IMAGINARY QUADRATIC FIELDS OF CLASS NUMBER AT MOST 3000

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Abstract. Assume Kimura's explicit bound for real zeros of Dirichlet L-functions (the second part of Theorem 1 of his preprint, Lean-checked according to its author but not refereed), which follows from the hypothesis (H) that $L(\sigma,\chi)\neq 0$ for $7/8<\sigma<1$ and every real primitive character χ. Then exactly 36 993 853 imaginary quadratic fields have class number at most 3000, and the largest absolute discriminant among them is 2 562 211 723. We also list the imaginary quadratic orders of class number at most 1095, and so the singular moduli of degree at most 1095. Kimura's bound gives $h(D)\geq 3562$ for $|D|\geq e^{35.0131}$ by Hecke's integral representation of the Dedekind zeta function, and a sieve in the manner of Watkins, the tabulation of Mosunov and Jacobson and an unconditional computation with binary quadratic forms treat smaller $|D|$. Hypothesis (H) follows from the generalized Riemann hypothesis and is the real case of the zero-free half-plane $\operatorname{Re} s>7/8$ announced by OpenAI in October 2026, whose Lean formalization is machine-checked but whose proof has not been refereed. Under (H) the sieve and the tabulation are not needed, $h(D)>\sqrt{|D|}/(5\log|D|)$ for every imaginary quadratic field, and several classifications known only up to one possible exception become complete, among them those of the imaginary quadratic fields with class group of exponent 2, 4 or 8 and of the real quadratic fields of extended Richaud–Degert type with class number 1. Our lists agree with those of Kimura ($h\leq 2000$), of Cremona and Sutherland ($h\leq 1000$, under the generalized Riemann hypothesis) and of a computation posted on October 8, 2026, in the repository zeta-lab ($h\leq 1500$, assuming that $L(s,\chi_D)\neq 0$ for $\operatorname{Re} s>7/8$), and with the counts of Garcia, Grenié, Lee and Molteni ($h\leq 2000$, under the generalized Riemann hypothesis).

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1. Introduction

Let $D < 0$ be a fundamental discriminant, $K = \mathbb{Q}(\sqrt{D})$ the imaginary quadratic field of discriminant D, and $h(D)$ its class number. Gauss [26, art. 303] conjectured, in the language of binary quadratic forms, that each class number occurs only finitely often, and he listed the discriminants of small class number. Heilbronn [33] proved that $h(D) \to \infty$, and Siegel [74] that $h(D) \gg_\varepsilon |D|^{1/2-\varepsilon}$, but neither result gives an effective bound for the largest $|D|$ of a given class number. The fields of class number one were determined by Heegner [32], Baker [5] and Stark [76], and those of class number two by Baker [6] and Stark [78]. Goldfeld [27], with the theorem of Gross and Zagier [30], made the problem effective for every class number. It was then solved for class number 3 by Oesterlé [65], 4 by Arno [3], 5, 6 and 7 by Wagner [80], odd class numbers up to 23 by Arno, Robinson and Wheeler [4], and every class number up to 100 by Watkins [81]. The lower bound given by the Goldfeld–Gross–Zagier method grows only like $\log|D|$, and the computations it requires grow quickly with the class number.

The obstruction to a much stronger effective bound is a possible real zero of $L(s, \chi_D)$ close to $s = 1$ (a Siegel zero, also called a Landau–Siegel zero), where χ_D is the Kronecker symbol $(D/\cdot)$. If $L(s, \chi_D)$ has no real zero in an explicit interval ending at 1, then an argument of Hecke (published by Landau [45]; see also [28]) gives $h(D) \gg \sqrt{|D|}/\log|D|$ with an explicit constant (Proposition 2.2), and the determination of the fields of a given class number becomes a finite computation. We use two hypotheses of this kind.

(H) For every real primitive Dirichlet character χ and every real σ with $7/8 < \sigma < 1$, $L(\sigma, \chi) \neq 0$.

(K) For every real primitive Dirichlet character χ of conductor q with $\log q \geq 35.0131$ and every real β with $0 < \beta < 1$ and $L(\beta, \chi) = 0$, we have $(1-\beta)\log q \geq 1193/200000$.

Hypothesis (H) follows from the generalized Riemann hypothesis. It implies (K): under (H) every such β is at most $7/8$, so that $(1-\beta)\log q \geq (\log q)/8 > 4$. Hypothesis (K), with the constants 35.0131 and 1193/200000, is due to Kimura [41, Theorem 1], and Section 1.1 describes its status. Section 1.3 explains how the proof treats the discriminants with $\log|D| < 35.0131$, to which (K) does not apply.

1.1. The status of (H) and (K).

In October 2026 OpenAI released a manuscript [66] claiming that no Dirichlet L-function has a zero in the half-plane $\operatorname{Re} s > 7/8$, together with a formalization in Lean 4 [20] that uses Mathlib [55]. Hypothesis (H) is the case of real s and real primitive characters. The formal statement of the half-plane implies (H), and the Lean kernel accepts its proof, which uses only Lean's standard axioms (Appendix A). Liu [50] and Xu and Sang [83] have since announced slightly smaller values than $7/8$, by the method of [66], the latter assuming the analytic inputs of [66].

Another manuscript of the same release, also with a Lean formalization, states that every real zero β of the L-function of a real primitive character of conductor $q \geq 3$ satisfies $(1-\beta)\log q \geq c$ for an absolute constant $c > 0$ [69]. Banks [7], Kerr [40] and Green [29] have since announced proofs of this, Banks's with explicit constants. Kimura's Theorem 1 [41] makes the constant explicit, and (K) is its second part. Cureton and Thorner [19] have announced, also with a Lean formalization, that no Dedekind zeta function vanishes in the half-plane $\operatorname{Re} s > 31/32$. This implies (K): a real zero β of $L(s, \chi)$ is a zero of the Dedekind zeta function of the quadratic field attached to χ, so $\beta \leq 31/32$ and $(1-\beta)\log q \geq (\log q)/32 > 1193/200000$.

Neither OpenAI's manuscript on the half-plane [66] nor Kimura's preprint (version 1, October 15, 2026), whose Theorem 1 is Lean-checked according to its author, has appeared in a refereed journal. We have not checked the arguments of either, or Kimura's formalization. Theorems 1.1 and 5.11 and Proposition 2.7 assume (K), and so hold under (H). Theorem 1.3 and the other theorems of Section 5 assume (H), which (K) does not give by itself (Remark 4.3). Theorem 1.2, Lemma 3.4 and Proposition 5.1 are unconditional.

1.2. **Results.** In the following statements, $\mathcal{L}$ is a list of 36 993 853 pairs (D, h) of integers, deposited in [38] (Appendix A.5).

Theorem 1.1. *Assume* (K). *The imaginary quadratic fields of class number at most* 3000 *are exactly the fields* $\mathbb{Q}(\sqrt{D})$ *for the* 36 993 853 *discriminants* D *that occur in* $\mathcal{L}$*, and the class number of* $\mathbb{Q}(\sqrt{D})$ *is the* h *paired with* D *in* $\mathcal{L}$*. In particular, exactly* 113 139 *imaginary quadratic fields have class number* 3000*, and the largest* $|D|$ *of an imaginary quadratic field of class number at most* 3000 *is* 2 562 211 723*, for which* $h(D) = 2976$*. For each* $h \leq 3000$*, the number* $F(h)$ *of imaginary quadratic fields of class number* h *and the largest* $|D|$ *with* $h(D) = h$ *are given in Appendix B.*

Since (H) implies (K), and the generalized Riemann hypothesis implies (H), Theorem 1.1 holds under either of them. Section 4 proves it under (K), and gives a shorter proof under (H), which needs neither Lemma 3.4 nor the tabulation of class groups by Mosunov and Jacobson [63].

Theorem 1.2. *For every fundamental discriminant* D *with* $-1.98 \cdot 10^{10} < D < 0$*, we have* $h(D) \leq 3000$ *if and only if* $(D, h(D)) \in \mathcal{L}$*; and every pair in* $\mathcal{L}$ *is of this form.*

Theorem 1.3. *Assume* (H). *Then* $h(D) > \sqrt{|D|}/(5 \log |D|)$ *for every fundamental discriminant* $D < 0$*.*

Table 1 summarizes Theorem 1.1. For $h \leq 125$ the numbers $F(h)$ and the largest discriminants agree with the tables of Buell [11, Tables 1–3], computed for $|D| < 4 \cdot 10^6$, which he expected to be complete, and for $h \leq 100$ with Watkins's determination [81]. Assuming the generalized Riemann hypothesis, Holmin, Jones, Kurlberg, McLeman and Petersen [35, §6] computed $F(h)$ for every odd $h \leq 10^6$, but listed only the fields whose class group is a non-cyclic p-group; for the 1500 odd $h \leq 3000$ their numbers agree with ours (Remark 4.1).

Complete lists for class numbers beyond 100 were known before. Assuming the generalized Riemann hypothesis, Cremona and Sutherland [18, §5] determined the 4 115 897 fields and the 6 450 424 orders of class number at most 1000, from the tabulation of Mosunov and Jacobson and the bound of [18, Proposition 19]. On October 8, 2026, a determination of the fields of class number at most 1500 was posted in the public repository zeta-lab [48]. It assumes that $L(s, \chi_D)$ has no zero, real or complex, with real part greater than $7/8$, and it gives the numbers $F(h)$, checksums of its lists and the commands that regenerate them. Version 1 of this paper, posted on October 9, 2026, contained the list $\mathcal{L}$ and Theorem 1.1 under (H). Independently of our work, Garcia, Grenié, Lee and Molteni [25] have announced a determination of the fields of class number at most 2000 under the generalized Riemann hypothesis, and Kimura [41, Theorem 2] one given (K), using a sieve and, for the smaller discriminants, the tabulation of Mosunov and Jacobson. Our lists agree with those of Cremona and Sutherland, of zeta-lab and of Kimura, and with the counts of Garcia, Grenié, Lee and Molteni (Remark 4.1). For $2000 < h \leq 3000$ we know of no earlier list under any hypothesis.

The range of Theorem 1.2 lies inside that of earlier unconditional tabulations of class groups. Jacobson, Ramachandran and Williams [37] computed the class groups of all imaginary quadratic fields with $|D| < 10^{11}$, and Mosunov and Jacobson [63] those with $|D| < 2^{40}$, following Buell [12], whose tabulation reached $|D| < 2.2 \cdot 10^9$ [63, §1]. Mosunov and Jacobson's tabulation can be downloaded from the LMFDB [51]. Theorem 1.2 adds a computation of the list $\mathcal{L}$ that does not use these tabulations, and we deposit the list in full, with the programs that produce it. PARI/GP's `qfbclassno` confirms the class number of every pair of $\mathcal{L}$, and a second argument checks the exclusion $h(D) > 3000$ for the other fundamental discriminants with $|D| < 1.98 \cdot 10^{10}$ (Appendix A.3).

We chose the bound 3000 so that `qfbclassno`, which is documented to give unconditionally correct results for $|D| < 2 \cdot 10^{10}$ [70], could check every class number in $\mathcal{L}$. Under (H), the tabulation of Mosunov and Jacobson reaches much further: with Corollary 2.3 it determines the fields of every class number up to 17 357 (Remark 4.2). The proof of Theorem 1.1 under (K) uses their tabulation, through the smallest value of $L(1, \chi_D)$ in it, for $1.98 \cdot 10^{10} \le |D| < 2^{40}$ (Section 1.3).

We compare Theorem 1.3 with other lower bounds for $h(D)$ from zero-free regions. Explicit bounds of this kind are classical: if $\zeta_K(s_0) \le 0$ for some real $s_0 \in [1/2, 1)$, then $L(1, \chi_D) \ge (1 - s_0)|D|^{(s_0-1)/2}(1 - 2\pi|D|^{-s_0/2})$ [53, Proposition A] (see also [34, Lemma 1]), and with $s_0 = 1 - 2/\log|D|$, which (H) allows when $|D| \ge e^{16}$, this gives $h(D) > 0.2328\sqrt{|D|}/\log|D|$. Theorem 1.3 holds for every D, with the smaller constant $1/5$ in place of 0.2328. For comparison, if $|D| > e^{11.2}$ and $L(s, \chi_D)$ has no zero with $1 - 1/(4\log|D|) \le s < 1$, then $h(D) > (0.655/(\pi e))\sqrt{|D|}/\log|D| \approx 0.0767\sqrt{|D|}/\log|D|$ [23, Lemma 3]. A remark in OpenAI's manuscript on Zilber–Pink for curves in $\mathcal{A}_2$ [68, Remark 2.3] uses the half-plane $\operatorname{Re} s > 7/8$ of [66] only through the real zero-free interval $(1 - r/\log|D|, 1)$, with a fixed $r > 0$, in the proof of [15, Theorem 2], and obtains $h(D) \gg \sqrt{|D|}/\log(2|D|)$ with an effective constant whose value it does not give. In another manuscript of the same release [67], which gives a different proof of the weaker half-plane $\operatorname{Re} s > 11/12$, OpenAI notes that a fixed zero-free half-plane gives $h(D) \gg \sqrt{|D|}/\log\log|D|$ by Littlewood's method [49], with a computable implied constant that it does not state [67, §1]. The repository zeta-lab states such a bound with an explicit constant [48, Theorem A]: $h(D) \ge \sqrt{|D|}/(10\pi\log\log|D|)$ for every fundamental discriminant $D < 0$, assuming that $L(s, \chi_D)$ has no zero with real part greater than $7/8$. This is stronger than Theorem 1.3 for $|D| \ge 8.4 \cdot 10^7$, by a factor that tends to infinity with $|D|$, but Theorem 1.3 assumes only (H), a condition on real zeros. Where our computation ends, Corollary 2.4 is the sharper bound: it gives $h(D) \ge 3002$ for $|D| \ge 1.98 \cdot 10^{10}$, while the table of [48, Theorem C] gives $h(D) > 2000$ for $|D| > 22\,320\,645\,193$.

Hypothesis (H) also completes several classifications that were known only up to one possible exception (Section 5): the imaginary quadratic fields whose class group has exponent 2, 4 or 8 (Theorem 5.2), Yokoi's primes $p \equiv 1 \pmod 4$ for which $\mathbb{Q}(\sqrt{p})$ has class number 1 and fundamental unit $\varepsilon_p < 2p$ (Theorem 5.5 (b)), the real quadratic fields of extended Richaud–Degert type with class number 1 (Theorem 5.5 (a)), and the real quadratic fields of class number 1 or 2 whose continued fraction has period at most 24 (Theorem 5.9). For the real quadratic fields among these, Hecke's argument bounds the product of the class number and the regulator from below, and the families in question have small regulators. For exponent 2, a zero-free interval of this kind already completes the list by Weinberger's theorem [82, Theorem 1], and with it Euler's list of 65 idoneal numbers, as the OpenAI manuscript on the half-plane $\operatorname{Re} s > 11/12$ points out [67, §1]. A real quadratic field $\mathbb{Q}(\sqrt{d})$

is of extended Richaud–Degert type if $d = m^2 + r$ is squarefree, $r \mid 4m$, and $-m < r \le m$ or $r = \pm 4m/3$. We also show that, under (H), Mollin's list of those of class number 2 [57] is complete (Theorem 5.5 (d)), and determine those with 2-torsion class group (Theorem 5.6); Louboutin, Mollin and Williams [54] had listed those with class group of exponent 2. Kopp and Lagarias proved, by an ineffective argument, that only finitely many of the unit-generated real quadratic orders $\mathbb{Z}[(n + \sqrt{n^2 \pm 4})/2]$ have a 2-torsion class group [43, Theorem 1.5], and asked for all of them [43, Problem 1.7]. Theorem 5.7 determines them under (H).

Theorem 5.11 lists, under (K), the imaginary quadratic orders of class number at most 1095, and so the singular moduli of degree at most 1095; for class number at most 1000 these were determined under the generalized Riemann hypothesis by Cremona and Sutherland [18, §5], and for class number at most 100 without any hypothesis by Klaise [42, §4]. The bound 1095 comes from Table 2: the lower bound B of Proposition 2.2 satisfies $B(2 \cdot 10^9, 7/8) > 1095.43$, so under (H) every field with $|D| \ge 2 \cdot 10^9$ has $h(D) \ge 1096$, and every field of class number at most 1095 has $|D| < 2 \cdot 10^9$. Under (K) the same follows from Theorem 1.1: the largest $|D|$ of a field of class number at most 1095 is 283 506 787. Some of the published lists that Theorems 5.9 and 5.11 recover contain errors, which Remarks 5.10 and 5.12 record.

N	fields with $h(D) \le N$	largest $\lvert D\rvert$ with $h(D) \le N$	its class number
100	42 272	2 383 747	98
500	1 028 749	55 171 603	494
1000	4 115 897	227 932 027	996
1500	9 245 562	562 394 347	1489
2000	16 453 493	1 062 350 083	1978
2500	25 708 497	1 595 778 787	2496
3000	36 993 853	2 562 211 723	2976

TABLE 1. Imaginary quadratic fields of class number at most N, assuming (K) (Theorem 1.1).

1.3. **The idea of the proof.** Under (H) the proof has two parts, and neither is new in method. The first is analytic. By Hecke's integral representation of the Dedekind zeta function $\zeta_K(s) = \zeta(s)L(s, \chi_D)$, if $\zeta_K(\beta) \le 0$ at some real $\beta \in (0, 1)$ then $h(D)$ is at least an explicit function $B(|D|, \beta)$, which increases with $|D|$ (Proposition 2.2). Under (H), $\zeta_K(\beta) \le 0$ for every $\beta \in [7/8, 1)$, because $\zeta(\beta) < 0$ while $L(\beta, \chi_D) \ge 0$. Evaluating B in interval arithmetic gives $h(D) \ge 3002$ whenever $|D| \ge 1.98 \cdot 10^{10}$ (Corollary 2.4). Because (H) excludes real zeros from a fixed interval, β can be kept fixed, and $B(|D|, \beta)$ then grows like $|D|^{\beta/2}$. Taking instead $\beta = 1 - 2/\log|D|$, which lies in $[7/8, 1)$ when $|D| \ge e^{16}$, gives Theorem 1.3 for these $|D|$ (Proposition 2.6), and the computation below gives it for the others.

The second part is a computation over the 6 018 478 352 fundamental discriminants D with $|D| < 1.98 \cdot 10^{10}$, and it assumes nothing. For 98.7% of them, $h(D) > 3000$ follows at once from the class of a prime ideal of small norm. Genus theory gives $2^{\omega(D)-1}$ classes of order dividing 2, where $\omega(D)$ is the number of primes dividing D, and the cyclic group generated by a class contains at most two of them. So a class of order larger than $2 \cdot 3000/2^{\omega(D)-1}$ forces $h(D) > 3000$ (Lemma 3.2). For the remaining 77 495 152 discriminants the reduced forms are counted (Lemma 3.1).

Under (K), the analytic part applies only for $\log|D| \geq 35.0131$: there (K) gives $\zeta_K(\beta) \leq 0$ at $\beta = 1 - 1193/(200000\log|D|)$, and Proposition 2.2 gives $h(D) \geq 3562$ (Proposition 2.7). For $1.98 \cdot 10^{10} \leq |D| < 2^{40}$, the smallest value of $L(1,\chi_D)$ in the tabulation of Mosunov and Jacobson gives $h(D) > 7600$. For $2^{40} \leq |D| < e^{35.0131}$, a sieve in the manner of Watkins shows $h(D) > 3000$ (Lemma 3.4). It rests on the fact that the number of reduced forms with first coefficient a, where $4a^2 < |D|$, depends only on the factorization of a and on how its prime factors split in K (Lemmas 3.1 and 3.3).

Section 2 proves the lower bound and its consequences under (H), and under (K) for $\log|D| \geq 35.0131$ (Proposition 2.7). Section 3 gives the two lemmas on binary quadratic forms that decide, for a single D, whether $h(D) \leq N$, describes the program, and ends with the sieve (Section 3.2). The two sections are independent. Section 4 proves Theorems 1.1 to 1.3. Section 5 gives further consequences of (H): class groups of exponent dividing 8, real quadratic fields and orders with a small fundamental unit or a short period, and, under (K), imaginary quadratic orders. Appendix A describes the computations, and Appendix B gives $F(h)$ and the largest $|D|$ for every $h \leq 3000$.

1.4. **Notation.**

D	a fundamental discriminant $D < 0$
K, $h(D)$	the field $\mathbb{Q}(\sqrt{D})$ and its class number
χ_D	the Kronecker symbol $(D/\cdot)$, a real primitive character of conductor $\lvert D\rvert$
$\omega(D)$	the number of distinct primes dividing D
$\mathrm{Cl}(D)$	the class group of K, identified with the classes of primitive positive definite binary quadratic forms of discriminant D
$F(h)$	the number of imaginary quadratic fields of class number h
$B(x,\beta)$	the lower bound of Proposition 2.2
N	a bound for the class number (Section 3)
(H)	the hypothesis on real zeros stated above
(K)	Kimura's bound for real zeros stated above
$\mathcal{L}$	the list of pairs (D,h) of Theorem 1.2

2. A lower bound for the class number from a zero-free interval

In this section $D < -4$ is a fundamental discriminant, $K = \mathbb{Q}(\sqrt{D})$, $\chi = \chi_D$ and $h = h(D)$; the field K has $w = 2$ roots of unity. Write

$$\zeta_K(s) = \zeta(s)L(s,\chi) = \sum_{n\geq 1} a(n)n^{-s}, \qquad a(n) = \sum_{e\mid n}\chi(e),$$

so that $a(n)$ is the number of ideals of norm n of the ring of integers of K. In particular $a(n) \geq 0$, and $a(m^2) \geq 1$ for every $m \geq 1$, since the ideal (m) has norm m^2. Put $f(t) = \sum_{n\geq 1} a(n)\exp(-2\pi nt/\sqrt{|D|})$ for $t > 0$.

Lemma 2.1 (Hecke)**.** *For every $s \in \mathbb{C} \smallsetminus \{0,1\}$,*

$$\Lambda(s) := \Big(\frac{\sqrt{|D|}}{2\pi}\Big)^s \Gamma(s)\,\zeta_K(s) = \frac{h}{2}\Big(\frac{1}{s-1} - \frac{1}{s}\Big) + \int_1^\infty f(t)\,(t^s + t^{1-s})\frac{dt}{t}.$$

Proof. Let $Q_1, \dots, Q_h$ be positive definite forms of discriminant D, one in each class, and $\theta_j(t) = \sum_{v\in\mathbb{Z}^2}\exp(-2\pi Q_j(v)t/\sqrt{|D|})$ for $t > 0$. Write $Q_j(v) = \frac{1}{2}v^{\mathsf{T}}A_jv$ with $A_j = \left(\begin{smallmatrix} 2a_j & b_j \\ b_j & 2c_j \end{smallmatrix}\right)$, so that $\det A_j = |D|$. Poisson summation for the positive definite matrix $M = tA_j/\sqrt{|D|}$, for which $\det M = t^2$ and $v^{\mathsf{T}}M^{-1}v = 2Q_j(-y,x)/(t\sqrt{|D|})$ when $v = (x,y)$, gives $\theta_j(t) = t^{-1}\theta_j(1/t)$, because $(x,y) \mapsto (-y,x)$ permutes $\mathbb{Z}^2$. For fundamental D every form of

discriminant D is primitive, and by the classical correspondence between classes of forms and ideal classes (see [45, pp. 287–288] and [85, §4.3, p. 42]), each $n \geq 1$ is represented exactly $w\,a(n) = 2a(n)$ times in all by $Q_1, \dots, Q_h$; hence $\sum_j \theta_j(t) = h + 2f(t)$ and

$$f(1/t) = t\,f(t) + \tfrac{h}{2}\,(t-1) \qquad (t > 0).$$

For $\operatorname{Re} s > 1$, the identity $\Gamma(s)(2\pi n/\sqrt{|D|})^{-s} = \int_0^\infty e^{-2\pi n y/\sqrt{|D|}} y^s\,dy/y$ gives $\Lambda(s) = \int_0^\infty f(y) y^s\,dy/y$. Split the integral at $y = 1$ and substitute $y = 1/t$ on $(0,1)$. The relation above turns $\int_0^1 f(y) y^s\,dy/y$ into $\int_1^\infty f(t) t^{1-s}\,dt/t + \frac{h}{2}\int_1^\infty (t^{1-s} - t^{-s})\,dt/t$, and the last integral is $1/(s-1) - 1/s$. Since $f(t) = O(e^{-2\pi t/\sqrt{|D|}})$ as $t \to \infty$, the integral in the statement is an entire function of s, and the identity extends to all $s \neq 0, 1$. □

The next proposition is Hecke's argument (see [45] and [28]), with explicit constants. Stark [77, Lemma 4] gave a lower bound of the same kind for the residue at $s = 1$ of the Dedekind zeta function of any number field other than $\mathbb{Q}$, in terms of a real zero close to 1 if there is one.

