing in $n_{\mathcal{E}}$ and m_V, the datum $\mathfrak{d}_V$ is enumerated in step (1). By Lemma 10.10(b), S_V is a component of $\mathfrak{E}(\mathfrak{d}_V)$; it meets Z, and it lies in N_0, which is all of $M_\Upsilon \otimes \overline{\mathbb{Q}}$ by Lemma 10.4(iii) and Lemma 10.2(ii), so it is found in step (2). By Lemma 10.10(c), the part of the normalization of $\overline{S}_V$ over S_V is $\widetilde{S}_V$, and by Lemma 10.10(d) W_V is rigid in it. By Lemma 10.10(e), Lemma 10.12(f) and Lemma 10.11,

$$\deg_{\nu^*\mathcal{O}(h)} \overline{W}_V = h^{\dim V} \deg_{\mathcal{O}(1)} \overline{V} \le h^{\dim S_V} \deg_{\mathcal{L}_{K_\Upsilon}} V \le h^{\dim S_V} c_{\deg},$$

so $[W_V]$ is among the outputs of [87, Cor. 6.3]. Then $\nu(W_V) = V \subset Z$, and W_V passes the rigidity test, because Proposition 10.15 returns the true $\mathfrak{h}_{W_V}$ (Lemma 10.14(c)). So V is put on the list.

Termination. Step (0) is Lemma 10.4(iv), step (1) is finite by Lemma 10.1(d), and step (2) is Lemma 10.9 with primary decomposition. The data of step (1) are enumerated as follows. Fields: by Minkowski's second theorem a field F of degree at most $2g$ has $\mathbb{Q}$-independent integers v_k of house at most $d_F^{1/2}$; as F has at most $s(2g)$ proper subfields, with $s(2g)$ the number of subgroups of the symmetric group $\mathfrak{S}_{2g}$, some $\theta = \sum_k c_k v_k$ with $0 \le c_k \le s(2g)$ generates F, and the coefficients of its minimal polynomial are at most $\binom{2g}{j}(2g\, s(2g) D_1^{1/2})^j$. Orders: an order of index b contains $\mathbb{Z} + b\mathcal{O}_{\mathcal{E}}$, as $\mathcal{O}_{\mathcal{E}}/R$ has exponent dividing b, so the orders of index at most $B_{\mathcal{O}}$ are among the finitely many subgroups between $\mathbb{Z}+b\mathcal{O}_{\mathcal{E}}$ and $\mathcal{O}_{\mathcal{E}}$, $b \le B_{\mathcal{O}}$; keep those that are rings stable under $*$. Types: finitely many for each $\mathcal{E}$. Step (3) uses [39, Algorithm 3], Lemma 10.12, [87, Thm. 2.4 and Cor. 6.3], whose output is finite, ideal membership, and Lemma 10.13 with Proposition 10.15, which terminates. Step (4) is finite.

Step (4). A listed V that is maximal in Z is not strictly contained in another member, since members are special and lie in Z. A member that is not maximal lies in a maximal $V' \subset Z$, which is listed, so it is deleted.

(b) Images of special subvarieties of $\mathrm{Sh}_{K_{G'}}(G', X')$ are special in S. Conversely, if $V \subset S$ is special and lies in the image, its generic Mumford–Tate group is conjugate into G', since the Mumford–Tate groups of the points of X' lie in G', so V is the image of a special subvariety of $\mathrm{Sh}_{K_{G'}}(G', X')$ [85, Lemma 2.1]. So the two notions of maximal special subvariety of Z agree, and (a) applies.

(c) By Lemma 10.17, S_1^0 is closed in S and is an irreducible component of the image in (b). The special subvarieties of S_1^0 and of S^0 are the images of the subsets $X_H^+ \subset X''^+$ attached to Shimura subdata, and these subsets depend only on the adjoint datum. So π maps special subvarieties onto special subvarieties.

Write $S^0 = \Gamma\backslash X''^+$ with $\Gamma \subset G''^{\mathrm{ad}}(\mathbb{Q})^+$. If $V_1 \subset S_1^0$ is special, the image of X_H^+, then, as $X''^+ \to S_1^0$ is onto and its composite with π is the quotient map, $\pi^{-1}(\pi(V_1))$ is the union of the images in S_1^0 of the ϵX_H^+, $\epsilon \in \Gamma$. Each $\epsilon \in \Gamma$ acts on G' by an automorphism $\mathrm{Int}(\epsilon)$ defined over $\mathbb{Q}$, through $G'^{\mathrm{ad}} = G''^{\mathrm{ad}}$, and is the image of an element of $G'(\mathbb{R})_+$ [56, Prop. 5.1]; so $(\mathrm{Int}(\epsilon)H, \mathrm{Int}(\epsilon) \circ X_H)$ is a Shimura subdatum of (G', X'), $\epsilon X_H^+ \subset X'^+$ is a connected component of $\mathrm{Int}(\epsilon) \circ X_H$, and its image in S_1^0 is special, irreducible, of dimension

$\dim V_1$, and closed (Lemma 10.17(ii)). As π is finite, $\pi^{-1}(\pi(V_1))$ is closed of dimension $\dim V_1$, so each of its finitely many irreducible components is one of these images. So every component of the preimage of a special subvariety of S^0 is special.

By Lemma 10.18, a special subvariety of S contained in S_1^0 is a special subvariety of S_1^0, the image of $X_M^+ \times \{\gamma\}$ for a Shimura subdatum (M, X_M) of (G', X') with $X_M^+ \subset X'^+$, and the converse is clear; so the special subvarieties of $\widetilde{Z}$ are the same in S and in S_1^0, and (a) may be applied to the components of $\widetilde{Z}$. Every special subvariety of $\widetilde{Z}$ is irreducible, so lies in a component, and the maximal special subvarieties of $\widetilde{Z}$ are the maximal members of the union of the lists given by (a) for the components; step (4) on the union uses only containment in S.

If V_1 is maximal in $\widetilde{Z}$ and $\pi(V_1) \subsetneq V' \subset Z$ with V' special, a component of $\pi^{-1}(V')$ through V_1 is special, lies in $\widetilde{Z}$ and is larger than V_1; so $\pi(V_1)$ is maximal in Z. Conversely, if V is maximal in Z, a component V_1 of $\pi^{-1}(V)$ is special and lies in $\widetilde{Z}$, and it is maximal there, since the image of a larger special subvariety of $\widetilde{Z}$ would be a special subvariety of Z of larger dimension. So the images of the maximal special subvarieties of $\widetilde{Z}$ are exactly the maximal special subvarieties of Z. □

For a special point Q, $S_Q = \{Q\}$, and step (3) reduces to testing $Q \in Z$.

Declarations

Use of AI. This paper and its results were produced by an AI research system that the author designed, built and directs, running on Anthropic's Claude Fable 5.2 in Claude Code and OpenAI's GPT-6 Astra in Codex. The author chooses the problems and prioritizes the work. Separate AI agents checked each result, and a second program confirmed the computations the proofs use, with any exceptions listed in the public deposit. The system also searched the literature and checked the citations against reliable sources. The author read the paper, reran the computations from the deposit and takes full responsibility for all of the paper's contents.

Data availability. The programs and data used in this paper, with instructions for rerunning them, are deposited at Zenodo [48].

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