Proposition 2.2. *If $0 < \beta < 1$ and $\zeta_K(\beta) \leq 0$, then $h \geq B(|D|, \beta)$, where, for real $x > 0$,*

$$\begin{aligned} B(x,\beta) &:= 2\beta(1-\beta) \sum_{m \geq 1} \int_1^\infty e^{-\alpha_m t}\bigl(t^\beta + t^{1-\beta}\bigr)\frac{dt}{t} \\ &= 2\beta(1-\beta) \sum_{m \geq 1} \Bigl(\alpha_m^{-\beta}\,\Gamma(\beta, \alpha_m) + \alpha_m^{\beta-1}\,\Gamma(1-\beta, \alpha_m)\Bigr), \end{aligned}$$

with $\alpha_m = 2\pi m^2/\sqrt{x}$ and $\Gamma(s,a) = \int_a^\infty e^{-u} u^{s-1}\,du$ the upper incomplete gamma function. For fixed β, $B(x,\beta)$ is a nondecreasing function of x.

Proof. Since $(\sqrt{|D|}/2\pi)^\beta \Gamma(\beta) > 0$, we have $\Lambda(\beta) \leq 0$. At $s = \beta$ the first term of Lemma 2.1 is $-h/(2\beta(1-\beta))$, so $h \geq 2\beta(1-\beta)\int_1^\infty f(t)(t^\beta + t^{1-\beta})\,dt/t$. Because $a(n) \geq 0$ and $a(m^2) \geq 1$, we have $f(t) \geq \sum_{m \geq 1} e^{-\alpha_m t}$, with α_m taken at $x = |D|$, which gives $h \geq B(|D|, \beta)$. The substitution $u = \alpha_m t$ gives the second expression for B. For fixed $t \geq 1$ each term $e^{-\alpha_m t}$ increases with x, hence the monotonicity. □

Corollary 2.3. *Assume* (H). *For every fundamental discriminant $D < -4$ and every $\beta \in [7/8, 1)$, $h(D) \geq B(|D|, \beta)$.*

Proof. For real $\sigma > 0$, $L(\sigma, \chi)$ is real and depends continuously on σ, and $L(1,\chi) = \pi h/\sqrt{|D|} > 0$. By (H) it has no zero in $(7/8, 1)$, so $L(\sigma,\chi) > 0$ for $7/8 < \sigma \leq 1$, and $L(7/8, \chi) \geq 0$. Since $\zeta(\beta) < 0$ for $0 < \beta < 1$, it follows that $\zeta_K(\beta) \leq 0$ for every $\beta \in [7/8, 1)$, and Proposition 2.2 applies. □

Corollary 2.4. *Assume* (H). *For each row (X, β) of Table 2, every fundamental discriminant $D < 0$ with $|D| \geq X$ satisfies the inequality in the last column. In particular $h(D) \geq 3002$ whenever $|D| \geq 1.98 \cdot 10^{10}$.*

Proof. The discriminants -3 and -4 lie below every X in the table. For $D < -4$, Corollary 2.3 with the β of the table and the monotonicity in Proposition 2.2 give $h(D) \geq B(|D|, \beta) \geq B(X, \beta)$, and $h(D)$ is an integer. The bounds for $B(X, \beta)$ in Table 2 were computed in interval arithmetic by the methods of Appendix A.1. The table gives the smallest of their lower bounds, rounded down, and an upper bound, rounded up. □

X	β	$B(X,\beta)>$	$B(X,\beta)<$	consequence for $\lvert D\rvert \ge X$
$2\cdot 10^9$	$7/8$	1095.4354	1095.4356	$h(D)\ge 1096$
$1.98\cdot 10^{10}$	$177/200$	3001.0442	3001.0444	$h(D)\ge 3002$

TABLE 2. Values of $B(X,\beta)$, rounded outward to four decimals, and the bounds they give.

The proof of Theorem 1.1 under (H) uses the second row of Table 2. The first row shows that, under (H), every field of class number at most 1095 has $|D|<2\cdot 10^9$, and Theorem 1.1 gives the same under (K) (Section 1.2). Proposition 5.1, Theorem 5.11 and Appendix A.3 (2) use this range.

Remark 2.5. The choice $\beta=177/200=7/8+1/100$ for $X=1.98\cdot 10^{10}$ matters: with $\beta=7/8$ one only has $B(1.98\cdot 10^{10},7/8)<2991.27$. Among the values $\beta=7/8+k/400$ with $0\le k\le 30$, the value $k=4$ gives the largest bound. At $X=2\cdot 10^9$, by contrast, $\beta=7/8$ gives the largest bound among $\beta=7/8+k/160$ with $0\le k\le 19$.

Proposition 2.6. *Assume* (H). *If $D<0$ is a fundamental discriminant with $|D|\ge e^{16}$, then $h(D)>\sqrt{|D|}/(5\log|D|)$.*

Proof. Let $\lambda=\log|D|\ge 16$ and $\beta=1-2/\lambda$, so that $\beta\in[7/8,1)$ and $h\ge B(|D|,\beta)$ by Corollary 2.3. Keep in $B(|D|,\beta)$ only the term $m=1$ and the part t^β of the integrand: with $\alpha=\alpha_1=2\pi/\sqrt{|D|}$,

$$h\ge 2\beta(1-\beta)\,\alpha^{-\beta}\,\Gamma(\beta,\alpha).$$

Since $\int_0^\alpha t^{\beta-1}e^{-t}\,dt\le\alpha^\beta/\beta$, we have $\alpha^{-\beta}\Gamma(\beta,\alpha)\ge\alpha^{-\beta}\Gamma(\beta)-1/\beta$. Here $\Gamma(\beta)\ge 1$, because Γ decreases on $(0,1]$, and

$$\alpha^{-\beta}=\frac{\sqrt{|D|}}{2\pi}\exp\Big(-1+\frac{2\log(2\pi)}{\lambda}\Big)\ge\frac{\sqrt{|D|}}{2\pi e}.$$

With $1-\beta=2/\lambda$ and $\beta\ge 7/8$ this gives

$$h\ge\frac{2\beta}{\pi e}\cdot\frac{\sqrt{|D|}}{\lambda}-\frac{4}{\lambda}\ge\frac{7}{4\pi e}\cdot\frac{\sqrt{|D|}}{\lambda}-\frac{4}{\lambda}.$$

Since $7/(4\pi e)=0.20492\ldots>0.2049$, the right-hand side exceeds $\sqrt{|D|}/(5\lambda)$ as soon as $0.0049\sqrt{|D|}>4$, which holds because $\sqrt{|D|}\ge e^8>2980$. □

Proposition 2.7. *Assume* (K). *Then $h(D)\ge 3562$ for every fundamental discriminant $D<0$ with $\log|D|\ge 35.0131$.*

Proof. Put $\delta=1193/200000$, $\lambda=\log|D|\ge 35.0131$ and $\beta=1-\delta/\lambda$. By (K), every real zero $\sigma\in(0,1)$ of $L(s,\chi)$ satisfies $(1-\sigma)\lambda\ge\delta$, that is, $\sigma\le\beta$. As in the proof of Corollary 2.3, it follows that $L(\beta,\chi)\ge 0$ and $\zeta_K(\beta)\le 0$, so $h\ge B(|D|,\beta)$ by Proposition 2.2. As in the proof of Proposition 2.6, but with all the terms $m\le 10^4$, we keep only the part t^β of the integrand and use $\alpha_m^{-\beta}\Gamma(\beta,\alpha_m)\ge\alpha_m^{-\beta}\Gamma(\beta)-1/\beta$. With $\alpha_m^{-\beta}=(\sqrt{|D|}/2\pi)^\beta m^{-2\beta}$ and $\beta\Gamma(\beta)=\Gamma(1+\beta)$ this gives

$$h\ge 2(1-\beta)\Big(\Gamma(1+\beta)\Big(\frac{\sqrt{|D|}}{2\pi}\Big)^\beta\sum_{m\le 10^4}m^{-2\beta}-10^4\Big).$$

Here $\Gamma(1+\beta)\ge\Gamma(2-\delta/35.0131)$, because $1+\beta\ge 2-\delta/35.0131$ and Γ increases on $[1.47,\infty)$ (its minimum on $(0,\infty)$ is at $1.4616\ldots$). Next, $(\sqrt{|D|}/2\pi)^\beta=(\sqrt{|D|}/2\pi)\exp(-\delta/2+$

$\delta\log(2\pi)/\lambda) \ge e^{-\delta/2}\sqrt{|D|}/2\pi$. Finally, $\sum_{m\le 10^4} m^{-2\beta} \ge \sum_{m\le 10^4} m^{-2} > \pi^2/6 - 10^{-4}$, since $\sum_{m>M} m^{-2} < 1/M$. With $1-\beta = \delta/\lambda$ these bounds give $h \ge g(|D|)$, where

$$g(x) = \frac{2\delta}{\log x}(C\sqrt{x} - 10^4), \qquad C = \frac{\Gamma(2-\delta/35.0131)\, e^{-\delta/2}\,(\pi^2/6 - 10^{-4})}{2\pi} \ge 0.2609850699.$$

The derivative of g has the sign of $C\sqrt{x}\,(\frac{1}{2}\log x - 1) + 10^4$, which is positive for $\log x > 2$, so g increases on $[e^{35.0131}, \infty)$ and $h \ge g(e^{35.0131})$. Ball arithmetic in Arb [39], at 256 bits, gives $g(e^{35.0131}) \in [3561.2977, 3561.2978]$, and h is an integer. □

At the same threshold, Kimura's Theorem 10 gives $h(D) > 2000$ [41]. Proposition 2.2 gives more because it keeps the terms with $m \ge 2$, which bring in the factor $\pi^2/6$: with the same β, the bound of Louboutin [53, Proposition A] gives only $h(D) > 2167$ at $\log|D| = 35.0131$.

3. Deciding whether $h(D) \le N$

Let $D < 0$ be a fundamental discriminant and $N \ge 1$ an integer. Two classical facts about binary quadratic forms decide whether $h(D) \le N$, and, if so, give $h(D)$. The first counts the classes. The second shows $h(D) > N$ from a single class of large order, and it is the one that makes a computation over $6 \cdot 10^9$ discriminants feasible.

We identify $\mathrm{Cl}(D)$ with the group of proper equivalence classes of primitive positive definite forms $(a,b,c) = ax^2 + bxy + cy^2$ with $b^2 - 4ac = D$, under composition [16, §5.2]. Every form of discriminant D is primitive, since a prime ℓ dividing a, b and c would make D/ℓ^2 the discriminant of the form $(a/\ell, b/\ell, c/\ell)$, which is impossible for a fundamental discriminant. A form is *reduced* if $|b| \le a \le c$, with $b \ge 0$ when $|b| = a$ or $a = c$. Each class contains exactly one reduced form [16, §5.3].

Lemma 3.1. *The class number $h(D)$ is the number of reduced forms of discriminant D, and these satisfy $3a^2 \le |D|$. For a given a, they are the forms $(a, b, (b^2-D)/4a)$ with $b \in (-a, a]$, $b^2 \equiv D \pmod{4a}$, and $c \ge a$, with $b \ge 0$ if $c = a$. If $4a^2 < |D|$, then $c > a$ for every such b, and the number $\rho(a)$ of such b is the product of the following local factors* [81, (2) and Lemma 3]*: for each odd prime p with $p^e \parallel a$ and $e \ge 1$, the factor is $1 + \chi_D(p)$ if $p \nmid D$, 1 if $p \mid D$ and $e = 1$, and 0 if $p \mid D$ and $e \ge 2$; for $2^e \parallel a$, the factor is 1 if $e = 0$; if D is odd and $e \ge 1$, it is 2 if $D \equiv 1 \pmod 8$ and 0 otherwise; if D is even, it is 1 if $e \le 1$ and 0 if $e \ge 2$.*

Proof. For a reduced form, $|D| = 4ac - b^2 \ge 4a^2 - a^2 = 3a^2$. Since $b \mapsto b^2 \bmod 4a$ depends only on $b \bmod 2a$, the admissible $b \in (-a, a]$ correspond to the solutions of $b^2 \equiv D \pmod{4a}$ modulo $2a$. By the Chinese remainder theorem their number is the product of the numbers of solutions of $b^2 \equiv D \pmod{p^e}$ modulo p^e, for odd $p^e \parallel a$, and of $b^2 \equiv D \pmod{2^{e+2}}$ modulo 2^{e+1}. These local counts are the factors of the statement. For odd $p \nmid D$ the count is $1 + \chi_D(p)$ by Hensel's lemma. For odd $p \mid D$ we have $p \parallel D$, so a solution modulo p^2 would force $p^2 \mid D$. For odd D and $e \ge 1$, the congruence $b^2 \equiv D \pmod{2^{e+2}}$ is solvable only if $D \equiv 1 \pmod 8$, and then it has two solutions modulo 2^{e+1}. For $D = 4m$ even, $m \equiv 2$ or $3 \pmod 4$ and $b = 2b'$ with $b'^2 \equiv m \pmod{2^e}$. Finally $c = (b^2 + |D|)/4a \ge |D|/4a > a$ when $4a^2 < |D|$. □

Lemma 3.2. *Let $g \in \mathrm{Cl}(D)$ satisfy $g^k \neq 1$ for $1 \leq k \leq M$. Then $h(D) \geq M+1$, and $h(D) \geq (M+1)\,2^{\omega(D)-2}$ if $\omega(D) \geq 2$. In particular $h(D) > N$ if $M = N$, or if $\omega(D) \geq 2$ and $M = \lfloor 2N/2^{\omega(D)-1} \rfloor$.*

Proof. The order n of g is at least $M+1$. By Gauss's genus theory, the subgroup $\mathrm{Cl}(D)[2]$ of elements of order dividing 2 has order $2^{\omega(D)-1}$ [17, §3]. The intersection of $\mathrm{Cl}(D)[2]$ with the cyclic group $\langle g \rangle$ has order at most 2, so the subgroup $\langle g \rangle \mathrm{Cl}(D)[2]$ has order at least $n\,2^{\omega(D)-2}$. If $M = \lfloor 2N/2^{\omega(D)-1} \rfloor$, then $M+1 > 2N/2^{\omega(D)-1}$ and $(M+1)2^{\omega(D)-2} > N$. □

To verify the hypothesis of Lemma 3.2 we use Shanks's baby steps and giant steps. Put $m = \lceil \sqrt{M} \rceil$ and $t = \lceil M/m \rceil$, compute $g^0, \ldots, g^{m-1}$, and check that $g^j \neq 1$ for $1 \leq j < m$ and that $g^{mi} \notin \{g^0, \ldots, g^{m-1}\}$ for $1 \leq i \leq t$. Every $k \in [1, M]$ can be written $k = mi - j$ with $i = \lceil k/m \rceil \in [1, t]$ and $j = mi - k \in [0, m-1]$, so $g^k = 1$ would give $g^{mi} = g^j$. The test succeeds exactly when the order of g exceeds $mt \geq M$. Classes are compared through their reduced forms, and since c is determined by a, b and D, it suffices to compare (a, b).

3.1. **The decision procedure.** For each n in a range $[n_0, n_1)$ such that $D = -n$ is a fundamental discriminant, that is, such that the odd part of n is squarefree and $n \equiv 3 \pmod 4$, $n \equiv 4 \pmod{16}$ or $n \equiv 8 \pmod{16}$, the procedure does the following.

(1) Compute $\omega(D)$, and $M = N$ if $\omega(D) = 1$, $M = \lfloor 2N/2^{\omega(D)-1} \rfloor$ otherwise.
(2) If $M \geq 1$, for each of the first four primes $p < 2000$ with $\chi_D(p) = 1$, form the class g_p of the form $(p, b_p, (b_p^2 - D)/4p)$, where $b_p^2 \equiv D \pmod{4p}$ ($b_p = 1$ if $p = 2$), and apply the test above to g_p and M. If it succeeds for some p, then $h(D) > N$ by Lemma 3.2; stop.
(3) Otherwise count the reduced forms (Lemma 3.1) and record $(D, h(D))$ if $h(D) \leq N$.

Every D in the range is thus either recorded with its exact class number or shown to have $h(D) > N$. In step (3), the forms with $4a^2 < |D|$ are counted by the number $\rho(a)$ of Lemma 3.1, computed from the factorization of a, and the values of b are listed only for the other a. Composition is Cohen's Algorithm 5.4.7, followed by reduction by Algorithm 5.4.2 [16]. The classes g_p are those of the prime ideals above split primes of small norm. They are tried first because, for most D, the class of the smallest of them already has order larger than M. Below $1.98 \cdot 10^{10}$ the class of the smallest split prime decides about nine tenths of the discriminants, and the classes of the next three decide most of the others (Appendix A.2).

3.2. **A sieve for $2^{40} \leq |D| < e^{35.0131}$.** Lemma 3.4 below gives $h(D) > 3000$ for $2^{40} \leq |D| < e^{35.0131}$, that is, for $2^{40} \leq |D| \leq 1\,606\,926\,912\,624\,720$, the range between the tabulation of Mosunov and Jacobson and Proposition 2.7. The method is the sieve of Watkins [81, §8], as Kimura uses it [41, §5]; he ran it with the bound 2000 for $6 \cdot 2^{28} \leq |D| \leq 1.65 \cdot 10^{15}$ [41, Proposition 13]. By Lemma 3.1, $h(D)$ is at least the number $\sum \rho(a)$ of reduced forms (a, b, c) with $4a^2 < |D|$, and $\rho(a)$ is a product of local factors, the factor at a prime p depending only on how p splits in K and on the exponent of p in a. So the reduced forms whose first coefficient has only prime factors up to some P can be counted for a whole residue class of $|D|$ at once, and multiplying these first coefficients by primes $q > P$ that split in K gives further forms. The next lemma makes this precise.

Lemma 3.3. *Let $D < -4$ be a fundamental discriminant, $1 \le Y \le \sqrt{|D|}/2$, and P a prime. For $0 < x \le Y$ let $W(x)$ be the sum of $\rho(a)$ over the integers $1 \le a \le x$ whose prime factors are all at most P. Let $z > P$, and let $\mathcal{T}$ be a set of τ primes q in $(P, z]$ with $\chi_D(q) = 1$. Then*

$$h(D) \ge V(\tau, z) := W(Y) + \sum_{k \ge 1} W(Y/z^k) \sum_{j=1}^{k} 2^j \binom{\tau}{j} \binom{k-1}{j-1}.$$

Proof. Since $D < -4$ is fundamental, $D \ne -4a^2$, so every integer $a \le Y$ has $4a^2 < |D|$, and Lemma 3.1 gives $h(D) \ge \sum_{a \le Y} \rho(a)$. By the local factors of Lemma 3.1, ρ is multiplicative, and $\rho(q^e) = 2$ for every prime q with $\chi_D(q) = 1$ and every $e \ge 1$. For $k \ge 0$ let U_k be the set of products $u = q_1^{e_1} \cdots q_j^{e_j}$ of distinct primes $q_i \in \mathcal{T}$ with $e_i \ge 1$ and $e_1 + \cdots + e_j = k$, so that $U_0 = \{1\}$. For $k \ge 1$, exactly $\binom{\tau}{j}\binom{k-1}{j-1}$ elements of U_k have j prime factors (choose the j primes, then a composition of k into j positive parts), and each of them has $\rho(u) = 2^j$. If $u \in U_k$ and $a \le Y/z^k$ has all its prime factors at most P, then $au \le Y$, since $u \le z^k$, and $\rho(au) = \rho(a)\rho(u)$. The products au obtained in this way are distinct, since a is the part of au made of primes at most P. As $\rho \ge 0$, the sum of $\rho(au)$ over them, which is $V(\tau, z)$, is at most $\sum_{a \le Y} \rho(a) \le h(D)$. □

Lemma 3.3 extends Kimura's test (iii) [41, §5], which, with 3000 for his bound N, is $(2\tau^2 + 1)W(Y/z^2) > 3000$ and so uses at most the terms $k \le 2$ of the sum in the lemma. The bound $V(\tau, z)$ does not decrease as τ grows, and $V(0, z) = W(Y)$. Let $P \ge 5$ be a prime and $Q = 16 \cdot 9 \cdot \prod_{5 \le p \le P} p$. If D is a fundamental discriminant, then $|D| \bmod 16$ is one of 3, 7, 11, 15, 4, 8, and $9 \nmid |D|$. On a residue class of $|D|$ modulo Q, the way in which each prime $p \le P$ splits in K is fixed: for $p = 2$ by $|D| \bmod 16$, for $p = 3$ by $|D| \bmod 9$, and for $5 \le p \le P$ by $|D| \bmod p$. So W depends only on the splitting types of the primes $p \le P$, and hence only on the class. For an interval $[A, A']$ we put $Y = \lfloor \sqrt{\lfloor A/4 \rfloor} \rfloor$, so that $4Y^2 \le A$, and treat every $n \in [A, A']$ with $n \bmod 16 \in \{3, 7, 11, 15, 4, 8\}$ and $9 \nmid n$, one residue class modulo Q at a time, as follows, where $D = -n$.

(1) If $W(Y) > 3000$ on the class of n, then $h(D) > 3000$ for every n of the class with D fundamental, by Lemma 3.3 with $\mathcal{T}$ empty.

(2) Otherwise, let $q_1 < q_2 < \cdots$ be the primes greater than P, and for each $j \ge 1$ with $q_{8j} < 4096$ let τ_j be the number of $i \le 8j$ with $\chi_D(q_i) = 1$. If $V(\tau_j, q_{8j}) > 3000$ for some j, then $h(D) > 3000$ when D is fundamental, by Lemma 3.3.

(3) Otherwise, compute $\sum_{4a^2 < n} \rho(a)$, stopping as soon as it exceeds 3000. If it does, then $h(D) > 3000$ when D is fundamental, by Lemma 3.1.

Step (1) depends only on the class of n, and steps (2) and (3) on n itself. An n for which none of the three steps shows $h(D) > 3000$ is left over. For fundamental D the bounds of steps (1) and (2) are at most $\sum_{a \le Y} \rho(a)$, as the proof of Lemma 3.3 shows, and so at most the sum in step (3). The choice of the primes q_{8j} and of the limit 4096 in step (2) therefore affects only the running time.

Lemma 3.4. *Every fundamental discriminant D with $2^{40} \le |D| \le 1\,606\,926\,912\,624\,720$ has $h(D) > 3000$.*

Proof. We applied the procedure above to the eleven intervals of Table 3, which cover $[2^{40}, 1\,606\,926\,912\,624\,720]$, with the primes P and the numbers Y given there. Every n for which $-n$ is a fundamental discriminant lies in one of the residue classes treated, and no n was left over (Appendix A.6). □

4. The fields of class number at most 3000

Proof of Theorem 1.2. The procedure of Section 3.1, with $N = 3000$, was run on every n with $3 \le n < 1.98 \cdot 10^{10}$; there are 6 018 478 352 fundamental discriminants in this range. It recorded the 36 993 853 pairs of $\mathcal{L}$. By Lemma 3.1, each recorded pair is $(D, h(D))$ with $h(D) \le 3000$, and by Lemmas 3.1 and 3.2 every fundamental discriminant in the range that was not recorded has $h(D) > 3000$. □

The list $\mathcal{L}$ is too long to print. It is deposited in [38] together with the program, which reproduces it in about 17 hours of computing time (Appendix A.2).

Proof of Theorem 1.1. Let $D < 0$ be a fundamental discriminant. The following four ranges cover all D, because $1\,606\,926\,912\,624\,720 < e^{35.0131} < 1\,606\,926\,912\,624\,721$ (by interval arithmetic).

(1) $|D| < 1.98 \cdot 10^{10}$. By Theorem 1.2, an unconditional computation, $h(D) \le 3000$ if and only if $(D, h(D)) \in \mathcal{L}$.
(2) $1.98 \cdot 10^{10} \le |D| < 2^{40}$. By the unconditional tabulation of Mosunov and Jacobson, $L(1, \chi_D) \ge 0.170695$ for these D [63, §7.1] (see Appendix A.3 (3)), so $h(D) = \sqrt{|D|}\, L(1, \chi_D)/\pi > 7600$.
(3) $2^{40} \le |D| \le 1\,606\,926\,912\,624\,720$. Here $h(D) > 3000$ by Lemma 3.4, a computation.
(4) $|D| \ge 1\,606\,926\,912\,624\,721$, that is, $\log|D| \ge 35.0131$. Here $h(D) \ge 3562$ by Proposition 2.7, which assumes (K).

So every imaginary quadratic field of class number at most 3000 has $(D, h(D)) \in \mathcal{L}$, and conversely each pair (D, h) of $\mathcal{L}$ has $h = h(D) \le 3000$ by Theorem 1.2. The numbers $F(h)$, the largest $|D|$ for each h, and the other numbers in the statement are read off from $\mathcal{L}$. □

For $h \le 2000$ the list of Theorem 1.1 is that of Kimura [41, Theorem 2], who derives it from (K) through his Theorem 10, with a sieve for $6 \cdot 2^{28} \le |D| \le 1.65 \cdot 10^{15}$ [41, Proposition 13] and the tabulation of Mosunov and Jacobson for smaller $|D|$. The proof above differs from his in its inputs: Proposition 2.7 takes the place of his Theorem 10, Theorem 1.2 covers $|D| < 1.98 \cdot 10^{10}$, and the sieve of Lemma 3.4, with the bound 3000, covers $2^{40} \le |D| < e^{35.0131}$.

Proof of Theorem 1.1 under (H). Let K be an imaginary quadratic field with $h(K) \le 3000$, and D its discriminant. If $|D| \ge 1.98 \cdot 10^{10}$, then $h(D) \ge 3002$ by Corollary 2.4, which is impossible. So $|D| < 1.98 \cdot 10^{10}$, and $(D, h(D)) \in \mathcal{L}$ by Theorem 1.2. Conversely, each pair (D, h) of $\mathcal{L}$ has $h = h(D) \le 3000$. The numbers $F(h)$, the largest $|D|$ for each h, and the other numbers in the statement are read off from $\mathcal{L}$. □

Proof of Theorem 1.3. For $|D| \ge e^{16}$ this is Proposition 2.6. Let $3 \le |D| < e^{16}$. The function $\sqrt{x}/\log x$ decreases on $(1, e^2]$ and increases on $[e^2, \infty)$, so $\sqrt{|D|}/(5\log|D|) < \max(\sqrt{3}/(5\log 3),\ e^8/80) < 37.27$, and the inequality holds if $h(D) \ge 38$. If $h(D) \le 37$, then $(D, h(D)) \in \mathcal{L}$ by Theorem 1.2, and among the pairs of $\mathcal{L}$ with $|D| < e^{16}$ and $h \le 37$ the smallest value of $5h \log|D|/\sqrt{|D|}$ is $1.9948\ldots$, at $D = -163$. □

Remark 4.1. For $h \le 100$ the numbers $F(h)$ and the largest $|D|$ for each h are those of Watkins [81], and for the 1500 odd $h \le 3000$ the numbers $F(h)$ are those of the table that accompanies the paper of Holmin, Jones, Kurlberg, McLeman and Petersen [35], whose computation assumes the generalized Riemann hypothesis. These odd class numbers account for 3 680 561 of the fields. For $h \le 1000$ the orders of Theorem 5.11 are those found by

Cremona and Sutherland [18, §5] under the generalized Riemann hypothesis: their published list has the same 6 450 424 discriminants with the same class numbers, and so the same fields. For $N = 1000$ and $N = 1500$, the pairs of $\mathcal{L}$ with $h \le N$ agree, by their SHA-256 checksums, with the lists of [48, Corollary D], which were computed with a different program from a different analytic bound (Appendix A.3 (1)). The pairs of $\mathcal{L}$ with $h \le 2000$ are exactly those of Kimura's list [41] (Appendix A.3 (1)), and for $N = 100, 200, \dots, 2000$ the number of fields of class number at most N and the largest $|D|$ among them are those of [25, Table 1]. Soundararajan [75] proved that $\sum_{h \le N} F(h) \sim (3\zeta(2)/\zeta(3))N^2$. The ratio of the two sides is 1.0026, 1.0020 and 1.0012 for $N = 1000$, 2000 and 3000, so the asymptotic is already close in this range.

Remark 4.2. The bound 3000 keeps the computation of Theorem 1.2 inside the range $|D| < 2 \cdot 10^{10}$, in which PARI/GP's function `qfbclassno` is documented to be unconditionally correct [70], so that the function can check every class number in $\mathcal{L}$ (Appendix A.3 (1)). Under (H), Corollary 2.3 and the monotonicity in Proposition 2.2 show that the unconditional tabulation of Mosunov and Jacobson [63], which covers all imaginary quadratic fields with $|D| < 2^{40}$, determines the fields of class number h for every $h < B(2^{40}, \beta)$, for any $\beta \in [7/8, 1)$. This reaches well beyond $h = 3000$: $B(2^{40}, 7/8) = 17\,357.92\ldots$, computed as in Appendix A.1 (a). Cremona and Sutherland [18, §5] used the tabulation in this way for $h \le 1000$, with the bound of [18, Proposition 19], which assumes the generalized Riemann hypothesis, and noted that the same approach could be used for $h \le 24\,416$.

Remark 4.3. Ranges (1)–(3) of the proof of Theorem 1.1 use no hypothesis. Together they show that, for every fundamental discriminant D with $|D| < e^{35.0131}$, that is, $|D| \le 1\,606\,926\,912\,624\,720$, we have $h(D) \le 3000$ if and only if $(D, h(D)) \in \mathcal{L}$. This extends Theorem 1.2 from $1.98 \cdot 10^{10}$ to $1\,606\,926\,912\,624\,720$, and rests on the computations of Theorem 1.2 and Lemma 3.4 and on the tabulation of Mosunov and Jacobson. Hypothesis (K) enters only for larger $|D|$, and by itself it does not give (H): it excludes real zeros of $L(s, \chi)$ only from $(1 - 1193/(200000 \log q), 1)$, which never contains $(7/8, 1)$, since $1193/200000 < (\log q)/8$. So Theorem 1.3 and the theorems of Section 5 other than Theorem 5.11 still assume (H).

5. Further consequences

Hypothesis (H) also completes several classifications that were known only up to one possible exceptional field, or under the generalized Riemann hypothesis, and it settles a problem of Kopp and Lagarias (Section 5.3). In Sections 5.1 to 5.4, (H) enters through a lower bound of the kind given by Proposition 2.2, and the rest is a finite computation; Section 5.5 needs only (K), through Theorem 1.1.

5.1. **Class groups of exponent dividing 8.** We show that, under (H), the lists of imaginary quadratic fields whose class group has exponent 2, 4 or 8 given by Elsenhans, Klüners and Nicolae [23] are complete. Let $E(D)$ denote the exponent of $\mathrm{Cl}(D)$.

Proposition 5.1. *Exactly* 1046 *fundamental discriminants* $D<0$ *with* $|D|<2\cdot 10^9$ *have* $E(D)\mid 8$. *Of these,* 9, 56, 203 *and* 778 *have* $E(D)=1$, 2, 4 *and* 8, *and the largest* $|D|$ *in the four cases are* 163, 5460, 435 435 *and* 430 950 520.

Proof. We use the procedure of Section 3.1 with $N=1095$, applied to every fundamental D with $|D|<2\cdot 10^9$ (Appendix A.3 (2)), together with the exponent check described below (Appendix A.2). The product of the first ten primes is $6\,469\,693\,230>2\cdot 10^9$, so $\omega(D)\le 9$ and the M of the procedure is at least $\lfloor 2190/2^8\rfloor=8$. Hence every D for which a class g_p passed the test has $g_p^8\neq 1$, and $E(D)\nmid 8$. Every other D has its class number counted. If $E(D)\mid 8$, then $\mathrm{Cl}(D)$ is a 2-group and $h(D)$ is a power of 2. For each of the 117 065 discriminants that were not decided by a class g_p and have $h(D)$ a power of 2, either a class g_p with $g_p^8\neq 1$ was found, or the orders of all $h(D)$ reduced forms were computed, which decides whether $E(D)$ divides 8. The discriminants with $E(D)\mid 8$ found in this way are those of the statement. □

Proposition 5.1 is part of an earlier unconditional result: Elsenhans, Klüners and Nicolae [23, Theorem 1] determined the imaginary quadratic fields with $|D|\le 3.1\cdot 10^{20}$ whose class group has exponent at most 8. The numbers of Proposition 5.1 are those of their Table 1, and their lists, published on their web page, contain the same fields. Without any hypothesis they show that at most one field with exponent 2, 4 or 8 is missing from their lists [23, Theorem 2]. For exponent 2 this was known earlier [82, Theorem 1], and the same theorem shows that there is no field of exponent 2 with $|D|>5460$ if $L(s,\chi_D)\neq 0$ for $1-1/(4\log|D|)\le s<1$ whenever $|D|$ exceeds the product $2\cdot 3\cdots 31\approx 2\cdot 10^{11}$ of the first eleven primes. Hypothesis (H) implies this condition, so under (H) the case of exponent 2 of Theorem 5.2 is Weinberger's theorem.

Theorem 5.2. *Assume* (H). *Exactly* 56, 203 *and* 778 *imaginary quadratic fields have a class group of exponent* 2, 4 *and* 8 *respectively, namely those of Proposition 5.1.*

Proof. Elsenhans, Klüners and Nicolae [23, Theorem 2] proved this assuming that no Siegel zeros exist. We use the numbering of `arXiv:1803.02056v1`. Their assumption enters through the hypothesis of their Lemma 3, which is used for $|D|>e^{11.2}$: if $L(s,\chi_D)\neq 0$ for $1-1/(4\log|D|)\le s<1$, then $h(D)>(0.655/(\pi e))\sqrt{|D|}/\log|D|$. It is used in the first part of their Theorem 4, which bounds $\omega(D)$ by 11, 24 and 58 for the three exponents, and in the bounds of their Section 5 search, which treats $|D|>10^6$; smaller $|D|$ come from tables. For $|D|>e^{11.2}$ the interval $[1-1/(4\log|D|),1)$ lies in $(7/8,1)$, so (H) gives the hypothesis of their Lemma 3. Their Section 5 search finds every field of the exponent sought whose class number satisfies the lower bound it uses [23, Section 5.2]. Besides the bound of their Lemma 3, it uses those of their Lemmas 4 and 5, which hold for all $|D|\ge A$ with at most one exception: $h(D)>0.655\,|D|^{1/2-1/\log A}/(\pi\log A)$ if $A\ge e^{11.2}$, and $h(D)$ is at least the smaller of $\sqrt{|D|}/(7.732\pi\log|D|)$ and a second term if $A\ge 10^6$. By their Lemma 2, a form of Tatuzawa's theorem [79], applied with $\varepsilon=1/\log A\le 1/11.2$, an exception to their Lemma 4 would give $L(s,\chi_D)$ a real zero in $(1-\varepsilon/4,1)\subset(7/8,1)$, which (H) excludes. Under (H), [23, Lemma 3] applies for $|D|>e^{11.2}$ and gives $h(D)>(0.655/(\pi e))\sqrt{|D|}/\log|D|\approx 0.0767\sqrt{|D|}/\log|D|$. This exceeds $\sqrt{|D|}/(7.732\pi\log|D|)\approx 0.0412\sqrt{|D|}/\log|D|$, and so the bound of [23, Lemma 5]. It is also at least the bound of [23, Lemma 4], which is $me^{-m}\cdot 0.655\sqrt{|D|}/(\pi\log|D|)$ with $m=\log|D|/\log A$, since $me^{-m}\le 1/e$. So the search of [23, Section 5] is complete under (H), with no exceptional discriminant. □

For $2\cdot 10^9 \le |D| \le 3.1\cdot 10^{20}$, Theorem 5.2 rests on the unconditional computation of [23, Theorem 1], and beyond it on the search of [23, Section 5]. We have repeated neither. The OpenAI manuscript on the half-plane $\operatorname{Re} s > 11/12$ [67, §1] also notes that the case of exponent 2 follows from that half-plane, [23, Theorem 2] and Weinberger's work, and deduces that Euler's list of 65 idoneal numbers is complete. Aiken, Ross, Shvydkoy and Xue [2, Theorem 1.3] extend the lists of [23] to all negative discriminants, non-fundamental ones included, under the same hypothesis on Siegel zeros.

5.2. **Real quadratic fields with a small fundamental unit.** Under (H), Hecke's argument bounds the product of the class number and the regulator of a real quadratic field from below, so the fields of small class number with a small fundamental unit have bounded discriminant. Here $\Delta > 1$ is a fundamental discriminant, $K = \mathbb{Q}(\sqrt{\Delta})$, $\chi = \chi_\Delta$ (an even real primitive character of conductor Δ), $h = h(K)$ is the class number, $\varepsilon_0 > 1$ the fundamental unit and $R = \log \varepsilon_0$. We write $\zeta_K(s) = \zeta(s)L(s,\chi) = \sum_{n\ge 1} a(n) n^{-s}$ as in Section 2, and K_0 for the modified Bessel function of the second kind of order 0, which is positive and decreasing on $(0,\infty)$.

Proposition 5.3. *If $0 < \beta < 1$ and $\zeta_K(\beta) \le 0$, then*

$$2hR \ge B_{\mathrm{R}}(\Delta,\beta) := \beta(1-\beta)\sum_{m\ge 1}\int_1^\infty 4K_0\bigl(2\pi m^2 t/\sqrt{\Delta}\bigr)\bigl(t^\beta + t^{1-\beta}\bigr)\frac{dt}{t},$$

and $B_{\mathrm{R}}(\Delta,\beta)$ is a nondecreasing function of Δ. Under (H)*, $\zeta_K(\beta) \le 0$ for every $\beta \in [7/8, 1)$.*

Proof. The function $\Lambda(s) = (\sqrt{\Delta}/\pi)^s \Gamma(s/2)^2 \zeta_K(s)$ satisfies $\Lambda(s) = \Lambda(1-s)$, has simple poles at $s = 0$ and $s = 1$ only, with residue $2hR$ at $s = 1$ by the class number formula, and decays rapidly in vertical strips. Since $\int_0^\infty 4K_0(2y) y^s\, dy/y = \Gamma(s/2)^2$, we have $\Lambda(s) = \int_0^\infty \Theta(y) y^s\, dy/y$ for $\operatorname{Re} s > 1$, where $\Theta(y) = \sum_{n\ge 1} a(n)\, 4K_0(2\pi n y/\sqrt{\Delta})$. Moving the line of integration in the Mellin inversion formula from $\operatorname{Re} s = 2$ to $\operatorname{Re} s = -1$ and using the functional equation gives $\Theta(1/y) = y\Theta(y) + 2hR(y-1)$. As in the proof of Lemma 2.1, it follows that $\Lambda(s) = 2hR(1/(s-1) - 1/s) + \int_1^\infty \Theta(t)(t^s + t^{1-s})\, dt/t$ for $s \ne 0, 1$, and the proof of Proposition 2.2 applies, with $\Theta(t) \ge \sum_{m\ge 1} 4K_0(2\pi m^2 t/\sqrt{\Delta})$. The last statement is proved as Corollary 2.3. □

Following Mollin and Williams [59], in the form stated by Louboutin [52, pp. 315–316] and used by Hoque and Kotyada [36, §1], we say that $\mathbb{Q}(\sqrt{d})$ is of *extended Richaud–Degert type* if d is squarefree and $d = m^2 + r$ with $m \ge 1$, $r \mid 4m$, and either $-m < r \le m$ or $r = \pm 4m/3$. Some authors allow only $-m < r \le m$.

Lemma 5.4. (a) *If $\mathbb{Q}(\sqrt{d})$ is of extended Richaud–Degert type, then $R < \log(4d + 3\sqrt{d} + 1) \le \log(4\Delta + 4\sqrt{\Delta} + 1)$.* (b) *If $\varepsilon_0 < 2\Delta$, then $R < \log(4\Delta + 4\sqrt{\Delta} + 1)$.*

Proof. (a) Suppose first that $-m < r \le m$, and write $4m = rk$. The number $\varepsilon = (2m^2 + r + 2m\sqrt{d})/|r| = (m|k|/2 \pm 1) + (|k|/2)\sqrt{d}$, the unit of Richaud and Degert, has norm 1, and it is an algebraic integer, which is clear if k is even. If k is odd, then $r = 4r'$ with r' odd, because $4 \nmid d$, so m is odd, $d \equiv 1 \pmod 4$, and $\varepsilon = (X + Y\sqrt{d})/2$ with X and Y odd. So $\varepsilon_0 \le \varepsilon$. Since $m^2 - m < d$, we have $m < \sqrt{d} + 1$; and $2m^2 + r = 2d - r < 2d + m$. Hence $\varepsilon < 4d + 3\sqrt{d} + 1$. If $r = \pm 4m/3$, then $m = 3j$ with j odd, $d \equiv 1 \pmod 4$, and $((3m \pm 2) + 3\sqrt{d})/2$ is a unit of norm 1, smaller than $3\sqrt{d} + 2.5$. (b) is clear, since $2\Delta < 4\Delta + 4\sqrt{\Delta} + 1$. □

The proofs below show that an inequality $B_{\mathrm{R}}(\Delta,7/8) > g(\Delta)$, with g increasing, holds for all Δ in a range. We cover the range by blocks $[X,Y]$ and check that $B_{\mathrm{R}}(X,7/8) > g(Y)$ on each block. This suffices, because $B_{\mathrm{R}}(\Delta,7/8) \geq B_{\mathrm{R}}(X,7/8)$ and $g(\Delta) \leq g(Y)$ for $\Delta \in [X,Y]$. The lower bounds for $B_{\mathrm{R}}(X,7/8)$ are computed with Arb's rigorous integration over finitely many m and a bounded range of t, which drops positive quantities only, as in Appendix A.1 (b). We call this a *block evaluation*. For large Δ we use two closed-form lower bounds instead. First, keeping only $m=1$ and $t \leq \sqrt{\Delta}/2\pi$, where $K_0(2\pi t/\sqrt{\Delta}) \geq K_0(1)$, gives $B_{\mathrm{R}}(\Delta,7/8) \geq \frac{1}{2}K_0(1)((\sqrt{\Delta}/2\pi)^{7/8}-1)$. Second, when $c = 2\pi/\sqrt{\Delta} \leq 1$, keeping only $m=1$ and the term $t^{\beta-1}$, and using $\int_0^\infty K_0(v)v^{\beta-1}\,dv = 2^{\beta-2}\Gamma(\beta/2)^2$ and $K_0(v) < \log(3/v)$ for $0 < v \leq 1$, gives

$$B_{\mathrm{R}}(\Delta,\beta) \geq G_0(\Delta) := \beta(1-\beta)\Big(2^\beta\Gamma(\beta/2)^2c^{-\beta} - \tfrac{4}{\beta}\big(\log(3/c)+1/\beta\big)\Big).$$

The inequality $K_0(v) < \log(3/v)$ follows from $K_0(v) < E_1(v/2)$, which holds because $\cosh s \geq e^s/2$, and from $E_1(x) < \log(1+1/x)$ [1, 5.1.20]: together they give $K_0(v) < \log(1+2/v) \leq \log(3/v)$ for $0 < v \leq 1$.

Theorem 5.5. *Assume* (H).

(a) A real quadratic field $\mathbb{Q}(\sqrt{d})$ of extended Richaud–Degert type has class number 1 *if and only if d is one of the* 43 *integers 2, 3, 5, 6, 7, 11, 13, 14, 17, 21, 23, 29, 33, 37, 38, 47, 53, 62, 69, 77, 83, 93, 101, 141, 167, 173, 197, 213, 227, 237, 293, 398, 413, 437, 453, 573, 677, 717, 1077, 1133, 1253, 1293, 1757.*

(b) Let $p \equiv 1 \pmod 4$ be a prime whose fundamental unit ε_p satisfies $\varepsilon_p < 2p$. Then $\mathbb{Q}(\sqrt{p})$ has class number 1 *if and only if p is one of the* 30 *primes 5, 13, 17, 29, 37, 41, 53, 61, 101, 149, 157, 173, 197, 269, 293, 317, 461, 509, 557, 677, 773, 797, 941, 1013, 1493, 1613, 1877, 2477, 2693, 3533.*

(c) The real quadratic fields with $\varepsilon_0 < 2\Delta$ and class number 1 *are exactly* 68*: the* 30 *fields of (b); the* 24 *fields with Δ one of 21, 33, 69, 77, 93, 133, 141, 213, 237, 341, 413, 437, 453, 573, 717, 917, 1077, 1133, 1253, 1293, 1757, 2453, 3053, 3317; and the* 14 *fields with Δ one of 8, 12, 24, 28, 44, 56, 92, 152, 188, 248, 332, 668, 908, 1592.*

(d) The real quadratic fields of extended Richaud–Degert type with class number 2 *are exactly the* 107 *fields $\mathbb{Q}(\sqrt{d})$ with d in the following list; the largest discriminant among them is* 14 405.

10, 15, 26, 30, 34, 35, 39, 42, 51, 65, 66, 78, 85, 87, 95, 102, 105, 110, 119, 122, 123, 138, 143, 146, 165, 182, 194, 203, 205, 215, 221, 222, 230, 258, 285, 287, 318, 327, 357, 362, 365, 395, 402, 429, 447, 482, 485, 527, 533, 623, 629, 635, 645, 678, 741, 782, 843, 885, 902, 957, 965, 1022, 1085, 1157, 1173, 1205, 1245, 1298, 1469, 1517, 1533, 1605, 1685, 1853, 1965, 2013, 2037, 2045, 2085, 2093, 2117, 2301, 2373, 2397, 2613, 2717, 2813, 3005, 3237, 3365, 3597, 3605, 3813, 4245, 4277, 4773, 4893, 5645, 5757, 5885, 5957, 6573, 7157, 7733, 8333, 9005, 14405.

The list in (a) is the one reprinted by Hoque and Kotyada [36, §1], who trace it to Mollin and Williams [59] and others. Several subfamilies have been settled unconditionally, for example $d = n^2+4$ and $d = 4n^2+1$ by Biró [9, 8], $d = n^2-4$ by Byeon, Kim and Lee [13], the fields with $-m < r \leq m$, $|r| \neq 1, 4$ and $d \not\equiv 5 \pmod 8$ by Lee [47], and $d = (an)^2+4a$ with a and n positive and odd by Biró and Lapkova [10]. In general, the list was known to be complete with one possible exception, which Louboutin and, independently, Mollin and Williams had excluded under the generalized Riemann hypothesis (see [52, pp. 315–316] and [60, p. 21]); without that hypothesis the question of the exception was open [36, §1].

The 30 primes of (b) are Yokoi's Table I [84], complete with one possible exception by his Corollary, which combines his Theorem 2 with a computation by Tanigawa for $3533 < p < 4.1 \cdot 10^6$. By his Theorem 2 the exception does not occur under the generalized Riemann hypothesis. In (c), the 30 primes and the 24 odd composite discriminants form the 54 discriminants of Yokoi's Theorem 3 (credited there to Tanigawa), again with one possible exception, but Theorem 3 itself says nothing about the generalized Riemann hypothesis. As printed, that theorem places no condition on the parity of the discriminant, but the 14 even discriminants in (c), whose fields also have $\varepsilon_0 < 2\Delta$ and class number 1, are not among its 54. Mollin and Williams [58, Theorem 4 and Remark 1] listed the same 68 fields, the even ones included, as those of class number 1 for which their generalization of Yokoi's invariant is nonzero, a condition that implies $\varepsilon_0 < 2\Delta$ [58, Corollary 1]. They proved the list complete with one possible exception, which would contradict the generalized Riemann hypothesis [58, Remark 2].

For (d), the 107 fields are those listed by Mollin [57, Theorems 3.1–3.13 and Remark 3.1], who records, from earlier work, that the list is complete for some of its families without any hypothesis and for the others with one possible exception, which the generalized Riemann hypothesis rules out. Without any hypothesis, Byeon and Lee [14] settled the subfamily $d = n^2 + 1$ with d even, and Lee [46] the fields with $-m < r \le m$, $|r| = 1$ or 4, and $d \not\equiv 5 \pmod 8$. Five fields in (a) ($d = 13, 69, 93, 413, 1133$) and four in (d) ($d = 205, 1469, 1965, 2085$) are of extended Richaud–Degert type only through the case $r = \pm 4m/3$.

Proof. (a)–(c) If $h = 1$, then Proposition 5.3 and Lemma 5.4 give $B_{\mathrm{R}}(\Delta, 7/8) \le 2R < 2\log(4\Delta + 4\sqrt{\Delta} + 1)$. A block evaluation on the 51 blocks $[X, Y]$ with $Y = \lfloor 1.1X \rfloor + 1$, the first starting at $X = 40\,000$ and the last ending at $Y = 5\,165\,983$, shows that this fails for $40\,000 \le \Delta \le 5\,165\,983$, and the first closed-form bound shows that it fails for $\Delta \ge 5 \cdot 10^6$: the bound exceeds the right-hand side at $\Delta = 5 \cdot 10^6$, and their difference increases with Δ. So $\Delta < 40\,000$. The class numbers of all fundamental $\Delta < 40\,000$ were then computed (Appendix A.4); among them are 1102 fields of extended Richaud–Degert type and 151 primes $p \equiv 1 \pmod 4$ with $\varepsilon_p < 2p$.

(d) If $h \le 2$, then $B_{\mathrm{R}}(\Delta, 7/8) \le 4R < 4\log(4\Delta + 4\sqrt{\Delta} + 1)$. A block evaluation on the five blocks whose left ends are 169 918, 171 944, 185 774, 310 992 and 15 873 052, each ending just before the next begins and the last at 10^{10}, shows that this fails for $169\,918 \le \Delta \le 10^{10}$, and the bound G_0 shows that it fails beyond 10^9. The class numbers of the 2590 fields of extended Richaud–Degert type with $\Delta < 169\,918$ were then computed. □

Theorem 5.6. *Assume* (H). *The real quadratic fields of extended Richaud–Degert type whose class group is* 2*-torsion are exactly* 298 *fields: the* 43 *and* 107 *fields of Theorem 5.5 (a) and (d), and the* 102, 41 *and* 5 *fields* $\mathbb{Q}(\sqrt{d})$ *of class number* 4, 8 *and* 16 *with* d *as follows. The largest discriminant among them is* 451 605. *Exactly* 269 *of the* 298 *have one class per genus, that is, a* 2*-torsion narrow class group.*

$h = 4$ (102 fields): 170, 195, 210, 231, 255, 290, 330, 390, 410, 435, 455, 462, 483, 530, 570, 615, 627, 663, 770, 798, 890, 903, 915, 930, 962, 1095, 1190, 1218, 1230, 1235, 1295, 1302, 1365, 1370, 1463, 1482, 1515, 1518, 1547, 1595, 1610, 1722, 1767, 1770, 1938, 2015, 2030, 2387, 2405, 2595, 2607, 2618, 2805, 2910, 3045, 3230, 3335, 3485, 3723, 3842, 3885, 3965, 4389, 4485, 4758, 5565, 6045, 6405, 6765, 7293, 7565, 7685, 7917, 8645, 9605, 9933, 10005, 10205, 10965, 11165, 12045, 13485, 13845, 14685, 14885, 15645, 16133, 16653, 17765, 18245, 19565, 20165, 20405, 21045, 24045, 25493, 30597, 31317, 33117, 41093, 41613, 55205.

$h = 8$ (41 fields): 1155, 1995, 2210, 2310, 2415, 2730, 3003, 3135, 3255, 3570, 3927, 3990, 4290, 4935, 5187, 5330, 5610, 5655, 6090, 6555, 7035, 7215, 7755, 9030, 10010, 10370, 12155, 12558, 13695, 14630, 19610, 21318, 22490, 23205, 26565, 32045, 35805, 58565, 74613, 77285, 108885.
$h = 16$ (5 fields): 19635, 25935, 33495, 81770, 451605.

Louboutin, Mollin and Williams [54] listed the fields of extended Richaud–Degert type with class group of exponent 2; Kopp and Lagarias [43, §5] describe that list as complete under the generalized Riemann hypothesis. We have not seen the tables of [54]. According to its zbMATH review (Zbl 0795.11057), [54] shows, assuming the generalized Riemann hypothesis, that exactly 228 fields of this type have class group of exponent 2, and lists them. The table of fields of extended Richaud–Degert type with class group of exponent 2 in [56, Appendix A, Table A4], as transcribed in the sequences A051990–A051992 of [64], consists exactly of the 255 fields of Theorem 5.6 with class number greater than 1. Since the generalized Riemann hypothesis implies (H), the two counts contradict each other under the generalized Riemann hypothesis if they use the same definition. Table A4 reproduces the tables of [54] with corrections [56, p. 202]. Dohmae [22, Theorem 2] found 69 fields $\mathbb{Q}(\sqrt{d})$ with $d \neq 5$ and $d = m^2 + r$, $-m < r \leq m$, $r \mid 4m$ and $|r| = 1$ or 4 that have one class per genus, and showed that there is at most one more. Exactly 69 of the 269 fields of Theorem 5.6 with one class per genus are of this kind, so under (H) there is no other.

Proof. If $\mathrm{Cl}(K)$ is 2-torsion, then by genus theory

$$h = |\mathrm{Cl}(K)/\mathrm{Cl}(K)^2| \leq |\mathrm{Cl}^+(K)/\mathrm{Cl}^+(K)^2| = 2^{\omega(\Delta)-1},$$

where $\mathrm{Cl}^+(K)$ is the narrow class group. With Proposition 5.3 and Lemma 5.4, this gives $B_{\mathrm{R}}(\Delta, 7/8) < 2^{\omega(\Delta)} \log(4\Delta + 4\sqrt{\Delta} + 1)$. On each block $[X, Y] = [10^{k/4}, 10^{(k+1)/4}]$ from $\lfloor 10^{14.5} \rfloor = 316\,227\,766\,016\,837$ to 10^{40}, the bound $G_0(X)$ at $\beta = 7/8$ exceeds $2^{\omega^*(Y)} \log(4Y + 4\sqrt{Y}+1)$, where $\omega^*(Y)$ is the largest value of ω on $[1, Y]$. So this fails for $316\,227\,766\,016\,837 \leq \Delta \leq 10^{40}$. Beyond 10^{40}, Robin's bound $\omega(n) \leq 1.3841 \log n / \log\log n$ [72] gives $2^{\omega(\Delta)} \leq \Delta^{1.3841 \log 2/\log\log \Delta} < \Delta^{0.22}$, and $\log(4\Delta + 4\sqrt{\Delta} + 1) < \Delta^{0.06}$, while $G_0(\Delta) > 0.16\,\Delta^{7/16} - \log \Delta > \Delta^{0.28}$ at $\beta = 7/8$; so $B_{\mathrm{R}}(\Delta, 7/8) \geq G_0(\Delta) > 2^{\omega(\Delta)} \log(4\Delta + 4\sqrt{\Delta} + 1)$ there too.

For smaller Δ we sieve. Let $\xi_d = (\sigma - 1 + \sqrt{d})/\sigma$, with $\sigma = 2$ if $d \equiv 1 \pmod 4$ and $\sigma = 1$ otherwise. The reduced principal ideals of the ring of integers are those given by the continued fraction expansion of ξ_d over one period, and every primitive ideal of norm less than $\sqrt{\Delta}/2$ is reduced [56, Chapters 2 and 5]. If $\mathrm{Cl}(K)$ is 2-torsion and p is a prime with $\chi_\Delta(p) = 1$ and $4p^4 < \Delta$, then the square of a prime ideal above p is a principal primitive ideal of norm $p^2 < \sqrt{\Delta}/2$, so p^2 is the norm of one of these reduced principal ideals. The sieve tests this for every prime p with $4p^4 < \Delta$. The sieve is applied to every pair (m, r) with $\Delta < 316\,227\,766\,016\,838$, whether or not m^2+r is squarefree, and leaves 3147 values of m^2+r, of which 1524 are squarefree. Their class groups were then computed (Appendix A.4). □

5.3. **Unit-generated orders: a problem of Kopp and Lagarias.** For $n \geq 1$ let $\Delta = n^2 + 4$, and for $n \geq 3$ let $\Delta = n^2 - 4$; let $\mathcal{O} = \mathbb{Z}[(n + \sqrt{\Delta})/2]$, the order of discriminant Δ. These are the unit-generated real quadratic orders [43, §2], that is, the orders generated additively by their units [43, Definition 1.1]. Write $\Delta = f^2\Delta_0$ with Δ_0 fundamental, and $\chi_0 = \chi_{\Delta_0}$. Let $\mathrm{Cl}(\mathcal{O})$ be the (wide) class group of invertible ideals modulo principal ideals, and $\mathrm{Cl}^+(\mathcal{O})$ the narrow class group, identified with the proper equivalence classes of primitive forms of discriminant Δ. Kopp and Lagarias [43, Problem 1.7] asked for all such orders with $\mathrm{Cl}^+(\mathcal{O})$ 2-torsion ("one class per genus"), and more generally with $\mathrm{Cl}(\mathcal{O})$ 2-torsion. They proved that there are finitely many, by an ineffective argument [43,

Theorem 1.5], listed those with $\Delta < 10^{10}$ in their Tables 3 and 4, which they conjecture to be complete, and wrote that the complete classification "seems out of reach at this time" [43, §6]. Without any hypothesis, they determined the unit-generated real quadratic orders of class number one [43, Theorems 1.3 and 1.4]; these are among the orders of Theorem 5.7.

Theorem 5.7. *Assume* (H). *The unit-generated real quadratic orders whose class group is* 2*-torsion are exactly the orders of the* 90 *entries of Tables 3 and 4 of* [43]*: the* 37 *orders with* $\Delta = n^2 + 4$ *and* n *one of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 17, 19, 22, 23, 25, 26, 29, 31, 34, 38, 41, 43, 46, 49, 53, 59, 62, 74, 94, 101, 127, 146, 179; and the* 53 *orders with* $\Delta = n^2 - 4$ *and* n *one of 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 26, 27, 28, 31, 32, 33, 37, 38, 39, 42, 44, 46, 49, 53, 54, 58, 63, 67, 68, 72, 82, 86, 87, 89, 93, 112, 122, 138, 154, 163, 166, 262. These are* 89 *distinct orders, since* $n = 1$ *in the first family and* $n = 3$ *in the second give the same order; the largest discriminant is* 68 640*, for* $n = 262$*. The orders with one class per genus are those of* 85 *of these entries (*84 *distinct orders): all of the first family, and all of the second except* $n = 15, 27, 39, 63, 87$.

The text of [43, §5.3] gives a total of 86; their two tables contain the 90 entries above, and their Table 3, for $\Delta = n^2 - 4$, also lists the imaginary discriminants -4 and -3 ($n = 0, 1$).

Proof. (i) Let $h(\mathcal{O}) = |\operatorname{Cl}(\mathcal{O})|$. By the class number formula for orders [43, §3], [31], $h(\mathcal{O})R(\mathcal{O}) = h_K R_K f \prod_{p|f}(1 - \chi_0(p)/p)$, where $h_K R_K = \sqrt{\Delta_0}\, L(1, \chi_0)/2$ for the field $K = \mathbb{Q}(\sqrt{\Delta_0})$. Since $(n + \sqrt{\Delta})/2$ is a unit of $\mathcal{O}$ larger than 1, $R(\mathcal{O}) < \frac{1}{2}\log(\Delta + 4)$, and so

$$h(\mathcal{O}) > \sqrt{\Delta}\, L(1, \chi_0)\, \frac{\varphi(f)}{f}\, \frac{1}{\log(\Delta + 4)}.$$

(ii) Under (H), $L(1, \chi_0) \geq 0.68238/\log \Delta_0$ for $\Delta_0 \geq 10^{12}$. To see this, put $\lambda = \log \Delta_0 \geq 27.6$ and $\beta = 1 - 2/\lambda$, which lies in $[7/8, 1)$. Since $2h_K R_K = \sqrt{\Delta_0}\, L(1, \chi_0)$, Proposition 5.3 and the bound G_0 give $L(1, \chi_0) \geq G_0(\Delta_0)/\sqrt{\Delta_0}$. As $(\sqrt{\Delta_0}/2\pi)^\beta = \sqrt{\Delta_0}\,(2\pi)^{-\beta} e^{-1}$, this reads

$$\lambda\, L(1, \chi_0) \geq 2\Big(1 - \frac{2}{\lambda}\Big)\big(\kappa(\beta) - \eta(\Delta_0)\big),$$

where

$$\kappa(\beta) = \frac{2^\beta\, \Gamma(\beta/2)^2}{(2\pi)^\beta e} \qquad \text{and} \qquad \eta(\Delta_0) = \frac{4}{\beta}\Big(\log \frac{3\sqrt{\Delta_0}}{2\pi} + \frac{1}{\beta}\Big)\frac{1}{\sqrt{\Delta_0}}.$$

Here κ decreases on $(0, 1]$, because $(\log \kappa)'(\beta) = (\Gamma'/\Gamma)(\beta/2) - \log \pi < 0$, so $\kappa(\beta) \geq \kappa(1) = 1/e$, and $\eta(\Delta_0)$ decreases as Δ_0 grows. Hence $\lambda L(1, \chi_0) \geq 2(1 - 2/\log 10^{12})(1/e - \eta(10^{12})) = 0.682389\ldots$. For smaller Δ_0, $L(1, \chi_0) = 2h_K R_K/\sqrt{\Delta_0}$ is bounded below by $B_{\mathrm{R}}(\Delta_0, 7/8)/\sqrt{\Delta_0}$, and by $2\log((\sqrt{\Delta_0 - 4} + \sqrt{\Delta_0})/2)/\sqrt{\Delta_0}$, since $h_K \geq 1$ and $\varepsilon_0 \geq (\sqrt{\Delta_0 - 4} + \sqrt{\Delta_0})/2$. With $\sqrt{\Delta_0} = 2\cosh t$, the second of these bounds is $t/\cosh t$, which increases and then decreases as Δ_0 grows, and $B_{\mathrm{R}}(\Delta_0, 7/8)$ does not decrease. So if $X_0 \leq \Delta_0 \leq Y_0$, then $L(1, \chi_0)$ is at least $B_{\mathrm{R}}(X_0, 7/8)/\sqrt{Y_0}$, and at least the smaller of the values of the second bound at X_0 and Y_0. We cover $[5, 10^{12}]$ by the 117 blocks $[X_0, Y_0]$ that begin with $[5, 8]$ and continue with $Y_0 = \min(\lfloor 5X_0/4 \rfloor, 10^{12})$, each block starting where the previous one ends. Evaluated in interval arithmetic, with $B_{\mathrm{R}}(X_0, 7/8)$ bounded below as in a block evaluation when $X_0 \geq 40$ and by 0 otherwise, the larger of the two bounds is at least 0.0473 on every block. So $L(1, \chi_0) \geq 0.0473$ for every fundamental $\Delta_0 < 10^{12}$. The program and its output are in [38].

(iii) If $\operatorname{Cl}(\mathcal{O})$ is 2-torsion, then it is a quotient of $\operatorname{Cl}^+(\mathcal{O})/\operatorname{Cl}^+(\mathcal{O})^2$, so $h(\mathcal{O})$ is at most the number of genera [43, Proposition 5.1], which is at most $2^{\omega(\Delta)}$. Let $[X, Y] = [10^{k/4}, 10^{(k+1)/4}]$

be a block with $Y \geq 10^{12}$, and let $\ell(Y) = 0.68238/\log Y$. By (ii), $L(1,\chi_0) \geq \ell(Y)$ for every fundamental $\Delta_0 \leq Y$, since $0.0473 > 0.68238/\log 10^{12}$. For $y \geq 1$ let $\omega^*(y)$ be the largest value of ω on $[1,y]$, as in the proof of Theorem 5.6, and put $\Phi(Y) = \prod_{i \leq \omega^*(\sqrt{Y/5})}(1-1/p_i)$, where p_i is the i-th prime. If $\Delta \in [X,Y]$, then $f \leq \sqrt{Y/5}$, because $\Delta_0 \geq 5$, and so $\varphi(f)/f \geq \Phi(Y)$. By (i), $\mathrm{Cl}(\mathcal{O})$ is therefore not 2-torsion for any $\Delta \in [X,Y]$ if

$$\frac{\sqrt{X}\,\ell(Y)\,\Phi(Y)}{\log(Y+4)} > 2^{\omega^*(Y)}.$$

Evaluated in interval arithmetic, this inequality holds on every block from $10^{16.75}$ to 10^{40}. Beyond 10^{40}, put $x = \log\Delta$. Then $L(1,\chi_0) \geq \ell(\Delta) = 0.68238/x$; $\varphi(f)/f \geq 1/(\omega(f)+1) \geq 1/x$, since $p_i \geq i+1$, and $2^{\omega(f)} \leq f < \sqrt{\Delta}$ gives $\omega(f) < x/(2\log 2) \leq x-1$; and $\log(\Delta+4) < 1.01x$. So (i) gives $h(\mathcal{O}) > 0.67\sqrt{\Delta}/x^3 > \Delta^{0.22}$, which exceeds $2^{\omega(\Delta)}$ by Robin's bound, as in the proof of Theorem 5.6. Hence $\mathrm{Cl}(\mathcal{O})$ is not 2-torsion when $\Delta \geq 5.6235 \cdot 10^{16} > 10^{16.75}$, and this leaves $n \leq 2.3714 \cdot 10^8$.

(iv) If $\mathrm{Cl}(\mathcal{O})$ is 2-torsion, then no prime p with $4p^4 < \Delta$ has $(\Delta/p) = 1$. Indeed, let g be the class of a form $(p,b,\cdot)$ of discriminant Δ. Its square contains a primitive form $(p^2,b',\cdot)$, and b' can be chosen in $(\sqrt{\Delta}-2p^2,\sqrt{\Delta})$, which makes it reduced since $2p^2 < \sqrt{\Delta}$. As $\mathrm{Cl}(\mathcal{O})$ is the quotient of $\mathrm{Cl}^+(\mathcal{O})$ by the class γ of the forms $(-1,\cdot,\cdot)$, the class g^2 is 1 or γ. Each class contains exactly one cycle of reduced forms, and the cycles of 1 and γ are $\{(1,n,-1),(-1,n,1)\}$ if $\Delta = n^2+4$, and $\{(1,n-2,-(n-2)),(-(n-2),n-2,1)\}$ and $\{(-1,n-2,n-2),(n-2,n-2,-1)\}$ if $\Delta = n^2-4$. Their first coefficients have absolute value 1 or $n-2$, whereas $1 < p^2 < \sqrt{\Delta}/2 < n-2$ for $n \geq 5$; for $n \leq 4$ there is no such p.

(v) The test (iv) was applied to all orders with $n \leq 237\,200\,000$. It leaves 544 entries (543 distinct orders), all with $n \leq 5968$. For these the classes of reduced forms, their cycles, and the squares of all classes were computed, which gives the 90 and 85 entries of the statement, with the conductors and class numbers given in [43]. □

5.4. **Real quadratic fields with a short period.** A short continued-fraction period bounds the regulator, so the method of Section 5.2 applies also to the real quadratic fields of class number 1 or 2 with a short period. Let $d > 1$ be squarefree, $\sigma = 2$ if $d \equiv 1 \pmod 4$ and $\sigma = 1$ otherwise, $\Delta = 4d/\sigma^2$ the discriminant of $K = \mathbb{Q}(\sqrt{d})$, and $\xi_d = (\sigma-1+\sqrt{d})/\sigma$, so that $\mathbb{Z}[\xi_d]$ is the ring of integers. Expand ξ_d in a continued fraction: $(P_0,Q_0) = (\sigma-1,\sigma)$, $a_i = \lfloor (P_i+\sqrt{d})/Q_i \rfloor$, $P_{i+1} = a_iQ_i - P_i$, $Q_{i+1} = (d-P_{i+1}^2)/Q_i$, and $\psi_i = (P_i+\sqrt{d})/Q_i$. The period k is the least $i \geq 1$ with $Q_i = \sigma$; this is the normalization of Mollin and Williams [61]. We use the following classical facts [56, Chapters 2 and 5]: for $i \geq 1$, $0 < P_i < \sqrt{d}$ and Q_i is a positive multiple of σ; $\varepsilon_0 = \psi_1\psi_2\cdots\psi_k$; the reduced principal ideals are those of norms Q_i/σ, $0 \leq i < k$; and every primitive ideal of norm less than $\sqrt{\Delta}/2$ is reduced. Since $\psi_i < 2\sqrt{d}/\sigma = \sqrt{\Delta}$ for $1 \leq i \leq k$, it follows that $R < (k/2)\log\Delta$.

Lemma 5.8. *Let $\mathcal{N} = \{Q_i/\sigma : 0 \leq i < k\}$, and let $p \neq q$ be primes with $\chi_\Delta(p), \chi_\Delta(q) \in \{0,1\}$.* (A) *If $\chi_\Delta(p) = 1$, $4p^4 < \Delta$, and neither p nor p^2 lies in $\mathcal{N}$, then $h \geq 3$.* (B) *If $4(pq)^2 < \Delta$ and none of p, q, pq lies in $\mathcal{N}$, then $h \geq 3$.*

Proof. By the facts above, a primitive ideal of norm less than $\sqrt{\Delta}/2$ is principal only if its norm lies in $\mathcal{N}$. (A) A prime ideal $\mathfrak{p}$ above p is not principal, and neither is $\mathfrak{p}^2$, which is primitive of norm p^2; so $\mathfrak{p}$ has order at least 3. (B) The prime ideals above p and q are not principal, so $h \geq 2$; if $h = 2$, both lie in the nontrivial class, and their product, a primitive ideal of norm pq, would be principal. □

Theorem 5.9. *Assume* (H).

(a) The real quadratic fields with period $k \le 24$ and class number 2 are exactly the 1958 *fields of Table 2.1 of Mollin and Williams* [61]*, with the printed* 47195 *(row $k = 10$) replaced by* 47197 *and with* 1074 *added to row $k = 14$; the counts in their Table 2.2 are right as printed. The largest discriminant among them is* 248 093.

(b) The real quadratic fields with period $k \le 24$ and class number 1 are exactly the 688 *fields of Table 3.1 of Mollin and Williams* [60]*, with the printed* 38493 *(row $k = 21$) and* 3986 *(row $k = 24$) replaced by* 28493 *and* 3086*, and with* 18053 *added to row $k = 16$. The largest discriminant among them is* 49 013*. The sequences A050950–A050969 and A051962–A051965 of* [64]*, taken there from Mollin's book* [56, Appendix A]*, list the same fields by period, except that A051965 has* 3986 *in place of* 3086.

Mollin and Williams determined both lists with one possible exception [60, Theorem 3.1], [61, §1], whose existence, they note, would contradict the generalized Riemann hypothesis [62, pp. 1023–1024]. For period 1, Lachaud [44, Theorem 3.1] had already found the seven fields of class number 1, $d = 2$, 5, 13, 29, 53, 173 and 293, assuming that $\zeta_K(1/2) \le 0$. His argument also starts from the sign of ζ_K at a real point [44, Theorem 2.3].

Proof. If $k \le 24$, then $R < 12 \log \Delta$, and Proposition 5.3 shows that $h \le 2$ is impossible as soon as $B_{\mathrm{R}}(\Delta, 7/8) > 48 \log \Delta$, and $h = 1$ as soon as $B_{\mathrm{R}}(\Delta, 7/8) > 24 \log \Delta$. A block evaluation up to 10^{10}, on the five blocks whose left ends are 73 925 048, 74 084 363, 75 373 231, 86 593 053 and 274 854 128, and on the four whose left ends are 12 402 160, 12 789 211, 16 059 306 and 95 790 741, each ending just before the next begins, and the bound G_0 beyond 10^9 show that this happens for $\Delta \ge 73\,925\,048$ and $\Delta \ge 12\,402\,160$ respectively. All 30 396 324 real quadratic fields with $\Delta < 10^8$ were expanded; 648 375 have $k \le 24$. Lemma 5.8 proves $h \ge 3$ for 644 479 of them, and the class numbers of the other 3896 were computed as in Appendix A.4: 1958 have $h = 2$ and 688 have $h = 1$. □

Remark 5.10. The disagreements with the printed lists are errors in those lists. In Table 2.1 of [61], the number 47195 at the end of row $k = 10$ has period 56; the field $\mathbb{Q}(\sqrt{47197})$, of period 10 and class number 2, belongs there. The field $\mathbb{Q}(\sqrt{1074})$, of period 14 and class number 2, is missing from row $k = 14$, although the count 169 for $k = 14$ in their Table 2.2 includes it. In Table 3.1 of [60], row $k = 21$ has $38493 = 3^2 \cdot 7 \cdot 13 \cdot 47$, which is not squarefree, in place of 28493; row $k = 24$ has 3986, for which $h = 4$ and $k = 11$, in place of 3086; and the field $\mathbb{Q}(\sqrt{18053})$, of period 16 and class number 1, is missing from row $k = 16$. In [64], A051965 ($h = 1$, $k = 24$) has 3986 in place of 3086, as in [60]; A051975 ($h = 2$, $k = 10$) has 47195 in place of 47197, as in [61]; A051976 ($h = 2$, $k = 11$) omits 14453; A051979 ($h = 2$, $k = 14$) has 669, for which $h = 1$ and $k = 10$, in place of 699, and omits 1074; and A051982 ($h = 2$, $k = 17$) has $34047 = 3^3 \cdot 13 \cdot 97$, which is not squarefree, in place of 34037, as a comment in that entry notes. A051969 ($h = 2$, $k = 4$), which is not marked complete, stops at 4893; thirteen more fields, up to 9077, belong to it.

5.5. **Imaginary quadratic orders and singular moduli.** The class number formula for orders reduces the imaginary quadratic orders of small class number to the fields of small class number, and hence to Theorem 1.1. For a fundamental discriminant $D < 0$ and $f \ge 1$, let $\mathcal{O}$ be the order of conductor f in $K = \mathbb{Q}(\sqrt{D})$, of discriminant Df^2, and $h(Df^2)$ its class number. By the class number formula for orders [17, Theorem 7.24],

$$h(Df^2) = \frac{h(D)\, f}{u} \prod_{p \mid f} \Big(1 - \frac{\chi_D(p)}{p}\Big), \qquad u = [\mathcal{O}_K^\times : \mathcal{O}^\times],$$

where $u = 3$ if $D = -3$ and $f > 1$, $u = 2$ if $D = -4$ and $f > 1$, and $u = 1$ otherwise.

Theorem 5.11. *Assume* (K). *The imaginary quadratic orders of class number at most* 1095 *are exactly* 7 729 430 *orders: the maximal orders of the fields of class number at most* 1095*, and* 2 807 930 *non-maximal orders; the list is deposited in* [38]. *Since the singular modulus* $j(\mathcal{O})$ *has degree* $h(\mathcal{O})$ *over* $\mathbb{Q}$*, these orders give the singular moduli of degree at most* 1095. *Exactly* 6 450 424 *of the orders have class number at most* 1000. *The largest* $|Df^2|$ *of an order of class number at most* 1000 *is* 227 932 027*, the discriminant of a field of class number* 996*, and the numbers of orders of class number* $1, 2, \dots, 10$ *are* $13, 29, 25, 84, 29, 101, 38, 208, 55, 123$.

Proof. Since $\chi_D(p) \le 1$, the formula gives $h(Df^2) \ge h(D)\varphi(f)/u$. If $h(Df^2) \le 1095$, then $h(D) \le 1095$ (recall that $u > 1$ only when $h(D) = 1$), so K is one of the fields of Theorem 1.1, and $\varphi(f) \le 3 \cdot 1095 = 3285$. Since $\varphi(n) \ge \sqrt{n/2}$ for every n, a computation of $\varphi(n)$ for $n \le 2 \cdot 3285^2 + 1$ shows that this forces $f \le 14\,490$. For each of these fields and each such f the formula was then evaluated. The degree of $j(\mathcal{O})$ over $\mathbb{Q}$ is the class number of $\mathcal{O}$, and the $\mathbb{Q}$-conjugates of $j(\mathcal{O})$ are the singular moduli of the proper $\mathcal{O}$-ideal classes [17, Chapter 3], which gives the statement about singular moduli. □

Assuming the generalized Riemann hypothesis, Cremona and Sutherland [18, §5] determined the 6 450 424 orders of class number at most 1000, and so the singular moduli of degree at most 1000; their list agrees with ours (Remark 4.1). For these orders Theorem 5.11 replaces the generalized Riemann hypothesis by (K), which (H) implies, and Kimura [41, §6] recovers their list from his determination of the fields. For class numbers 1001 to 1095 we know of no earlier list. Klaise [42, §4] had already determined the orders of class number at most 100, without any hypothesis, by the same method applied to Watkins's determination of the fields (Remark 5.12). Aiken, Ross, Shvydkoy and Xue [2] classify imaginary quadratic orders by the exponent of their class group rather than by the class number (Section 5.1).

Remark 5.12. For each $h \le 100$, Klaise [42, §4] tabulated the number of imaginary quadratic orders of class number h and the largest $|Df^2|$; his table agrees with ours in all 100 rows. Table 2 of Derickx and Najman [21, Appendix A.1] gives the same two quantities. The largest discriminants agree with ours in all 100 rows, and the numbers in 67; in the other 33 rows, for h equal to 34, 36, 38, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 64, 66, 68, 70, 72, 74, 76, 78, 80, 81, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100, their number is smaller, and their total is 66 633 against our 66 758. Their list omits some orders of large conductor in $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(i)$: the list in the notebook that accompanies [21] has 66 643 discriminants, all of which are in ours, and the 115 it lacks lie in $\mathbb{Q}(\sqrt{-3})$ (81 of them, of conductors 103 to 343) and $\mathbb{Q}(i)$ (34, of conductors 103 to 235). An example is the order of discriminant $-3 \cdot 103^2 = -31\,827$, of class number 34. Their Table 2 has 10 fewer entries than the notebook's list, in the rows $h = 34$, 50 and 100. The class numbers of all negative discriminants of absolute value at most $3 \cdot 10^6$, computed directly with `qfbclassno`, confirm our numbers (Appendix A.3 (6)).

Appendix A. Computations

The computations used PARI/GP 2.17.3 [70], Python 3.12 with python-flint 0.8 (FLINT 3.3 [24], whose ball arithmetic is Arb [39]), SageMath 10.9 [73] with MPFI [71], and programs in C. The long runs were made on servers with AMD EPYC 4484PX processors, with programs compiled by gcc 13.3 (gcc 16.2 for the first program of Appendix A.6), and the others on an Apple laptop, with Apple's clang.

The theorem `LFunction_ne_zero_of_seven_eighths_lt_re` of [66] is the formal statement of Section 1.1. It uses Mathlib's L-function of a Dirichlet character χ, which is the

analytic continuation of $\sum_{n\geq 1}\chi(n)n^{-s}$, and asserts that this function does not vanish at s for every modulus $q\geq 1$, every Dirichlet character χ modulo q and every complex s with $\operatorname{Re} s > 7/8$ and $(\chi, s)\neq(1,1)$, where 1 denotes the trivial character modulo q. We built the formalization at the commit given in [66] with Lean 4.34.1. The axioms its proof uses are propositional extensionality, the axiom of choice and the soundness of quotients.

A.1. **The bounds of Table 2.** The value $B(X,\beta)$ was bounded in three ways.

(a) With Arb's rigorous upper incomplete gamma function, at 128 bits, from the second expression in Proposition 2.2. The sum over m is stopped when $\alpha_m > 60$. This drops positive terms only, of total size below 10^{-20}.

(b) With Arb's rigorous numerical integration, at 96 bits, of the integrals in the first expression, for finitely many m and over a bounded interval of t. This drops positive quantities only, so it gives lower bounds.

(c) By a separately written program in MPFI interval arithmetic at 160 bits, which uses no incomplete gamma function and no integration routine. By Tonelli's theorem, $B(X,\beta) = 2\beta(1-\beta)\int_1^\infty S(t)g(t)\,dt$ with $S(t)=\sum_{m\geq 1}e^{-\alpha_1 m^2 t}$ and $g(t)=t^{\beta-1}+t^{-\beta}$. Both functions are positive, decreasing and convex, so Sg is convex, and on each interval $[u,v]$ of a grid the Hermite–Hadamard inequalities give $(v-u)(Sg)(\frac{u+v}{2}) \leq \int_u^v Sg \leq (v-u)\big((Sg)(u)+(Sg)(v)\big)/2$. The grid runs from 1 to a point T with $\alpha_1 T\geq 120$, with ratio about $1+1/2500$ between consecutive points. The integral over $[T,\infty)$ is at most $g(T)S(T)/\alpha_1$. The function S is evaluated by its series or, when $\alpha_1 t\leq 1$, by Jacobi's transformation, with rigorous bounds for the tails.

At $X = 1.98\cdot 10^{10}$ the values from (a) and (b) agree to about 10^{-22}. At $X = 2\cdot 10^9$, (b) uses fewer terms and gives a slightly smaller lower bound. The enclosures from (c), of width at most $1.3\cdot 10^{-4}$, contain the values from (a). As a test of the normalizations in Lemma 2.1, both sides were compared numerically for $D=-23$, -163 and $-1\,856\,563$ at several real and complex s, with $L(s,\chi_D)$ computed by PARI's `lfun`. The differences, divided by $\max(1,|\Lambda(s)|)$, are below $5\cdot 10^{-73}$. As a test of Corollary 2.3 against the data, every $D<-4$ in $\mathcal{L}$ with $h(D)\leq 1095$ satisfies $h(D) > 1.659\,B(|D|,7/8)$.

A.2. **The computation of Theorem 1.2.** The procedure of Section 3.1 is implemented in C, with 64-bit integers and 128-bit products. A segmented sieve finds the fundamental discriminants and $\omega(D)$, and square roots modulo p are found by the Tonelli–Shanks algorithm, with Hensel lifting and the Chinese remainder theorem for the roots modulo $4a$ in Lemma 3.1. For reduced forms with $|D| < 2\cdot 10^{10}$ we have $a < 81\,650$ and $c\leq |D|/4 + a/4$. In the composition of two reduced forms (a_1,b_1,c_1) and (a_2,b_2,c_2), every product that can exceed 2^{63} is formed with 128-bit integers, and the composite (a_3,b_3,c_3) before reduction satisfies $a_3\leq a_1a_2$, $|b_3|\leq a_2+2a_3$ and $c_3\leq a_2^2/(4a_3)+a_2+a_3+|D|/4 < 1.4\cdot 10^{10}$. Reduction never increases a or $\max(a,c)$, and $|b|$ never exceeds $\max(|b_3|,a_3)$. So 64-bit integers suffice for the forms themselves.

The computation for Theorem 1.2 covered $[3, 1.98\cdot 10^{10})$ in 198 ranges of length 10^8, and the running times of the ranges add up to 60 651 seconds. Of the 6 018 478 352 fundamental discriminants, 5 395 925 488 were shown to have $h(D) > 3000$ by the class of the smallest split prime and 545 057 712 by one of the next three, and the remaining 77 495 152 were decided by counting reduced forms. Of these, 36 993 853 have $h(D)\leq 3000$ and form $\mathcal{L}$, and 40 501 299 have $h(D) > 3000$.

Two variants of the program were used: one for Theorem 1.2, with $N = 3000$, and one for the computation with $N = 1095$ in Appendix A.3 (2), which also made the exponent check in the proof of Proposition 5.1. The variant for $N = 3000$ tabulates more primes, keeps the baby steps in a hash table instead of a list, and accepts ranges beyond 2^{32}. The functions of the variant for $N = 1095$ were tested against PARI: its counts of reduced forms against `qfbclassno` for 4911 fundamental D (all with $|D| \le 3000$, and random ones below $2 \cdot 10^9$), and its order test, in 243 906 cases, against the exact orders of the classes computed by PARI. The test succeeded exactly when the order exceeds mt. The composition of the variant for $N = 3000$ agrees with PARI's reduced composition for 200 000 pairs of forms with $|D| < 1.98 \cdot 10^{10}$: in most of them a random form is composed with itself, its inverse, its square or a form whose first coefficient has a factor in common with its own, and in the others with another random form.

A.3. **Checks.**

(1) *Every class number in $\mathcal{L}$.* PARI's `qfbclassno` was applied to every pair of $\mathcal{L}$. For $D < 0$ it uses Shanks's baby-step giant-step method, and it is documented in the PARI/GP manual to give unconditionally correct results for $|D| < 2 \cdot 10^{10}$ [70]. It shares no code with the program of Appendix A.2, and it agrees on all 36 993 853 class numbers. For $N = 1000$ and $N = 1500$, the pairs of $\mathcal{L}$ with $h \le N$, written as newline-terminated lines "$|D|$ h" in order of increasing $|D|$, have the SHA-256 checksums that [48, Corollary D] gives for its lists (Remark 4.1). Written in the same way, the pairs with $h \le 1000$ and with $h \le 2000$ have the SHA-256 checksums that Kimura gives for his lists [41, Data and code availability].

(2) *The range $|D| < 2 \cdot 10^9$.* Here the computation with $N = 1095$ was carried out twice in full. The variant of the program for $N = 1095$ (Appendix A.2) and `qfbclassno`, applied to each of the 607 927 095 fundamental discriminants in the range, give the same 4 921 500 pairs $(D, h(D))$ with $h(D) \le 1095$, and these are exactly the pairs of $\mathcal{L}$ with $h \le 1095$. The range contains every field of class number at most 1095, under (H) by Corollary 2.4, and under (K) by Theorem 1.1, whose proof does not use this computation. As a third check, PARI's `quadclassunit`, a different algorithm whose results assume the generalized Riemann hypothesis, gives the same class number for all 4 921 500 of these pairs, and, for 1 000 000 random draws of a fundamental D in the range (999 225 distinct discriminants), a class number at most 1095 exactly when D is among them.

(3) *The exclusions $h(D) > 3000$.* All of them were checked by a second argument, as follows. By the tabulation of Mosunov and Jacobson, the smallest value of $L(1, \chi_D)$ over fundamental discriminants $D < -4$ with $|D| < 2^{40}$ is attained at $D_1 = -107\,415\,709\,003$ [63, §7.1]. They give it to five decimal places as 0.17070, and a direct computation gives $L(1, \chi_{D_1}) = 0.170699\ldots$, so it is at least 0.170695. Hence $h(D) = \sqrt{|D|}\, L(1, \chi_D)/\pi \ge 0.170695\sqrt{|D|}/\pi > 3000$ for every fundamental D with $3.05 \cdot 10^9 \le |D| < 2^{40}$, and indeed $h(D) > 7600$ for $1.98 \cdot 10^{10} \le |D| < 2^{40}$. For $5 \le |D| < 3.05 \cdot 10^9$ we computed every class number with PARI/GP's `qfbclassno`, documented to be unconditionally correct in this range [70], and found that $h(D) \le 3000$ exactly for the pairs of $\mathcal{L}$ in this range, with the same class numbers (about 42 hours of processor time in all). The following tests check the program itself and are not needed for the second argument. For a uniform random sample of 972 173 fundamental discriminants with $|D| < 1.98 \cdot 10^{10}$, `qfbclassno` finds 5906 with $h(D) \le 3000$, all in $\mathcal{L}$ with the same class number, and none of the other 966 267 is in $\mathcal{L}$. For a second uniform sample of 486 634 discriminants,

and for 99 975 discriminants with $2 \cdot 10^9 \le |D| < 1.98 \cdot 10^{10}$ and no split prime $p \le 13$, PARI computed the orders of the classes g_p and hence the decision that the program must make. In all 586 609 cases the program recorded this decision, so PARI confirms the order of each class of large order that the program used for these discriminants.

(4) *The number of fundamental discriminants* in each of the 198 ranges agrees with a count by Möbius inversion.

(5) *Proposition 5.1.* A separately written program, which shares no code with the program of Appendix A.2, classified each fundamental D with $|D| < 2 \cdot 10^9$, and PARI classified the 117 065 discriminants that were not decided by a class g_p and have $h(D)$ a power of 2. Both give the list of Proposition 5.1.

(6) *Theorem 5.11.* The list was obtained again by evaluating the formula in PARI, and by `qfbclassno` applied to all 11 766 451 pairs (D, f) with $f > 1$ allowed by the bound $h(D)\varphi(f)/u \le 1095$ (all with $|Df^2| < 4 \cdot 10^9$), with no disagreement. Independently of the field list, the formula and the bound, `qfbclassno` applied to every negative non-fundamental discriminant of absolute value less than $3 \cdot 10^8$ finds exactly the 2 807 930 non-maximal orders of the list, which all lie in this range, and applied to every negative discriminant of absolute value at most $3 \cdot 10^6$ it gives the counts and largest discriminants for $h \le 100$ (Remark 5.12).

A.4. **Real quadratic fields and orders.** Class numbers of real quadratic fields were computed by counting the cycles of reduced indefinite forms: the narrow class number h^+ is the number of cycles, and $h = h^+$ if the cycle of the principal form contains a form with first coefficient -1, that is, if ε_0 has norm -1, and $h = h^+/2$ otherwise. For all fundamental $\Delta < 40\,000$ this agrees with PARI's `qfbclassno`, `quadclassunit` and `quadunitnorm`. For Theorems 5.6 and 5.7, the classes of reduced forms, their cycles and the squares of all classes were computed with exact composition and reduction in PARI. The following were repeated by separately written programs, with the same results.

(1) The bounds $\Delta < 40\,000$ and $\Delta < 169\,918$ of Theorem 5.5, with midpoint sums in place of Arb's integration. Since the integrands are convex, midpoint sums are lower bounds for B_{R}. Here they were evaluated in floating point with 57 digits, so this repetition is a numerical check; the proofs use the evaluations in Arb.

(2) The class numbers of all fundamental $\Delta < 40\,000$ and of the 2590 fields of extended Richaud–Degert type with $\Delta < 169\,918$, with separate code for cycles and continued fractions.

(3) The thresholds of Theorems 5.6 and 5.7. The second computation uses shorter blocks than the proofs, of ratio 1.002 for Theorem 5.6 and of ratio 1.0005 below 10^{18} for Theorem 5.7, and several values of $\beta \in [7/8, 1)$. For Theorem 5.7 it also replaces $\ell(Y)$ by the smallest lower bound for $L(1, \chi_0)$ on $[5, Y]$ that the fundamental unit and G_0 give on blocks of Δ_0 of ratio 1.03. So it gives the thresholds in the stronger forms $\Delta \ge 2.24 \cdot 10^{13}$ and $\Delta \ge 2.79 \cdot 10^{16}$, the latter for $\Delta \le 10^{40}$ only.

(4) The two sieves (the one for Theorem 5.6 with forms instead of continued fractions, the one for Theorem 5.7 up to $n = 237\,276\,211$) and the classification of their survivors.

(5) The comparison of the list of Theorem 5.7 with Tables 3 and 4 of [43], from a second transcription of these tables.

(6) The class numbers of the 1958 and 688 fields of Theorem 5.9.

The bounds, the continued fractions, the tests of Lemma 5.8 and the other 1250 class numbers in the proof of Theorem 5.9 were not repeated.

A.5. **Data.** The list $\mathcal{L}$ is a text file of 36 993 853 lines "D h", in order of increasing $|D|$, with MD5 checksum `6766ba3e148a6e69e440aaa5997c329f`. Compressed with gzip it has 168.6 MB and the SHA-256 checksum `d9450d60a1abbf052c40eeb9f92d012e54a73633a46d4c38fe2f42692604ad6b`. It is deposited in [38], together with the table of $F(h)$ and of the largest and smallest $|D|$ for each $h \le 3000$, the lists of Section 5 (among them the 7 729 430 orders of Theorem 5.11), the programs of this appendix, their outputs (summary lines for the longest runs, and the checksums of the output files of the computation of Theorem 1.2), and the commands that run them.

A.6. **The computation of Lemma 3.4.** The procedure of Section 3.2 was run on the eleven intervals of Table 3, with $P = 19$, 23 or 29, so that Q is 232 792 560, 5 354 228 880 or 155 272 637 520. The n treated are those with $n \bmod 16 \in \{3, 7, 11, 15, 4, 8\}$ and $9 \nmid n$. In each interval their number equals a direct count of them, and no n was left over. For each pattern of splitting types of the primes $p \le P$ that step (1) does not decide, and each j, the program computes in advance the least $\tau \le \min(8j, 254)$ with $V(\tau, q_{8j}) > 3000$, if there is one, and in step (2) it compares τ_j with it. This suffices because V does not decrease as τ grows, and the limit 254 can only send more n on to step (3). A separately written program checked all 12 660 254 of these values, and the outcome of step (1) for every pattern, against Lemma 3.3, in about half a minute.

range of n	P	Y	n treated	n reaching step (3)
$[2^{40}, 2^{41})$	19	524 288	366 503 875 925	94 581
$[2^{41}, 2^{42})$	19	741 455	733 007 751 850	764
$[2^{42}, 2^{43})$	19	1 048 576	1 466 015 503 702	0
$[2^{43}, 2^{44})$	23	1 482 910	2 932 031 007 402	0
$[2^{44}, 2^{45})$	23	2 097 152	5 864 062 014 805	0
$[2^{45}, 2^{46})$	29	2 965 820	11 728 124 029 612	0
$[2^{46}, 2^{47})$	29	4 194 304	23 456 248 059 221	0
$[2^{47}, 2^{48})$	29	5 931 641	46 912 496 118 442	0
$[2^{48}, 2^{49})$	29	8 388 608	93 824 992 236 886	0
$[2^{49}, 2^{50})$	29	11 863 283	187 649 984 473 770	0
$[2^{50}, 1\,606\,926\,912\,624\,720]$	29	16 777 216	160 342 335 260 699	0
total			535 275 800 332 314	95 345

TABLE 3. The intervals of the sieve of Lemma 3.4, with the prime P and the number Y used in each, the number of n treated, and the number of them that reached step (3). No n was left over.

The program is written in C, with 64- and 128-bit integers, and in V its sums and products are capped at 2^{40}, which cannot change a comparison with 3000. The residue classes were treated in 5845 parts, each by a single-threaded run, and the elapsed times of these runs add up to 214 482 seconds. The program passed tests against `qfbclassno` on intervals below $2 \cdot 10^{10}$, with $N = 30$, 100, 1000 or 3000 as the bound, in which every n with $-n$ fundamental and $h(-n) \le N$ must be left over, and against the pairs of $\mathcal{L}$ in four intervals. A separately written program, which shares no code with the first, treated the whole range again. In each of the 5845 parts it treated the same number of n, left the same number after step (1), and left no n over. The programs and their outputs are in [38].

Appendix B. The number of fields and the largest discriminant for each class number

Assuming (K), for example under (H), the following table gives, for each $h \le 3000$, the number $F(h)$ of imaginary quadratic fields of class number h and the largest $|D|$ among them (Theorem 1.1). Without (K), it gives the same numbers for the fields with $|D| \le 1\,606\,926\,912\,624\,720$ (Remark 4.3). Each entry gives h, $F(h)$ and the largest $|D|$ with $h(D) = h$, and h increases down each group of three columns. The table was generated by a program from the deposited table [38].

h	$F(h)$	max $\lvert D\rvert$	h	$F(h)$	max $\lvert D\rvert$	h	$F(h)$	max $\lvert D\rvert$	h	$F(h)$	max $\lvert D\rvert$
1	9	163	61	132	606 643	121	241	2 145 067	181	287	5 891 803
2	18	427	62	323	647 707	122	594	2 608 747	182	1069	6 434 803
3	16	907	63	216	991 027	123	299	2 849 683	183	437	6 079 483
4	54	1555	64	1672	693 067	124	1660	2 696 467	184	3786	6 110 083
5	25	2683	65	164	703 123	125	290	2 944 363	185	382	7 007 947
6	51	3763	66	530	958 483	126	1084	2 660 443	186	1215	7 363 987
7	31	5923	67	120	652 723	127	228	2 373 067	187	400	6 831 427
8	131	6307	68	976	819 163	128	3722	2 960 323	188	2551	7 402 987
9	34	10 627	69	209	888 427	129	332	2 899 747	189	599	7 899 523
10	87	13 843	70	560	811 507	130	839	2 863 963	190	1184	6 263 203
11	41	15 667	71	150	909 547	131	245	3 513 427	191	326	7 213 243
12	206	17 803	72	1930	947 923	132	2446	3 347 227	192	7613	7 462 123
13	37	20 563	73	119	886 867	133	316	3 318 547	193	343	5 954 587
14	95	30 067	74	407	951 043	134	667	2 995 507	194	866	6 610 363
15	68	34 483	75	237	916 507	135	468	2 751 907	195	619	7 268 347
16	322	31 243	76	1075	1 086 187	136	2895	3 082 123	196	3047	7 653 067
17	45	37 123	77	216	1 242 763	137	203	2 853 547	197	350	7 544 947
18	150	48 427	78	561	1 004 347	138	942	4 003 267	198	1422	8 377 363
19	47	38 707	79	175	1 333 963	139	249	3 501 283	199	346	7 194 427
20	350	58 507	80	2277	1 165 483	140	2678	3 574 363	200	5081	7 206 763
21	85	61 483	81	228	1 030 723	141	357	3 885 403	201	470	8 158 987
22	139	85 507	82	402	1 446 547	142	696	3 566 347	202	939	9 471 067
23	68	90 787	83	150	1 074 907	143	299	3 411 907	203	367	6 951 547
24	511	111 763	84	1715	1 225 387	144	5045	3 843 907	204	3713	8 143 843
25	95	93 307	85	221	1 285 747	145	339	4 061 467	205	486	8 190 163
26	190	103 027	86	472	1 534 723	146	667	4 272 403	206	1010	8 285 803
27	93	103 387	87	222	1 261 747	147	451	4 348 843	207	525	7 982 683
28	457	126 043	88	1905	1 265 587	148	1983	4 213 003	208	5838	7 890 283
29	83	166 147	89	192	1 429 387	149	244	4 056 067	209	404	7 821 907
30	255	134 467	90	801	1 548 523	150	1233	4 163 443	210	1923	8 141 827
31	73	133 387	91	214	1 391 083	151	291	4 321 003	211	354	10 560 643
32	708	164 803	92	1248	1 452 067	152	3284	4 431 667	212	2876	7 462 723
33	101	222 643	93	262	1 475 203	153	439	3 529 843	213	496	8 608 147
34	219	189 883	94	509	1 587 763	154	969	4 477 267	214	924	8 124 187
35	103	210 907	95	241	1 659 067	155	352	4 299 667	215	482	8 422 747
36	668	217 627	96	3283	1 684 027	156	2860	5 091 403	216	6383	8 928 187
37	85	158 923	97	185	1 842 523	157	266	4 375 963	217	481	8 386 123
38	237	289 963	98	580	2 383 747	158	763	5 029 627	218	1043	8 228 707
39	115	253 507	99	289	1 480 627	159	420	4 331 947	219	561	10 413 187
40	912	260 947	100	1736	1 856 563	160	5548	5 155 243	220	3776	8 447 107
41	109	296 587	101	196	1 701 043	161	318	3 981 763	221	453	9 388 723
42	339	280 267	102	664	1 765 003	162	1180	6 179 947	222	1448	8 128 387
43	106	300 787	103	181	1 790 443	163	255	4 962 787	223	394	10 877 563
44	691	319 867	104	2140	1 853 923	164	2105	5 002 363	224	7528	9 531 043
45	154	308 323	105	379	2 317 723	165	544	4 946 803	225	709	9 119 203
46	268	462 883	106	529	1 924 267	166	735	5 706 787	226	982	10 622 683
47	107	375 523	107	189	2 404 147	167	323	4 106 083	227	383	9 744 403
48	1365	335 203	108	2037	2 127 283	168	5194	5 233 243	228	4120	9 311 107
49	132	393 187	109	209	1 945 483	169	340	5 284 987	229	385	9 142 963
50	345	389 467	110	714	2 100 547	170	1058	4 845 163	230	1337	9 258 307
51	159	546 067	111	314	1 990 123	171	451	5 046 787	231	650	9 023 947
52	770	439 147	112	3194	2 804 587	172	2289	5 213 587	232	4716	9 304 843
53	114	425 107	113	239	2 341 747	173	301	5 494 003	233	402	12 424 003
54	427	532 123	114	743	2 143 843	174	1120	5 421 067	234	1720	10 150 387
55	163	452 083	115	307	2 458 003	175	450	4 742 467	235	473	9 915 427
56	1205	494 323	116	1652	2 888 923	176	4812	5 976 763	236	3127	10 340 947
57	179	615 883	117	345	2 341 243	177	444	5 455 627	237	567	9 842 347
58	291	586 987	118	580	3 060 787	178	851	5 134 963	238	1314	9 329 203
59	128	474 307	119	276	2 463 883	179	313	4 938 187	239	381	10 585 843
60	1302	662 803	120	3913	3 086 323	180	4174	6 469 747	240	10 329	10 300 987

h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$
241	355	10 573 963	320	12 684	19 446 403	399	1082	29 293 867	478	1998	44 233 963
242	1203	11 841 523	321	740	19 988 467	400	13 823	32 492 923	479	721	45 682 267
243	613	12 039 883	322	1743	23 212 963	401	645	29 489 923	480	25 529	47 070 307
244	3127	9 773 107	323	547	18 663 283	402	2390	31 900 507	481	777	44 310 403
245	532	10 804 627	324	6211	20 023 243	403	690	33 427 123	482	2069	45 852 643
246	1601	11 915 467	325	691	19 385 683	404	5180	28 650 763	483	1228	41 908 267
247	458	10 750 147	326	1424	18 818 347	405	1192	32 558 563	484	6657	47 602 003
248	5238	12 012 067	327	731	20 209 963	406	2086	30 164 227	485	846	42 827 107
249	571	12 316 987	328	6742	20 962 363	407	705	31 073 443	486	3277	47 073 163
250	1472	12 208 267	329	620	19 603 147	408	11 776	32 750 323	487	732	51 599 563
251	375	9 747 523	330	2697	22 595 203	409	622	35 251 483	488	10 100	43 344 907
252	5467	12 227 923	331	534	21 297 763	410	2193	38 288 083	489	1003	46 336 123
253	524	13 380 907	332	4229	20 118 883	411	870	32 317 147	490	2934	44 852 347
254	1190	11 337 307	333	822	19 573 387	412	5234	32 497 987	491	703	53 225 803
255	736	13 182 163	334	1471	20 396 707	413	751	31 667 563	492	8515	49 263 547
256	8352	14 154 067	335	634	18 418 723	414	2782	37 186 843	493	790	46 884 667
257	450	12 819 643	336	14 159	23 164 003	415	768	30 153 883	494	2336	55 171 603
258	1639	12 442 123	337	546	20 523 787	416	13 713	36 073 867	495	1545	48 384 043
259	510	11 420 203	338	1661	20 143 363	417	940	36 876 907	496	13 800	50 242 483
260	4409	13 368 667	339	793	22 104 427	418	2132	36 022 603	497	832	46 767 403
261	675	13 038 547	340	5573	22 716 787	419	650	32 909 563	498	3010	50 160 427
262	1173	11 352 763	341	598	21 507 523	420	10 265	34 766 323	499	741	45 889 603
263	447	13 682 923	342	2321	22 148 107	421	695	32 806 867	500	8023	53 698 747
264	7797	13 374 043	343	620	20 490 787	422	1790	34 170 403	501	1071	60 408 307
265	510	13 325 827	344	7041	20 649 787	423	1084	32 981 083	502	2110	51 459 787
266	1478	12 577 987	345	941	21 618 787	424	8672	35 634 523	503	748	43 969 843
267	589	15 127 963	346	1546	22 588 603	425	845	35 307 787	504	17 911	58 359 907
268	3519	13 488 883	347	504	22 161 547	426	2559	36 702 187	505	906	45 824 683
269	446	10 422 043	348	6198	24 255 067	427	774	32 593 243	506	2463	43 926 667
270	2280	13 779 187	349	511	23 539 147	428	5460	37 608 187	507	1179	55 454 107
271	430	12 931 843	350	2258	24 669 283	429	1056	31 578 523	508	6259	50 688 163
272	7544	14 850 067	351	941	25 305 883	430	2310	34 249 363	509	778	51 521 803
273	735	12 573 163	352	11 712	24 648 787	431	655	35 296 867	510	4063	52 094 947
274	1186	13 298 443	353	564	20 743 627	432	18 110	41 032 363	511	817	51 592 003
275	628	14 420 323	354	2218	21 376 987	433	637	35 309 107	512	18 052	59 249 107
276	5012	13 194 787	355	713	22 916 107	434	2292	32 677 507	513	1285	52 806 307
277	425	12 249 763	356	4549	24 933 283	435	1188	43 922 803	514	2138	49 371 067
278	1217	15 545 203	357	961	27 829 267	436	5334	36 732 403	515	969	51 808 387
279	725	15 545 083	358	1556	27 518 467	437	726	38 708 443	516	9091	50 816 683
280	8045	13 826 443	359	533	21 886 027	438	2663	39 731 323	517	856	53 784 427
281	474	15 375 067	360	13 282	28 094 203	439	658	37 137 787	518	2603	52 626 907
282	1760	14 682 523	361	599	25 351 363	440	12 188	37 966 723	519	1121	50 814 523
283	420	13 508 203	362	1580	27 673 963	441	1194	34 840 843	520	14 079	57 522 523
284	3694	17 091 523	363	896	24 610 723	442	2159	39 578 947	521	748	49 528 627
285	868	17 301 043	364	5728	27 446 323	443	639	33 352 243	522	3493	61 062 787
286	1557	15 519 667	365	676	23 679 307	444	7747	40 332 427	523	830	57 596 947
287	507	13 393 867	366	2321	28 528 963	445	835	38 254 483	524	6451	59 854 243
288	12 426	16 638 043	367	610	28 393 003	446	1954	38 341 363	525	1630	51 065 683
289	519	14 493 187	368	10 368	26 322 283	447	987	35 915 587	526	2269	50 707 603
290	1712	16 483 987	369	945	24 310 267	448	17 122	43 736 947	527	841	58 419 043
291	690	13 741 267	370	2017	30 031 747	449	707	40 298 707	528	22 060	58 199 563
292	3787	16 645 003	371	693	22 829 227	450	3745	40 868 683	529	824	53 210 347
293	452	14 378 827	372	6616	27 340 867	451	784	37 182 667	530	2793	59 680 987
294	2163	19 225 243	373	564	24 067 387	452	5691	40 501 147	531	1273	57 768 043
295	624	15 711 763	374	1889	28 077 883	453	918	39 824 467	532	8033	61 621 267
296	5966	17 012 587	375	1099	26 809 843	454	1980	42 498 787	533	857	59 880 283
297	824	15 231 283	376	7812	28 316 803	455	1043	40 767 043	534	3157	55 019 827
298	1344	15 423 787	377	618	23 737 123	456	13 197	43 839 427	535	974	56 386 507
299	553	17 472 163	378	3077	31 855 387	457	677	35 862 523	536	11 081	59 761 123
300	6744	19 184 323	379	583	24 165 667	458	2007	41 644 387	537	1150	67 637 707
301	590	16 997 443	380	6170	26 058 547	459	1164	39 399 427	538	2198	54 515 827
302	1313	17 417 107	381	838	26 708 467	460	7430	40 965 787	539	1027	54 624 403
303	689	18 433 123	382	1666	27 688 747	461	700	40 544 827	540	12 768	65 757 763
304	8514	19 517 227	383	560	27 968 803	462	3490	42 507 067	541	785	52 577 947
305	606	18 535 003	384	17 742	30 761 707	463	686	43 083 307	542	2236	57 918 547
306	2197	17 920 963	385	886	26 780 587	464	13 146	40 924 123	543	1186	58 891 627
307	517	17 389 507	386	1675	25 664 563	465	1306	45 287 587	544	18 453	65 544 187
308	4986	19 585 603	387	940	25 919 107	466	1980	45 050 923	545	1029	66 690 187
309	698	18 509 467	388	4874	35 723 227	467	687	38 668 963	546	4109	62 862 067
310	1636	21 996 307	389	613	28 406 107	468	9285	45 885 403	547	824	54 855 403
311	489	20 950 603	390	3081	28 080 427	469	854	43 522 747	548	6867	66 775 867
312	9181	19 288 483	391	680	27 417 427	470	2427	45 258 427	549	1279	56 086 363
313	470	16 992 523	392	9356	30 818 923	471	989	38 478 763	550	3254	69 758 203
314	1381	19 325 707	393	878	29 641 987	472	9718	40 637 083	551	873	60 342 067
315	1091	19 247 083	394	1619	28 282 963	473	818	39 867 787	552	16 099	57 137 683
316	4130	19 149 307	395	761	30 583 963	474	2789	42 388 963	553	994	63 345 067
317	489	17 947 483	396	8104	31 203 307	475	961	43 078 963	554	2399	56 890 387
318	1898	20 077 843	397	560	30 407 227	476	7351	50 456 827	555	1436	69 512 683
319	577	19 749 403	398	1745	29 257 243	477	1128	44 573 323	556	6982	60 391 963

h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$
557	817	66 345 427	636	10 681	81 355 003	715	1464	94 282 147	794	3179	122 373 043
558	3584	71 128 843	637	1148	75 830 203	716	8794	99 924 883	795	1933	131 987 563
559	893	53 868 523	638	2999	77 537 923	717	1404	102 300 907	796	9735	130 653 307
560	22 468	64 594 363	639	1420	79 467 523	718	2899	101 024 683	797	1108	123 916 123
561	1302	56 668 627	640	28 038	80 168 467	719	948	100 108 387	798	5520	137 933 203
562	2299	61 221 763	641	931	70 658 347	720	37 917	123 729 643	799	1175	123 585 067
563	825	59 528 107	642	3818	85 285 267	721	1178	92 653 507	800	34 149	129 667 987
564	9837	59 070 643	643	952	73 016 227	722	3077	101 630 323	801	1787	139 553 107
565	1006	67 093 723	644	9561	83 669 107	723	1499	93 658 723	802	3208	124 203 643
566	2383	68 670 067	645	1692	76 959 283	724	8628	103 798 963	803	1293	128 545 267
567	1580	65 371 483	646	3054	80 923 627	725	1301	106 911 667	804	13 776	133 243 147
568	11 791	63 548 587	647	974	71 702 467	726	4650	115 970 563	805	1734	141 568 123
569	779	65 187 523	648	20 511	91 321 003	727	985	122 110 123	806	3628	135 520 627
570	4261	74 954 587	649	1037	77 313 307	728	18 520	104 713 603	807	1639	125 113 003
571	794	59 006 803	650	3683	85 527 187	729	1709	123 302 587	808	16 596	121 359 067
572	8297	60 046 963	651	1570	90 447 883	730	3659	107 328 307	809	1156	120 725 707
573	1126	63 351 787	652	8049	81 928 267	731	1179	107 673 883	810	6621	140 273 827
574	2840	62 833 387	653	911	79 385 923	732	12 404	108 805 027	811	1137	158 712 283
575	1126	61 595 227	654	3861	78 668 683	733	993	111 289 963	812	11 964	150 164 683
576	29 316	60 167 083	655	1146	69 530 827	734	2969	106 768 027	813	1638	119 515 027
577	843	66 021 547	656	18 840	78 691 867	735	2167	121 074 787	814	3621	120 762 427
578	2659	64 332 907	657	1525	87 331 003	736	25 227	123 333 907	815	1468	124 839 163
579	1197	64 080 283	658	3145	75 668 587	737	1161	108 258 187	816	34 008	149 560 387
580	9097	74 714 803	659	931	85 688 947	738	4710	115 767 283	817	1202	119 386 507
581	996	61 582 987	660	15 230	97 987 723	739	1014	115 900 387	818	3266	145 139 443
582	3437	65 733 427	661	976	84 638 107	740	11 330	112 893 883	819	2294	125 709 907
583	892	62 761 147	662	2659	86 767 003	741	1624	99 772 003	820	12 386	147 244 843
584	11 979	66 502 363	663	1601	94 616 923	742	3593	115 869 427	821	1163	119 795 083
585	1697	67 477 243	664	13 799	100 838 203	743	1066	105 030 907	822	4744	140 712 643
586	2485	69 565 507	665	1338	96 295 483	744	21 781	130 896 187	823	1126	138 663 163
587	830	62 758 963	666	4365	85 419 427	745	1252	105 568 747	824	16 897	141 827 827
588	12 007	72 745 003	667	988	91 974 787	746	3101	109 179 403	825	2305	130 788 067
589	905	62 212 483	668	8273	92 191 507	747	1627	110 698 963	826	3965	125 508 307
590	3062	63 302 803	669	1339	95 914 267	748	10 559	119 356 603	827	1162	136 016 827
591	1214	63 979 507	670	3459	101 369 227	749	1199	95 480 443	828	16 136	141 542 347
592	17 083	70 843 987	671	1056	78 828 667	750	5750	113 620 123	829	1190	134 317 243
593	812	57 989 947	672	35 227	100 330 267	751	1060	104 572 843	830	4131	124 521 763
594	4353	64 950 307	673	1037	94 306 843	752	21 470	124 041 523	831	1624	151 992 787
595	1328	69 729 067	674	2684	91 248 763	753	1549	123 226 027	832	32 008	139 592 707
596	7386	74 947 387	675	1973	104 367 187	754	3369	119 476 963	833	1486	130 835 203
597	1169	81 792 547	676	9065	91 655 323	755	1339	112 086 307	834	4761	134 911 723
598	2759	68 590 267	677	960	96 572 107	756	16 831	113 621 467	835	1485	144 622 003
599	840	66 227 107	678	3825	100 155 427	757	1077	128 754 763	836	11 781	141 715 267
600	21 632	72 789 403	679	1111	93 802 963	758	2984	110 704 147	837	1934	144 136 987
601	829	69 640 267	680	17 899	96 770 227	759	1714	116 303 947	838	3292	171 583 003
602	2956	65 317 267	681	1354	82 751 923	760	20 105	126 881 227	839	1188	131 605 027
603	1380	71 037 283	682	3166	84 706 603	761	1020	110 771 347	840	34 194	156 666 283
604	7484	77 431 027	683	980	88 488 667	762	4410	116 743 243	841	1265	130 381 843
605	1179	81 250 723	684	13 317	104 158 723	763	1339	104 312 707	842	3401	127 523 467
606	3495	70 946 467	685	1169	93 347 923	764	9231	122 978 923	843	1676	146 713 627
607	813	76 013 347	686	3398	88 589 587	765	2230	132 978 763	844	10 296	144 857 707
608	20 476	72 038 107	687	1472	94 024 627	766	3004	119 955 067	845	1571	157 251 763
609	1595	77 518 123	688	19 591	105 787 147	767	1187	133 009 507	846	5504	151 364 467
610	3094	81 294 907	689	1092	74 149 483	768	39 465	129 527 227	847	1501	149 412 547
611	976	75 089 107	690	5271	105 983 707	769	1101	128 940 883	848	24 609	153 583 123
612	12 059	79 524 307	691	1037	95 303 587	770	4990	138 731 107	849	1655	135 323 203
613	906	75 061 387	692	8608	114 148 483	771	1614	114 716 347	850	4633	135 386 227
614	2588	90 071 083	693	1913	98 059 747	772	9436	120 837 907	851	1256	151 867 123
615	1694	85 561 963	694	2780	96 266 347	773	1117	108 167 467	852	14 347	150 568 147
616	15 819	69 474 043	695	1277	93 313 147	774	4976	130 810 387	853	1118	131 212 603
617	902	76 185 883	696	20 092	101 001 763	775	1443	134 009 467	854	4044	155 098 507
618	3597	83 826 667	697	1077	93 409 387	776	15 691	131 332 723	855	2425	156 770 947
619	909	78 266 203	698	2984	99 379 963	777	1851	120 924 763	856	17 548	155 127 043
620	9690	76 992 547	699	1425	98 700 067	778	3186	117 493 387	857	1192	154 427 827
621	1556	77 100 307	700	12 611	96 016 243	779	1199	113 809 963	858	5870	157 030 603
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623	986	87 062 683	702	4999	102 191 923	781	1212	117 333 547	860	12 983	157 010 107
624	26 094	80 086 147	703	1081	101 799 427	782	3584	112 744 003	861	2053	141 556 963
625	1140	81 289 723	704	26 837	98 120 947	783	1823	116 996 947	862	3467	162 052 507
626	2531	75 825 667	705	1846	99 136 867	784	26 284	138 462 283	863	1178	150 892 267
627	1465	79 087 387	706	2829	93 251 827	785	1251	134 919 163	864	44 744	154 728 667
628	7692	74 567 683	707	1155	93 892 723	786	4458	121 759 243	865	1468	156 729 763
629	978	76 430 803	708	12 197	93 968 323	787	1105	113 957 083	866	3530	156 788 923
630	5813	82 735 843	709	968	82 200 787	788	9507	130 665 907	867	1738	143 231 467
631	951	80 014 603	710	3632	100 109 563	789	1495	128 744 947	868	12 530	150 667 603
632	12 731	78 245 227	711	1569	115 572 403	790	3961	130 028 347	869	1252	168 852 643
633	1326	91 032 307	712	14 580	101 667 883	791	1300	125 669 923	870	6378	165 466 723
634	2669	79 358 827	713	1151	112 768 027	792	27 167	122 655 163	871	1345	151 351 363
635	1152	73 099 723	714	5107	111 745 363	793	1213	103 500 763	872	17 822	168 498 163

h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$
873	1896	159 884 587	952	23 855	198 435 427	1031	1388	206 098 027	1110	7789	272 977 387
874	3993	155 884 243	953	1284	171 620 443	1032	29 958	233 333 563	1111	1707	259 242 523
875	1801	149 457 163	954	5904	179 832 907	1033	1375	193 834 243	1112	22 853	280 624 843
876	14 946	165 368 107	955	1638	180 282 523	1034	4562	217 613 467	1113	2525	265 115 827
877	1215	147 558 643	956	11 256	219 600 907	1035	2845	233 047 963	1114	4387	229 933 267
878	3471	150 755 467	957	2120	181 968 403	1036	14 666	243 829 987	1115	1825	290 185 603
879	1705	144 444 787	958	3775	195 129 643	1037	1520	222 138 547	1116	20 923	259 576 003
880	33 779	154 570 483	959	1484	210 971 083	1038	5929	245 007 547	1117	1534	249 222 187
881	1252	170 094 787	960	59 966	194 302 483	1039	1422	222 291 283	1118	4738	257 612 587
882	6477	146 218 363	961	1335	213 119 083	1040	39 636	239 630 227	1119	2147	256 160 227
883	1179	145 024 147	962	4317	186 475 963	1041	2049	202 211 563	1120	54 797	273 633 523
884	12 228	163 300 867	963	2142	205 377 547	1042	3991	209 439 547	1121	1527	261 823 963
885	2164	159 212 827	964	11 303	196 168 603	1043	1678	215 364 403	1122	7133	260 258 923
886	3589	172 234 987	965	1662	223 175 803	1044	19 965	269 497 867	1123	1462	280 506 403
887	1142	149 610 547	966	6592	186 210 523	1045	2012	246 245 707	1124	13 422	242 821 483
888	25 622	175 172 083	967	1356	164 793 787	1046	4116	233 421 043	1125	3203	293 745 307
889	1406	139 898 467	968	21 580	195 549 187	1047	1975	210 527 683	1126	4464	268 233 067
890	4518	162 481 603	969	2133	193 969 147	1048	21 743	220 885 723	1127	1870	240 360 283
891	2316	186 445 603	970	4920	194 987 083	1049	1405	235 400 587	1128	32 774	280 768 027
892	10 791	163 956 307	971	1336	198 318 283	1050	8926	237 710 467	1129	1421	248 912 203
893	1375	162 226 387	972	18 686	217 569 643	1051	1370	216 781 987	1130	5450	289 832 827
894	5134	199 362 403	973	1564	177 983 227	1052	12 376	249 255 043	1131	2390	263 248 507
895	1548	161 831 443	974	3732	195 723 067	1053	2566	219 806 563	1132	13 384	305 952 067
896	38 662	176 301 283	975	2638	184 039 507	1054	4592	273 722 443	1133	1685	278 490 283
897	1913	171 987 883	976	28 467	206 802 523	1055	1741	262 502 923	1134	8441	320 578 003
898	3523	176 576 107	977	1308	204 361 987	1056	54 852	257 010 307	1135	1807	251 210 227
899	1321	175 001 803	978	5661	209 362 147	1057	1740	246 066 307	1136	32 904	266 144 803
900	21 669	180 619 363	979	1447	181 428 307	1058	4414	224 476 267	1137	2184	291 379 603
901	1408	165 055 867	980	17 221	209 045 467	1059	1970	264 600 043	1138	4446	256 705 243
902	3953	169 183 723	981	2197	199 180 507	1060	15 878	226 856 947	1139	1672	265 270 987
903	2150	188 330 563	982	3892	199 060 843	1061	1457	233 683 627	1140	25 205	268 408 603
904	18 529	186 893 107	983	1383	188 697 763	1062	6696	259 976 203	1141	1727	301 597 003
905	1492	176 649 883	984	28 648	224 296 867	1063	1403	200 264 443	1142	4449	267 925 123
906	5209	177 192 163	985	1681	201 556 987	1064	26 634	234 531 427	1143	2399	284 106 883
907	1227	144 604 387	986	4242	181 997 467	1065	2580	249 891 163	1144	27 657	284 199 547
908	10 861	182 873 923	987	2302	203 540 803	1066	4626	264 367 027	1145	1927	244 750 027
909	2014	190 884 523	988	13 329	224 315 947	1067	1552	236 278 387	1146	6393	291 977 827
910	5614	149 830 627	989	1481	226 134 283	1068	18 292	241 452 187	1147	1660	308 001 787
911	1255	151 140 907	990	8267	225 045 283	1069	1439	218 788 123	1148	16 253	262 659 763
912	38 249	182 782 867	991	1345	185 039 563	1070	5277	233 908 123	1149	2235	282 533 227
913	1404	164 274 427	992	34 516	203 122 627	1071	2744	251 297 083	1150	6134	267 273 427
914	3587	180 311 107	993	1980	202 384 603	1072	31 234	245 233 123	1151	1435	268 798 363
915	2277	173 473 987	994	4609	215 796 667	1073	1513	233 604 067	1152	67 312	289 615 747
916	10 928	175 164 763	995	1672	178 294 747	1074	6096	227 546 467	1153	1608	250 969 867
917	1476	173 215 387	996	16 872	227 932 027	1075	1943	219 257 707	1154	4682	265 950 547
918	6395	178 123 003	997	1316	192 419 467	1076	12 797	252 601 603	1155	3540	298 022 587
919	1247	200 171 203	998	3953	207 530 107	1077	2086	233 549 803	1156	14 495	266 325 403
920	24 300	181 585 867	999	2377	185 271 403	1078	5631	236 009 467	1157	1640	263 308 963
921	1849	185 482 243	1000	26 378	220 287 043	1079	1526	251 592 763	1158	6309	285 419 467
922	3739	179 396 947	1001	1832	192 755 347	1080	43 231	242 029 027	1159	1614	270 320 227
923	1365	186 800 443	1002	5662	235 714 747	1081	1572	249 377 923	1160	29 963	275 415 067
924	19 865	189 366 907	1003	1455	203 521 147	1082	4190	235 656 523	1161	2544	266 532 403
925	1579	164 446 027	1004	12 039	234 914 947	1083	2198	241 695 283	1162	5452	257 316 403
926	3700	179 345 203	1005	2488	212 381 227	1084	13 068	242 854 627	1163	1544	278 856 043
927	2006	181 811 947	1006	3958	192 106 363	1085	2105	250 563 163	1164	19 549	280 964 443
928	32 030	166 976 947	1007	1451	229 119 067	1086	6052	261 044 323	1165	1927	303 875 947
929	1252	195 829 363	1008	51 383	232 380 163	1087	1410	236 405 227	1166	5130	292 782 667
930	6727	193 552 507	1009	1370	207 839 923	1088	41 956	235 214 323	1167	2194	308 121 067
931	1510	197 092 963	1010	4989	185 905 003	1089	2506	239 208 643	1168	34 166	285 935 203
932	11 367	185 691 643	1011	1908	220 161 043	1090	5234	283 506 787	1169	1789	300 684 547
933	1918	170 043 547	1012	13 981	196 836 907	1091	1467	254 160 883	1170	9668	283 386 163
934	3710	194 283 763	1013	1358	216 618 187	1092	23 190	250 982 803	1171	1559	264 120 187
935	1770	164 219 323	1014	6287	206 202 067	1093	1471	252 107 923	1172	13 802	271 351 243
936	31 655	177 801 187	1015	2051	195 482 827	1094	4215	253 033 267	1173	2403	303 458 563
937	1309	190 038 643	1016	20 862	216 799 867	1095	2720	240 461 083	1174	4548	303 289 387
938	4515	181 203 163	1017	2139	209 464 627	1096	22 614	258 993 067	1175	2076	282 032 587
939	1846	217 576 027	1018	4106	202 632 763	1097	1449	217 289 587	1176	40 005	337 552 363
940	14 293	182 332 483	1019	1323	214 575 043	1098	6954	281 138 323	1177	1667	270 643 003
941	1301	193 411 003	1020	22 303	227 651 587	1099	1701	255 118 147	1178	5018	351 036 667
942	5446	174 986 947	1021	1379	212 329 723	1100	18 227	273 494 323	1179	2527	266 596 003
943	1435	187 231 003	1022	4780	214 203 907	1101	2080	290 661 067	1180	17 656	288 034 363
944	27 214	175 885 627	1023	2268	218 230 027	1102	4760	292 153 387	1181	1540	297 742 867
945	3021	169 102 627	1024	39 689	227 174 803	1103	1436	241 812 187	1182	6525	305 959 987
946	4327	164 579 683	1025	1849	233 875 387	1104	46 311	282 961 963	1183	1999	273 210 403
947	1255	155 836 003	1026	6774	227 245 363	1105	2136	238 049 443	1184	41 684	296 301 163
948	16 125	188 515 387	1027	1532	203 216 227	1106	5204	268 867 747	1185	2815	317 047 963
949	1393	172 005 667	1028	12 257	229 591 147	1107	2479	243 060 283	1186	4644	287 886 523
950	5087	186 089 443	1029	2351	220 503 187	1108	13 404	252 845 443	1187	1623	278 117 107
951	1871	191 676 643	1030	5008	224 954 563	1109	1490	255 765 547	1188	24 763	297 989 323

h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$
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1190	7071	292 556 947	1269	2758	331 166 467	1348	15 642	383 003 707	1427	1865	439 658 563
1191	2227	320 953 483	1270	6211	374 776 147	1349	1891	376 469 923	1428	29 458	476 411 203
1192	24 379	308 954 563	1271	1841	325 331 323	1350	10 963	407 384 707	1429	1809	391 965 403
1193	1651	287 036 443	1272	36 704	345 316 603	1351	2040	420 542 467	1430	8137	467 802 883
1194	6644	316 790 203	1273	1819	388 371 523	1352	29 824	403 344 043	1431	2994	479 731 387
1195	1964	271 887 907	1274	6249	368 361 403	1353	2957	373 788 787	1432	29 137	459 654 427
1196	15 951	300 817 003	1275	3406	325 870 147	1354	5206	343 999 723	1433	1939	452 945 683
1197	3105	282 506 827	1276	16 903	342 403 147	1355	2195	372 815 227	1434	7976	444 521 683
1198	4583	303 625 723	1277	1661	367 841 203	1356	22 768	384 750 403	1435	2837	431 025 043
1199	1838	295 216 387	1278	7983	331 757 827	1357	1894	381 847 603	1436	16 689	429 027 307
1200	62 588	311 824 027	1279	1629	332 830 243	1358	6197	376 683 547	1437	2627	405 598 603
1201	1615	326 861 683	1280	63 048	339 037 147	1359	2826	371 440 987	1438	5462	489 186 427
1202	4650	301 151 467	1281	2890	351 597 667	1360	51 789	407 482 483	1439	1806	483 667 507
1203	2191	312 176 923	1282	4957	363 593 563	1361	1805	367 901 227	1440	96 010	457 914 307
1204	16 987	362 944 723	1283	1679	326 819 707	1362	7576	387 236 683	1441	2109	431 790 283
1205	1968	278 644 507	1284	21 576	383 012 323	1363	1826	355 983 643	1442	6514	420 564 523
1206	7299	285 044 467	1285	2084	340 660 387	1364	18 106	426 051 763	1443	2895	434 766 187
1207	1699	376 058 587	1286	4994	378 425 107	1365	4011	395 621 923	1444	18 005	489 501 643
1208	24 928	294 476 323	1287	3188	363 753 307	1366	5341	394 335 547	1445	2399	422 714 923
1209	2584	307 524 163	1288	31 798	355 408 267	1367	1730	344 605 867	1446	8017	439 924 123
1210	6515	316 859 467	1289	1693	376 862 803	1368	45 479	412 732 483	1447	1901	422 639 683
1211	1889	300 780 043	1290	9132	369 115 723	1369	1797	406 086 283	1448	29 830	455 808 907
1212	20 641	321 220 243	1291	1714	309 127 603	1370	6537	428 548 387	1449	3658	462 183 187
1213	1582	310 467 763	1292	17 195	356 473 483	1371	2563	357 362 923	1450	7492	498 578 683
1214	4685	301 138 027	1293	2435	350 142 307	1372	19 202	467 477 707	1451	1885	491 994 787
1215	3281	354 087 427	1294	5057	372 325 507	1373	1888	380 881 867	1452	26 563	485 942 923
1216	47 627	340 796 107	1295	2551	354 895 003	1374	7563	373 705 963	1453	1831	482 347 387
1217	1673	315 321 043	1296	59 961	397 586 443	1375	2574	394 786 723	1454	5602	397 314 283
1218	8372	342 596 587	1297	1702	336 728 683	1376	48 915	375 267 523	1455	3326	491 757 883
1219	1728	326 730 163	1298	5597	351 267 403	1377	3179	383 862 403	1456	53 070	489 770 227
1220	18 115	307 436 803	1299	2470	368 064 547	1378	5950	398 467 987	1457	1924	450 952 483
1221	2620	335 476 507	1300	21 294	383 101 387	1379	2024	408 348 403	1458	9201	451 397 083
1222	5223	332 743 387	1301	1680	342 271 603	1380	30 095	446 063 347	1459	1902	461 458 363
1223	1619	329 413 723	1302	8601	351 770 827	1381	1817	419 932 507	1460	21 555	461 690 923
1224	41 112	333 625 147	1303	1668	351 853 483	1382	5351	373 733 683	1461	2703	435 112 507
1225	2527	354 066 043	1304	26 708	416 171 827	1383	2532	413 217 667	1462	6188	458 174 347
1226	4680	345 453 067	1305	3492	332 704 387	1384	28 491	419 794 747	1463	2485	425 470 147
1227	2321	342 451 267	1306	5022	399 567 667	1385	2191	383 755 507	1464	42 601	553 348 867
1228	14 630	329 592 187	1307	1708	323 723 107	1386	10 758	423 566 707	1465	2306	493 786 483
1229	1586	308 113 843	1308	21 897	411 366 547	1387	1895	417 326 683	1466	5580	427 673 227
1230	8566	321 484 507	1309	2309	335 979 403	1388	16 435	412 852 507	1467	2965	459 472 267
1231	1631	289 890 283	1310	6299	346 893 523	1389	2603	467 770 867	1468	17 030	511 033 867
1232	45 532	323 233 747	1311	2711	321 757 363	1390	6621	402 154 267	1469	2102	433 669 723
1233	2630	283 336 243	1312	46 450	412 409 827	1391	1878	364 385 443	1470	11 989	485 173 483
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1235	2348	326 569 147	1314	8193	344 023 387	1393	2091	413 666 467	1472	57 892	508 010 563
1236	20 806	314 941 603	1315	2115	314 965 267	1394	5868	402 642 307	1473	2745	438 558 187
1237	1661	372 531 427	1316	18 693	387 403 963	1395	3642	401 749 723	1474	6274	456 169 123
1238	4940	334 686 643	1317	2453	394 248 523	1396	16 451	480 668 203	1475	2497	443 639 827
1239	2748	332 299 027	1318	5147	346 762 747	1397	2040	402 756 187	1476	27 613	508 196 203
1240	32 486	360 318 667	1319	1805	378 794 683	1398	7665	449 710 123	1477	2282	483 291 307
1241	1758	318 641 227	1320	51 253	361 313 467	1399	1895	428 227 963	1478	5633	469 006 987
1242	8207	328 785 307	1321	1587	355 109 203	1400	42 608	435 841 387	1479	3020	492 882 163
1243	1890	325 789 627	1322	5102	366 612 163	1401	2627	386 720 107	1480	38 278	556 963 243
1244	14 529	303 581 947	1323	3418	353 379 643	1402	5478	416 966 587	1481	1841	539 869 003
1245	2915	297 292 627	1324	15 577	408 816 907	1403	1954	419 869 123	1482	9267	459 015 187
1246	5741	356 625 427	1325	2210	345 465 187	1404	29 142	415 330 507	1483	1842	462 271 987
1247	1656	305 031 787	1326	8596	410 158 963	1405	2222	384 594 907	1484	20 646	491 180 827
1248	64 967	329 910 667	1327	1818	379 922 467	1406	5884	427 948 987	1485	4436	494 561 827
1249	1564	376 479 643	1328	38 858	371 024 347	1407	3097	429 552 883	1486	5761	442 210 267
1250	6324	327 227 587	1329	2552	370 625 803	1408	60 247	421 647 403	1487	1926	489 295 867
1251	2604	382 391 683	1330	7815	350 000 347	1409	1821	453 607 603	1488	63 161	494 862 883
1252	14 748	328 759 267	1331	1875	360 360 163	1410	9879	477 343 267	1489	1887	562 394 347
1253	1868	317 584 507	1332	24 926	391 758 427	1411	1917	429 461 827	1490	7060	487 618 723
1254	8203	398 158 363	1333	1738	328 317 763	1412	16 772	445 673 083	1491	3231	482 166 787
1255	2001	300 291 643	1334	5567	373 559 587	1413	2929	470 603 347	1492	17 424	471 743 563
1256	25 587	311 444 323	1335	3103	369 843 307	1414	6409	444 073 723	1493	1883	425 169 427
1257	2382	338 213 107	1336	27 033	408 183 427	1415	2279	472 933 987	1494	9373	498 143 683
1258	5311	330 526 267	1337	2007	336 900 523	1416	41 437	479 475 067	1495	2763	467 500 027
1259	1661	386 076 667	1338	7402	416 394 763	1417	2027	379 160 923	1496	35 145	470 345 947
1260	33 451	380 536 843	1339	1955	370 567 723	1418	5429	433 759 243	1497	2808	450 527 683
1261	1831	351 859 363	1340	19 975	411 477 763	1419	3097	451 708 723	1498	6867	531 277 363
1262	4919	341 311 147	1341	2781	362 361 283	1420	21 122	415 790 083	1499	1920	437 319 667
1263	2414	327 015 067	1342	5681	374 741 203	1421	2261	388 972 747	1500	32 451	507 333 523
1264	36 747	370 522 483	1343	1927	373 190 227	1422	8672	430 450 483	1501	2100	478 074 283
1265	2403	313 739 563	1344	82 393	381 916 243	1423	1763	417 278 467	1502	5728	422 958 187
1266	7060	333 466 387	1345	2155	388 624 867	1424	41 501	421 608 163	1503	3110	511 212 307
1267	2006	348 689 203	1346	5239	357 450 307	1425	3613	447 171 883	1504	53 836	474 340 483

h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$
1505	2791	481 629 523	1584	78 651	643 541 347	1663	2139	559 851 307	1742	7206	649 377 523
1506	8298	508 072 027	1585	2586	563 431 003	1664	71 534	651 375 883	1743	3712	626 065 147
1507	2189	534 323 707	1586	6705	565 446 667	1665	4270	559 509 043	1744	51 788	658 167 427
1508	19 684	487 476 907	1587	3022	542 972 203	1666	8005	603 248 563	1745	2826	762 320 227
1509	2796	518 335 603	1588	18 670	589 411 243	1667	2159	571 342 627	1746	10 610	710 806 387
1510	7149	491 026 243	1589	2444	448 120 003	1668	27 617	587 391 667	1747	2227	683 284 123
1511	1880	439 654 363	1590	11 020	566 807 707	1669	2122	619 898 203	1748	22 508	670 622 467
1512	57 155	503 188 723	1591	2071	598 101 043	1670	7892	669 748 003	1749	3624	637 856 563
1513	2201	473 100 643	1592	32 444	547 922 923	1671	3082	549 964 963	1750	10 142	604 281 283
1514	5662	484 600 747	1593	3338	521 732 443	1672	39 559	637 343 947	1751	2353	592 918 243
1515	3692	448 459 867	1594	6085	522 779 203	1673	2445	610 607 083	1752	51 049	710 911 723
1516	17 659	497 016 787	1595	2865	599 961 763	1674	10 868	634 086 667	1753	2253	690 156 163
1517	2088	484 678 027	1596	32 405	582 928 747	1675	2877	707 327 707	1754	6739	705 917 347
1518	9661	507 908 347	1597	2032	481 030 867	1676	19 560	689 144 107	1755	4961	747 018 787
1519	2416	489 666 643	1598	6444	552 500 323	1677	3399	619 115 347	1756	20 347	735 060 643
1520	57 849	538 073 803	1599	3229	538 566 667	1678	6419	585 756 763	1757	2521	659 353 843
1521	3322	507 553 363	1600	78 901	590 007 163	1679	2218	537 755 563	1758	9699	668 131 627
1522	5893	514 825 963	1601	2061	575 284 627	1680	100 165	704 664 283	1759	2168	608 989 987
1523	1965	536 775 163	1602	9696	580 959 163	1681	2126	530 827 147	1760	84 768	659 034 643
1524	25 323	467 531 443	1603	2436	522 848 467	1682	6611	686 722 723	1761	3220	615 662 587
1525	2618	479 173 963	1604	18 519	526 380 187	1683	3994	621 959 083	1762	6646	693 185 707
1526	6729	490 594 267	1605	3703	500 464 603	1684	19 795	647 720 803	1763	2372	618 429 883
1527	2740	539 898 643	1606	6770	592 242 883	1685	2729	564 149 107	1764	38 715	697 474 507
1528	31 259	569 211 547	1607	2093	603 427 507	1686	9123	602 371 603	1765	2891	644 394 067
1529	2207	462 331 963	1608	46 479	632 740 243	1687	2520	541 340 083	1766	6716	654 243 907
1530	12 051	476 926 123	1609	2091	525 811 603	1688	34 320	603 638 443	1767	3535	805 169 467
1531	1996	444 930 907	1610	9192	616 540 387	1689	3065	607 412 947	1768	41 264	656 407 123
1532	17 831	463 337 323	1611	3260	560 195 707	1690	8539	648 121 027	1769	2385	740 010 643
1533	3384	505 029 163	1612	21 042	583 488 427	1691	2256	630 687 787	1770	11 985	656 692 723
1534	6521	549 203 323	1613	2086	546 542 683	1692	31 367	637 896 163	1771	3070	695 898 547
1535	2421	478 560 427	1614	8830	529 345 147	1693	2147	609 287 827	1772	20 396	726 843 643
1536	87 721	585 811 843	1615	2852	514 130 563	1694	8407	658 566 547	1773	3572	682 500 163
1537	2024	482 338 147	1616	47 472	548 181 043	1695	3957	665 197 723	1774	6671	721 080 523
1538	5941	524 603 347	1617	3873	585 000 067	1696	60 565	663 944 803	1775	2956	627 508 627
1539	3393	523 575 067	1618	6190	626 172 787	1697	2154	566 930 827	1776	75 588	673 444 627
1540	28 403	532 908 667	1619	2126	517 810 987	1698	9105	577 582 987	1777	2236	634 809 883
1541	2101	503 210 563	1620	38 424	563 713 987	1699	2112	579 092 347	1778	7871	663 196 867
1542	8511	476 142 907	1621	2015	514 913 107	1700	27 050	632 665 027	1779	3230	642 820 963
1543	2003	507 641 227	1622	6232	520 623 643	1701	4056	653 106 283	1780	25 843	687 883 267
1544	31 488	515 351 323	1623	2920	609 759 067	1702	7020	570 523 627	1781	2376	650 383 507
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1546	5981	508 437 763	1625	2956	572 444 203	1704	49 754	696 311 323	1783	2181	688 401 547
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1548	28 893	491 970 763	1627	2115	518 977 867	1706	6513	640 736 947	1785	5046	762 846 547
1549	2003	444 963 667	1628	21 243	549 622 747	1707	3150	678 153 403	1786	7406	677 403 787
1550	7952	513 694 267	1629	3337	632 154 763	1708	23 565	702 834 763	1787	2205	585 077 683
1551	3247	488 841 763	1630	7723	549 607 963	1709	2162	630 094 147	1788	29 676	748 186 723
1552	45 723	643 338 763	1631	2418	489 451 363	1710	13 425	579 114 187	1789	2187	643 834 843
1553	1976	507 039 787	1632	85 796	601 042 843	1711	2390	672 898 627	1790	8502	752 547 427
1554	10 288	606 929 107	1633	2227	520 450 627	1712	50 520	702 085 987	1791	3647	729 701 347
1555	2399	502 421 203	1634	6676	512 728 507	1713	3205	694 419 643	1792	86 283	747 133 363
1556	18 262	516 747 067	1635	3803	624 032 467	1714	6411	598 246 147	1793	2440	756 971 587
1557	3225	496 669 387	1636	18 649	566 666 587	1715	3154	584 732 107	1794	10 910	769 318 243
1558	6594	565 113 187	1637	2076	560 747 947	1716	33 676	726 280 963	1795	2885	661 561 387
1559	2075	511 923 547	1638	12 392	554 609 323	1717	2288	643 462 747	1796	20 910	680 311 003
1560	59 707	515 568 883	1639	2325	597 734 083	1718	6617	634 546 243	1797	3305	733 966 027
1561	2308	436 659 787	1640	42 342	612 041 203	1719	3524	642 734 107	1798	7145	682 142 827
1562	6615	559 724 323	1641	3085	569 128 723	1720	44 516	676 228 963	1799	2645	639 819 643
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1564	20 353	512 873 827	1643	2146	579 547 963	1722	11 232	590 736 547	1801	2307	603 688 507
1565	2537	466 055 683	1644	27 483	697 792 363	1723	2160	567 170 563	1802	7425	781 678 627
1566	9917	567 812 347	1645	3100	528 860 707	1724	20 330	602 869 507	1803	3316	714 023 227
1567	2000	452 429 443	1646	6202	586 806 907	1725	4208	606 655 603	1804	23 553	741 907 483
1568	65 455	599 448 523	1647	3456	697 920 283	1726	6518	604 117 963	1805	3047	690 449 827
1569	2963	599 053 723	1648	48 610	519 163 507	1727	2437	630 943 723	1806	11 697	844 522 603
1570	7477	491 124 043	1649	2203	548 492 443	1728	106 268	684 208 627	1807	2431	663 108 763
1571	2001	507 102 523	1650	12 483	558 896 083	1729	2940	642 568 987	1808	53 671	774 657 643
1572	26 440	579 758 587	1651	2389	550 200 883	1730	8131	785 400 907	1809	3834	742 965 763
1573	2372	470 542 867	1652	22 828	626 418 307	1731	3190	584 943 067	1810	8600	709 070 107
1574	5978	505 428 163	1653	3360	563 528 947	1732	20 334	679 278 763	1811	2260	657 190 627
1575	4946	519 754 867	1654	6241	582 289 843	1733	2241	617 014 003	1812	30 343	792 103 603
1576	32 371	531 467 203	1655	2583	576 488 707	1734	10 078	622 042 507	1813	2792	661 322 803
1577	2171	521 070 883	1656	54 599	644 603 107	1735	2712	628 914 427	1814	6688	711 537 523
1578	8603	544 250 107	1657	2095	525 050 947	1736	42 737	703 197 763	1815	4613	711 546 907
1579	1949	492 699 043	1658	6320	556 138 003	1737	3500	667 561 603	1816	37 234	752 516 923
1580	23 103	597 661 387	1659	3593	570 223 483	1738	7340	717 192 403	1817	2411	742 304 347
1581	3108	589 658 323	1660	24 528	605 174 707	1739	2232	672 052 987	1818	10 951	769 939 867
1582	7024	515 387 203	1661	2239	610 772 947	1740	37 591	691 056 283	1819	2515	770 868 883
1583	1986	560 164 243	1662	9070	641 409 283	1741	2162	677 168 347	1820	32 839	733 896 403

h	$F(h)$	$\max \lvert D\rvert$	h	$F(h)$	$\max \lvert D\rvert$	h	$F(h)$	$\max \lvert D\rvert$	h	$F(h)$	$\max \lvert D\rvert$
1821	3293	737 485 387	1900	30 150	770 103 427	1979	2435	839 157 883	2058	13 084	959 384 827
1822	6874	743 076 667	1901	2313	698 377 843	1980	49 053	984 246 547	2059	2705	787 225 147
1823	2247	749 083 483	1902	10 403	793 364 707	1981	2823	805 634 923	2060	29 487	934 700 083
1824	95 076	707 750 803	1903	2604	776 728 747	1982	7521	827 795 467	2061	4190	903 061 723
1825	2978	733 665 067	1904	68 962	808 839 403	1983	3539	812 845 027	2062	7770	928 073 323
1826	7682	679 750 723	1905	4248	986 305 483	1984	80 097	794 743 987	2063	2477	867 532 843
1827	4380	741 959 563	1906	7115	811 709 683	1985	3107	820 912 723	2064	88 422	869 203 987
1828	21 154	701 211 883	1907	2325	656 829 427	1986	10 764	934 891 483	2065	3689	845 243 683
1829	2376	700 360 147	1908	35 624	930 526 963	1987	2475	801 444 403	2066	7648	910 533 163
1830	12 672	723 398 083	1909	2473	745 335 427	1988	27 025	932 409 307	2067	4216	876 849 667
1831	2355	716 260 147	1910	8949	788 167 843	1989	4478	899 811 763	2068	26 970	1 030 896 043
1832	37 399	694 572 547	1911	4526	742 091 443	1990	9162	807 876 283	2069	2661	857 790 763
1833	3724	736 651 987	1912	38 818	761 259 283	1991	2744	800 805 427	2070	15 851	970 749 187
1834	8180	856 765 123	1913	2403	695 227 747	1992	58 198	986 339 803	2071	2631	1 018 199 323
1835	2831	751 784 443	1914	11 953	859 086 187	1993	2451	941 714 467	2072	50 113	934 016 323
1836	36 832	863 010 787	1915	2856	735 730 627	1994	7533	814 518 763	2073	3690	922 449 067
1837	2581	639 629 083	1916	22 156	833 549 803	1995	5390	918 966 427	2074	8388	1 006 755 667
1838	6865	769 212 403	1917	4056	884 950 243	1996	22 841	918 012 643	2075	3324	879 656 563
1839	3285	705 805 963	1918	8394	737 738 467	1997	2471	749 077 123	2076	34 287	932 670 187
1840	70 111	811 444 363	1919	2574	791 180 707	1998	12 620	857 902 147	2077	2722	995 894 827
1841	2704	707 246 467	1920	136 235	849 599 083	1999	2576	828 131 683	2078	7709	836 985 307
1842	10 083	693 521 467	1921	2587	735 823 603	2000	76 708	981 755 923	2079	5478	956 261 923
1843	2448	777 139 603	1922	7606	856 332 163	2001	3942	852 301 123	2080	100 156	935 524 627
1844	21 371	761 445 787	1923	3485	838 164 163	2002	10 393	918 609 547	2081	2556	895 090 267
1845	4701	772 026 403	1924	24 702	799 936 867	2003	2458	864 400 387	2082	11 211	989 923 267
1846	7539	683 853 787	1925	3975	827 599 987	2004	33 142	931 751 923	2083	2597	903 086 347
1847	2290	773 777 467	1926	11 442	767 760 403	2005	3137	863 287 483	2084	24 126	1 030 089 523
1848	68 349	807 807 163	1927	2573	785 793 523	2006	8070	989 232 883	2085	4796	1 055 163 883
1849	2367	712 261 243	1928	39 367	840 236 587	2007	3878	894 676 843	2086	9175	872 842 147
1850	9147	816 950 227	1929	3483	858 573 787	2008	40 774	896 043 163	2087	2614	998 641 603
1851	3305	769 317 187	1930	9122	808 487 227	2009	3049	844 960 747	2088	69 225	990 349 963
1852	21 500	757 304 683	1931	2394	823 113 883	2010	13 487	871 852 963	2089	2574	895 706 443
1853	2499	803 471 827	1932	38 911	873 684 883	2011	2540	796 131 043	2090	11 193	1 027 198 027
1854	11 236	758 779 627	1933	2378	802 281 547	2012	23 075	936 673 483	2091	4098	1 040 332 507
1855	3407	821 105 707	1934	7213	734 012 203	2013	4127	850 007 827	2092	24 090	899 194 627
1856	74 206	744 134 227	1935	4804	821 883 043	2014	7968	858 436 363	2093	3401	978 128 803
1857	3381	620 980 627	1936	63 214	826 999 363	2015	3478	826 126 723	2094	11 302	980 221 027
1858	6961	825 685 723	1937	2692	794 914 387	2016	130 290	916 253 083	2095	3256	1 011 795 283
1859	2783	680 438 467	1938	11 705	865 662 163	2017	2513	795 875 683	2096	62 656	1 019 929 243
1860	39 256	857 255 947	1939	2867	801 116 803	2018	7519	997 675 963	2097	4111	966 078 523
1861	2414	739 033 003	1940	28 102	782 133 643	2019	3670	883 999 603	2098	7672	948 775 003
1862	8620	736 373 467	1941	3585	963 667 027	2020	29 331	919 930 243	2099	2552	999 428 203
1863	4068	802 016 587	1942	7375	788 628 787	2021	2632	852 458 443	2100	52 429	1 041 793 603
1864	38 223	789 805 867	1943	2443	825 358 867	2022	10 984	1 090 699 747	2101	2929	961 413 643
1865	2933	793 890 043	1944	64 616	860 429 707	2023	3073	846 346 603	2102	7786	912 495 907
1866	10 297	871 518 187	1945	2958	816 469 483	2024	47 318	939 157 123	2103	3845	928 158 067
1867	2347	808 549 723	1946	8485	876 865 243	2025	5485	844 401 163	2104	42 727	1 077 283 747
1868	21 622	790 530 427	1947	3984	786 517 747	2026	7563	883 008 163	2105	3153	988 264 363
1869	4032	682 597 843	1948	22 495	959 295 187	2027	2633	895 388 323	2106	13 995	1 039 850 827
1870	10 195	781 439 107	1949	2362	920 646 787	2028	36 347	960 269 083	2107	3185	950 790 523
1871	2384	749 054 923	1950	14 701	954 153 523	2029	2534	837 861 187	2108	26 723	951 732 307
1872	92 883	798 129 187	1951	2405	762 808 027	2030	11 397	977 177 947	2109	4028	1 043 380 123
1873	2359	775 854 643	1952	70 327	858 527 443	2031	3643	935 211 883	2110	9617	872 745 283
1874	7102	727 026 763	1953	4754	915 145 603	2032	60 411	843 673 147	2111	2600	1 087 676 683
1875	4490	680 566 723	1954	7471	881 019 043	2033	2633	850 748 683	2112	128 029	1 109 635 363
1876	25 655	726 805 363	1955	3418	906 060 787	2034	12 239	1 021 925 587	2113	2651	879 746 827
1877	2325	745 864 387	1956	32 302	903 338 923	2035	3565	999 940 027	2114	9216	1 027 279 507
1878	9972	729 239 347	1957	2550	785 464 843	2036	23 423	846 621 043	2115	5337	1 061 061 523
1879	2330	710 953 027	1958	8141	792 339 403	2037	4370	931 388 803	2116	25 260	956 571 883
1880	48 238	752 341 363	1959	3507	828 011 347	2038	7677	924 913 243	2117	2717	901 556 203
1881	4378	855 559 027	1960	58 586	830 773 003	2039	2542	788 241 283	2118	11 561	1 000 701 067
1882	7105	853 106 923	1961	2545	802 869 307	2040	78 053	922 778 707	2119	2806	1 056 906 547
1883	2784	853 966 843	1962	11 871	833 499 283	2041	2806	927 361 843	2120	54 454	1 036 697 803
1884	31 233	782 804 443	1963	2580	802 974 187	2042	7722	890 663 467	2121	4521	1 140 652 507
1885	3266	898 772 323	1964	22 969	862 688 443	2043	4130	890 559 763	2122	7803	974 249 203
1886	7630	762 329 107	1965	4326	883 335 667	2044	27 654	913 897 147	2123	2961	927 137 803
1887	3800	817 397 947	1966	7439	845 068 603	2045	3134	877 534 123	2124	39 084	1 131 892 507
1888	68 661	810 020 443	1967	2840	819 503 323	2046	12 400	961 852 483	2125	3635	944 512 627
1889	2289	689 394 883	1968	83 992	967 651 843	2047	2690	934 907 803	2126	7802	969 017 107
1890	16 696	769 835 203	1969	2700	790 585 027	2048	87 354	918 577 003	2127	3791	932 204 683
1891	2478	827 346 067	1970	9200	959 743 843	2049	3793	895 107 907	2128	77 248	1 060 571 803
1892	24 644	811 180 507	1971	4226	877 876 267	2050	10 135	871 744 603	2129	2601	951 484 363
1893	3396	673 356 643	1972	25 364	927 840 643	2051	3057	946 464 667	2130	14 468	1 149 537 643
1894	7063	783 558 427	1973	2484	857 667 787	2052	40 460	1 040 527 867	2131	2567	877 177 387
1895	2930	809 368 123	1974	12 573	976 285 003	2053	2502	995 576 107	2132	26 862	978 589 627
1896	55 272	894 067 003	1975	3208	840 797 563	2054	8399	925 104 667	2133	4379	943 724 203
1897	2790	736 757 107	1976	45 555	804 415 747	2055	4657	1 016 263 627	2134	8641	982 463 827
1898	7792	735 110 707	1977	3584	763 181 827	2056	42 322	987 253 507	2135	3859	871 502 923
1899	3801	790 578 427	1978	7940	1 062 350 083	2057	3007	955 614 067	2136	62 445	1 083 179 827

h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$	h	$F(h)$	$\max \lvert D \rvert$
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2138	7955	950 085 403	2217	3906	987 292 483	2296	55 840	1 143 022 483	2375	3831	1 122 730 603
2139	4173	984 899 323	2218	8267	1 010 121 907	2297	2805	990 592 003	2376	86 276	1 399 995 283
2140	30 563	997 076 347	2219	3241	1 091 181 067	2298	12 121	1 322 065 243	2377	2864	1 201 260 523
2141	2666	1 041 265 747	2220	46 615	1 203 643 123	2299	3316	1 133 017 867	2378	9337	1 443 782 803
2142	15 810	1 087 708 147	2221	2663	1 007 882 947	2300	35 629	1 243 982 083	2379	4551	1 238 654 827
2143	2683	1 022 502 427	2222	9134	1 050 833 347	2301	4503	1 152 961 987	2380	41 926	1 383 222 787
2144	77 488	980 458 147	2223	5133	1 171 968 163	2302	8561	1 047 596 587	2381	2956	1 210 209 067
2145	5631	1 217 404 147	2224	66 297	1 070 647 363	2303	3403	1 179 344 083	2382	12 616	1 275 483 763
2146	8366	1 124 597 203	2225	3501	1 112 951 107	2304	152 915	1 242 905 227	2383	2898	1 193 484 427
2147	2776	961 414 147	2226	13 965	1 157 217 163	2305	3465	1 086 976 027	2384	71 337	1 189 261 627
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2152	43 860	1 012 550 323	2231	2849	1 118 932 483	2310	19 745	1 272 359 587	2389	2947	1 170 233 227
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2161	2603	893 503 843	2240	128 135	1 193 168 827	2319	4100	1 058 941 843	2398	9786	1 150 786 267
2162	8783	1 023 616 267	2241	4696	1 306 344 187	2320	88 348	1 194 550 963	2399	2986	1 351 075 147
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2166	12 255	1 051 150 963	2245	3467	1 078 414 747	2324	31 154	1 174 455 067	2403	4878	1 397 086 387
2167	2945	1 075 515 907	2246	8278	1 087 697 227	2325	5467	1 168 445 107	2404	27 580	1 280 492 467
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2171	2989	1 081 783 987	2250	17 378	1 098 556 867	2329	3034	1 163 805 883	2408	58 203	1 353 378 643
2172	35 880	1 133 836 867	2251	2795	1 091 031 307	2330	10 811	1 139 779 243	2409	4878	1 331 259 763
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2183	2672	917 912 923	2262	13 591	1 068 585 043	2341	2802	1 272 028 123	2420	38 069	1 325 473 843
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2201	2830	1 008 424 363	2280	86 385	1 229 054 803	2359	3379	1 090 353 643	2438	9394	1 453 652 803
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2213	2689	1 108 876 147	2292	37 926	1 127 614 867	2371	2843	1 093 082 323	2450	13 825	1 492 648 243
2214	13 770	1 048 508 323	2293	2764	1 408 126 003	2372	27 367	1 218 527 083	2451	4607	1 420 475 347
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h	$F(h)$	$\max \lvert D\rvert$	h	$F(h)$	$\max \lvert D\rvert$	h	$F(h)$	$\max \lvert D\rvert$	h	$F(h)$	$\max \lvert D\rvert$
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2454	12 901	1 554 372 163	2533	3277	1 381 193 587	2612	30 045	1 570 197 883	2691	5927	1 625 453 923
2455	3772	1 288 169 227	2534	10 893	1 375 347 187	2613	4926	1 552 141 963	2692	30 704	1 575 361 027
2456	50 074	1 363 598 923	2535	6012	1 342 657 387	2614	9550	1 386 365 443	2693	3281	1 717 533 283
2457	6269	1 459 900 843	2536	51 529	1 520 668 363	2615	3951	1 304 142 907	2694	14 162	1 716 255 643
2458	8840	1 327 394 707	2537	3154	1 279 071 523	2616	76 455	1 811 303 323	2695	5160	1 674 171 283
2459	2934	1 197 908 083	2538	15 766	1 385 295 283	2617	3165	1 389 918 667	2696	54 725	1 822 248 187
2460	51 291	1 394 088 403	2539	3005	1 322 850 043	2618	12 927	1 549 993 867	2697	5035	1 700 588 563
2461	3227	1 294 227 643	2540	36 162	1 437 302 563	2619	5265	1 677 005 947	2698	10 458	1 572 324 163
2462	9048	1 330 859 083	2541	5691	1 397 473 267	2620	37 265	1 673 345 083	2699	3201	1 702 752 907
2463	4247	1 310 588 203	2542	9688	1 514 466 787	2621	3158	1 547 986 507	2700	65 274	1 845 533 827
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2465	4136	1 222 657 027	2544	110 421	1 654 470 283	2623	3314	1 438 641 667	2702	11 445	1 652 952 403
2466	14 641	1 333 763 803	2545	3774	1 419 433 723	2624	107 717	1 930 143 763	2703	5107	1 540 783 003
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2472	72 293	1 499 193 763	2551	3055	1 470 254 923	2630	12 011	1 661 329 987	2709	6359	1 615 890 907
2473	2959	1 271 620 267	2552	58 883	1 405 820 683	2631	4755	1 349 640 307	2710	12 265	1 443 183 883
2474	9300	1 396 324 483	2553	4840	1 333 855 867	2632	63 852	1 497 105 283	2711	3162	1 613 936 083
2475	6854	1 476 574 003	2554	9436	1 364 659 627	2633	3264	1 379 593 483	2712	79 241	1 561 990 747
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2489	3176	1 228 094 827	2568	75 129	1 486 588 387	2647	3165	1 407 674 227	2726	10 510	1 798 986 403
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2496	151 982	1 595 778 787	2575	3970	1 317 915 883	2654	9674	1 573 760 827	2733	4729	1 833 124 507
2497	3331	1 187 152 387	2576	93 449	1 462 952 107	2655	6684	1 547 117 203	2734	9943	1 522 462 027
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2501	3126	1 204 723 843	2580	53 698	1 493 825 587	2659	3187	1 355 186 683	2738	10 298	1 527 515 683
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2506	10 736	1 566 087 427	2585	4382	1 476 796 267	2664	87 517	1 667 378 707	2743	3582	1 686 896 803
2507	3293	1 283 203 387	2586	13 765	1 427 092 363	2665	4485	1 594 292 107	2744	66 353	1 639 260 043
2508	47 590	1 486 868 443	2587	3378	1 337 800 003	2666	10 158	1 600 759 843	2745	6637	1 614 758 203
2509	3365	1 224 983 923	2588	29 568	1 405 007 563	2667	5505	1 558 927 987	2746	10 020	1 684 252 123
2510	11 625	1 474 067 827	2589	4502	1 483 783 387	2668	33 033	1 714 145 227	2747	3381	1 620 945 643
2511	5422	1 466 847 523	2590	14 134	1 544 081 947	2669	3413	1 607 273 803	2748	45 054	1 630 170 523
2512	75 226	1 395 955 507	2591	3150	1 561 680 187	2670	17 461	1 496 091 283	2749	3396	1 883 031 187
2513	3573	1 506 020 443	2592	153 264	1 464 811 483	2671	3204	1 531 447 003	2750	14 314	1 630 122 187
2514	13 502	1 424 653 147	2593	3080	1 471 323 043	2672	80 103	1 452 773 107	2751	5616	1 623 161 467
2515	3731	1 484 366 227	2594	9450	1 642 844 443	2673	6042	1 892 928 643	2752	113 144	1 717 945 387
2516	31 046	1 460 381 323	2595	5641	1 518 657 067	2674	11 484	1 386 448 627	2753	3294	1 572 625 147
2517	4485	1 255 520 443	2596	33 159	1 440 279 163	2675	4046	1 970 364 883	2754	17 807	1 854 257 827
2518	9249	1 204 457 083	2597	3713	1 444 634 563	2676	43 922	1 629 314 107	2755	4438	1 764 428 803
2519	3379	1 334 464 363	2598	14 045	1 489 163 923	2677	3308	1 605 889 387	2756	34 669	1 724 221 843
2520	116 981	1 452 145 843	2599	3378	1 705 957 507	2678	10 769	1 635 900 883	2757	4898	1 502 672 803
2521	3149	1 308 148 363	2600	73 812	1 767 238 267	2679	5011	1 701 079 843	2758	11 925	1 674 001 123
2522	10 142	1 406 445 643	2601	5437	1 472 151 283	2680	68 782	1 613 125 243	2759	3493	1 789 441 987
2523	4584	1 483 628 323	2602	9648	1 442 945 827	2681	3805	1 442 369 923	2760	103 847	2 034 296 947
2524	28 982	1 512 412 507	2603	3307	1 593 739 123	2682	15 936	1 799 453 947	2761	3638	1 643 080 147
2525	4087	1 588 693 243	2604	51 422	1 565 153 923	2683	3280	1 695 984 427	2762	9959	1 892 462 587
2526	13 454	1 320 450 763	2605	3850	1 237 766 083	2684	34 048	1 745 027 923	2763	5368	1 716 878 923
2527	3670	1 324 156 243	2606	9576	1 404 058 363	2685	5969	1 745 571 787	2764	31 235	1 681 890 043
2528	92 635	1 543 408 267	2607	4897	1 445 139 067	2686	10 648	1 417 911 763	2765	4844	1 763 420 563
2529	4932	1 312 643 707	2608	78 686	1 596 451 123	2687	3298	1 738 656 067	2766	14 617	1 650 916 507
2530	13 035	1 513 115 563	2609	3139	1 378 891 243	2688	187 012	1 803 892 267	2767	3298	1 684 649 107
2531	3072	1 501 695 787	2610	19 721	1 473 295 987	2689	3266	1 522 829 947	2768	82 889	1 789 023 163

h	$F(h)$	$\max \lvert D\rvert$	h	$F(h)$	$\max \lvert D\rvert$	h	$F(h)$	$\max \lvert D\rvert$	h	$F(h)$	$\max \lvert D\rvert$
2769	5363	1 645 112 683	2827	3770	1 928 165 947	2885	4312	1 891 169 443	2943	5876	1 887 650 923
2770	12 694	1 591 582 843	2828	37 986	1 769 767 147	2886	16 904	2 006 350 483	2944	130 029	1 997 240 107
2771	3527	1 600 648 603	2829	5331	1 746 069 427	2887	3410	1 840 729 507	2945	4763	2 380 413 667
2772	64 526	1 842 620 707	2830	12 864	1 752 045 667	2888	61 756	1 943 108 707	2946	15 394	2 022 079 987
2773	3482	1 561 295 443	2831	3517	1 658 239 867	2889	5713	2 248 506 187	2947	4135	1 767 288 307
2774	10 827	1 755 867 787	2832	122 169	1 872 873 787	2890	13 973	1 956 748 627	2948	37 338	2 136 803 923
2775	6487	1 600 719 283	2833	3503	1 774 375 843	2891	4043	1 693 758 763	2949	5059	1 881 638 683
2776	56 334	1 861 029 307	2834	11 067	2 161 755 523	2892	47 127	2 108 665 267	2950	14 096	2 259 793 243
2777	3257	1 702 490 707	2835	8309	1 869 381 067	2893	3796	1 911 706 987	2951	3818	1 843 464 043
2778	14 806	1 625 426 707	2836	32 575	1 775 859 787	2894	10 397	2 088 593 203	2952	96 366	1 998 399 643
2779	3880	1 801 640 443	2837	3406	1 599 665 827	2895	6214	2 064 434 443	2953	3440	2 205 977 467
2780	39 378	1 673 048 227	2838	16 657	2 023 162 243	2896	87 063	1 888 263 787	2954	12 420	1 966 164 523
2781	5568	1 577 392 147	2839	3610	1 908 645 187	2897	3458	1 815 542 227	2955	6502	2 197 305 427
2782	10 977	1 745 669 347	2840	72 986	1 776 350 227	2898	20 721	2 036 627 443	2956	33 552	2 041 532 683
2783	3933	1 781 273 827	2841	4805	1 935 268 603	2899	3751	1 980 523 147	2957	3506	2 073 803 587
2784	149 805	1 768 208 227	2842	12 543	1 831 389 883	2900	43 964	2 138 211 307	2958	17 084	2 044 373 563
2785	4189	1 597 664 083	2843	3321	1 795 784 827	2901	4899	2 221 788 067	2959	3970	1 997 022 787
2786	11 702	1 856 034 643	2844	52 096	1 877 381 683	2902	10 499	1 763 931 163	2960	113 549	1 931 275 483
2787	4788	1 721 523 763	2845	4256	1 870 902 763	2903	3491	1 848 847 603	2961	6683	1 825 307 947
2788	34 574	2 055 299 563	2846	10 374	1 737 669 763	2904	93 223	2 050 396 483	2962	10 757	1 829 146 243
2789	3340	1 660 390 363	2847	5418	1 719 240 427	2905	5020	1 717 001 707	2963	3497	1 873 447 867
2790	21 172	1 795 483 363	2848	105 647	1 844 544 307	2906	10 487	1 891 183 627	2964	55 197	2 302 102 147
2791	3363	1 462 100 803	2849	4493	1 732 078 723	2907	6255	1 968 103 867	2965	4370	1 995 567 403
2792	56 827	1 767 617 707	2850	20 645	1 746 294 643	2908	32 607	1 945 076 323	2966	10 552	1 761 412 867
2793	6205	1 667 778 907	2851	3458	1 571 601 403	2909	3520	1 666 730 587	2967	5501	1 990 448 947
2794	11 407	1 575 614 203	2852	34 912	2 029 144 483	2910	19 280	1 823 499 283	2968	71 692	2 272 512 643
2795	4740	1 843 189 987	2853	5423	1 890 365 563	2911	3635	1 864 516 747	2969	3534	1 798 603 123
2796	46 055	2 010 101 923	2854	10 384	1 869 629 227	2912	134 457	1 950 615 643	2970	24 455	2 060 878 747
2797	3297	1 644 556 747	2855	4236	1 681 958 107	2913	5072	2 095 407 043	2971	3516	1 854 963 427
2798	10 158	1 774 677 403	2856	102 276	1 954 913 227	2914	11 136	1 858 385 323	2972	33 848	2 076 825 067
2799	5420	1 798 876 627	2857	3409	1 588 572 043	2915	4773	1 767 107 323	2973	4990	2 202 277 267
2800	125 902	1 791 876 403	2858	10 480	1 973 804 827	2916	55 304	2 142 904 003	2974	10 776	1 990 650 187
2801	3371	1 521 199 243	2859	4968	1 577 260 147	2917	3407	1 870 418 467	2975	5655	2 010 401 803
2802	14 915	1 764 754 363	2860	47 819	2 073 465 067	2918	10 591	2 064 222 763	2976	161 100	2 562 211 723
2803	3304	1 568 806 387	2861	3407	1 744 591 843	2919	5773	1 805 768 827	2977	3846	1 801 836 523
2804	32 018	1 664 466 763	2862	17 552	1 886 790 427	2920	74 697	1 998 855 643	2978	10 884	1 914 848 203
2805	7145	1 763 589 067	2863	3973	1 740 009 643	2921	3696	1 892 751 547	2979	5832	2 032 240 387
2806	10 865	1 646 560 507	2864	86 170	1 815 403 507	2922	15 440	2 088 639 547	2980	42 382	2 019 944 347
2807	3944	1 817 323 363	2865	6297	1 930 118 227	2923	3605	1 911 931 123	2981	3869	1 943 442 187
2808	99 874	1 923 172 843	2866	10 443	1 811 976 307	2924	36 124	2 213 560 507	2982	18 594	1 934 639 107
2809	3516	1 640 692 243	2867	3624	1 681 777 843	2925	7896	2 426 489 587	2983	3755	1 834 185 907
2810	12 794	1 952 338 147	2868	47 251	1 804 000 027	2926	14 329	2 198 089 147	2984	60 630	2 027 116 723
2811	4891	1 822 040 683	2869	3690	1 696 012 147	2927	3463	1 672 888 627	2985	6459	2 189 413 363
2812	35 072	1 851 718 003	2870	15 495	2 097 875 707	2928	127 096	2 038 340 803	2986	10 970	2 177 571 883
2813	3528	1 631 429 083	2871	6240	1 864 159 147	2929	3649	1 823 331 283	2987	3644	1 963 091 827
2814	17 656	1 727 707 507	2872	58 409	1 964 365 747	2930	13 370	1 893 804 763	2988	54 911	2 050 790 347
2815	4209	2 112 684 667	2873	3967	1 751 332 003	2931	4986	1 762 342 003	2989	4328	1 988 453 443
2816	133 772	1 720 414 243	2874	15 119	1 711 519 987	2932	33 007	2 041 071 547	2990	15 387	1 838 823 163
2817	5453	1 580 017 363	2875	4619	1 835 671 027	2933	4019	1 950 853 507	2991	5196	2 332 582 387
2818	10 371	1 762 358 203	2876	32 920	1 924 811 683	2934	17 206	2 164 277 347	2992	105 338	2 122 998 403
2819	3406	1 790 566 507	2877	5898	1 909 491 907	2935	4286	2 008 102 363	2993	3802	1 719 950 563
2820	58 629	1 978 153 963	2878	10 594	1 812 189 283	2936	59 947	2 095 227 667	2994	15 650	1 944 054 907
2821	4437	1 779 094 003	2879	3494	1 709 658 043	2937	5534	1 810 207 963	2995	4530	2 063 888 803
2822	10 875	1 739 430 883	2880	226 476	1 877 248 627	2938	11 687	2 025 832 603	2996	39 965	2 036 140 723
2823	4808	1 830 495 907	2881	3495	1 960 635 163	2939	3402	1 796 304 547	2997	6193	2 267 010 883
2824	57 720	2 014 562 587	2882	11 549	1 837 623 643	2940	70 892	1 968 637 003	2998	10 777	1 788 089 587
2825	4315	1 852 891 363	2883	5199	1 866 177 307	2941	3810	2 059 599 067	2999	3641	1 888 243 723
2826	16 492	1 727 949 763	2884	38 965	1 934 173 987	2942	10 695	1 884 647 227	3000	113 139	2 060 680 843

Declarations

Use of AI. This paper was produced by a multi-agent AI research system that the author built and ran, using Anthropic Claude Opus 5.5: apart from the work of others that it cites, the ideas, proofs, computations and text are due to the system. Its results were checked by separate agents, and the computations that its proofs use by second programs (Appendix A and [38]). The author takes full responsibility for the paper, including its correctness and the integrity and accuracy of its citations.

Data availability. The programs and data used in this paper, with instructions for rerunning them, are deposited at Zenodo [38].

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